UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Mechanics, Automatic Control and Robotics** Vol.2, No 7/2, 1997 pp. 429 - 447
Editor of series: Katica (Stevanovi}) Hedrih, e-mail: katica@masfak.masfak.ni.ac.yu
Address: Univerzitetski trg 2, 18000 Niš, YU, Tel: (018) 547-095, Fax: (018)-547-950
http:// ni.ac.yu/Facta

VECTOR ANALYSIS OF THE MASS INERTIA MOMENT STATE

UDC: 531.3

Katica (Stevanović) Hedrih

18000 - Niš, Vojvode Tankosića 3/22
Telefax: (381 18) 41-663
e-mail: katica@masfak.masfak.ni.ac.yu

Abstract. *This paper introduces the vector* $\mathrm{J}_{\vec{n}}^{(N)}$ *of the body mass inertia moment at the point* N *for the axis oriented by the unit vector* $\vec{n}$. *The vector is used for interpretation of the rigid body kinetic characteristics as well as the mass moment state in the body point. The change of the vector of the rigid body mass inertia moment is determined in the transition from one space point to another when the axis retains its orientation which represents the Huygens-Steiner theorem translated for the defined body mass inertia moment vector. Then the change of the vector of the body mass inertia moment is defined at the given point in the case of the axis changing its orientation in the way analogous to the Cauchy equations in the Elasticity theory. Then the interpretation of the main inertia asymmetry are defined. The relation between the axis deviation load vector by the body mass inertia moment for the octahedron axis and the inertia asymmetry axis is analyzed.*

1. INTRODUCTIONS

The idea for this paper appeared during my considerations of some analogies between the models in the stress theory and the strain theory of the stressed and strained deformable bodies as they are studied or as they can be studied in the Elasticity Theory. See Ref.[10], [11], [26], [21], [34] and [20]. While considering this analogy as well as the analogy between the stress tensor matrices, the relative deformation tensor-strain tensor and the body mass inertia tensor it occurred to me to introduce the concept of the vector $\vec{\delta}_{\vec{n}}^{(N)}$ of the total relative deformation-total relative strain, at the point N and for the line element drawn from that point and oriented by unit vector $\vec{n}$, as well as the concept of the vector $\vec{\mathrm{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N, and for the axis oriented by the unit vector $\vec{n}$. For more details see the references [21], [34] and [20].

Received September 16, 1997

In further consideration of the dynamic parameters of the rigid and deformable bodies as well as of the possibility of their interpretation by means of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N for the axis oriented by the unit vector $\vec{n}$, I came to the ideas and conclusions as well as interpretations given in my papers [21], [34] and [20]. The question always asked was if something like that already existed in some classic literature or not? The literature available to me which is quoted in the appendix of this paper contains no such interpretation of the rigid deformable bodies dynamic parameters by means of the mass inertia moment vector fixed to the point and to the axis.

This paper defines three dynamic vectors fixed to a certain point and axis passing through the given rigid body point. These are: the vector $\vec{M}_{\vec{n}}^{(N)}$ of the body mass at the point N for the axis oriented by the unit vector $\vec{n}$; the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ of the body mass static (linear) moment at the point N for the axis oriented by the unit vector $\vec{n}$; and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N for the axis oriented by the unit vector $\vec{n}$.

The rigid body kinetic parameters are interpreted by these vectors.

The change of the mass inertia moment vector in the transition from one rigid body point to another is determined when the axis retains its orientation which represents the modification of the Huygens-Steiner theorem expressed by means of the defined mass inertia moment vector. Then the change of the mass inertia moment vector is determined in the case of the axis changing its orientation in the way analogous to the Cauchy equations for the total stress vector in the elasticity theory. Then the interpretation of the main inertia directions are derived as well as of the main inertia asymmetry are derived. The relation between the axis deviation load vector by the material line mass inertia moment for the octahedron axis and the inertia asymmetry axis is analyzed.

Further interpretation of the kinetic parameters of the of the body by means of the body mass inertia moment vector and by means of the body mass linear (static) moment vector for the axis and the point refers to the description of the motion quantity (linear momentum) as well as motion quantity moment (angular momentum) and kinetic energy as the function of the mass moment vectors for the axis and the point and the momentary angular velocity and referential point velocity.

2. BODY MASS MOMENTS VECTORS AT POINT FOR THE AXIS

In studying the dynamics of a rigid and solid body, geometry of mass plays an important part. In the References [2] and [3] there is a conclusion that it is not necessary to know all the details about the mass distribution and the masses internal structures in order to study the rigid body translatory motion under the action of the force. The properties necessary for the study of the rigid body motion as a material system are the rigid body dynamic properties. The values determining the dynamic properties are called the rigid body dynamic parameters (See Ref. [2]).

According to the given reference these parameters are taken to be: mass M of the rigid body; position vector $\vec{\rho}_C$ of the body mass center, the point C with respect to a certain point O and $\mathbf{J}^{(C)}$ the body mass inertia moment tensor for the point C which is determined with six scalar dynamic parameters, In this way in the general case the dynamic rigid body characteristic ten independent scalar dynamic parameters are required. By means of these ten dynamic parameters of the rigid body the sixth order matrix of the following shape is formed:

$$\mathrm{J}^{(O)} = \begin{pmatrix} \mathrm{M} & 0 & 0 & 0 & \mathrm{M}z_C & -\mathrm{M}y_C \\ 0 & \mathrm{M} & 0 & -\mathrm{M}z_C & 0 & \mathrm{M}x_C \\ 0 & 0 & \mathrm{M} & \mathrm{M}y_C & -\mathrm{M}x_C & 0 \\ 0 & -\mathrm{M}z_C & \mathrm{M}y_C & \mathrm{J}_x & \mathrm{D}_{yx} & \mathrm{D}_{zx} \\ \mathrm{M}z_C & 0 & -\mathrm{M}x_C & \mathrm{D}_{xy} & \mathrm{J}_y & \mathrm{D}_{zy} \\ -\mathrm{M}y_C & \mathrm{M}x_C & 0 & \mathrm{D}_{xz} & \mathrm{D}_{yz} & \mathrm{J}_z \end{pmatrix} \tag{1}$$

and this matrix is given in the references [2] and [3] as the rigid body mass inertia matrix for the given point *O* and the given trihedron. This is the matrix of the tensor expanded in an appropriate way. The mass inertia moment matrix changes its coordinates according to the change of the reference trihedron.

In the Reference [1] the mass linear polar moment $\vec{\mathrm{M}}^{(O)}$ of the material system or the vector static system mass moment is defined with respect to the pole *O* in the form:

$$\vec{\mathrm{M}}^{(O)} = \iiint\limits_V \vec{\rho}dm = \vec{\rho}_C\,\mathrm{M} \tag{2}$$

where $\vec{\rho}$ is the vector of the rigid body points position with respect to the common pole *O*, *V* is the space region that the observed body occupies.

There are two important properties of a certain body mass: the mass center position of a material body does not depend on the pole choice but only on the body mass distribution and the mass linear polar moment $\vec{\mathrm{M}}^{(C)}$ with respect to the body mass center is equal to zero.

Since our aim is to consider a possibility of the interpretation of the rigid body dynamic parameters in a modified shape we are going to set, as a reference, the pole *O* as well as the axis oriented by the unit vector $\vec{n}$. Considering that the general case the rigid body motion can be represented by one rotation around momentary axis, that is, by the translation of the center velocity and the rotation around the axis through the given center we are led to the idea to define the rigid body dynamic parameters by means of the pole *O* as the referential point through we position an axis parallel to the momentary rotation axis.

Therefore we define the following (See Fig. No. 1a):

1* Vector $\vec{M}_{\vec{n}}^{(O)}$ of the body mass at the point *O* for the axis oriented by the unit vector $\vec{n}$:

$$\vec{M}_{\vec{n}}^{(O)} = \iiint\limits_V \vec{n}dm = \mathrm{M}\,\vec{n} \tag{3}$$

which does not depend on the mass distribution in the body, that is, on the density. For all the space points and parallel axes it has the same values and it changes only with the axis orientation change. It is determined only with the mass quantity and the axis orientation.

2* Vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point *O* for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathrm{S}}_{\vec{n}}^{(O)} = \iiint_V [\vec{n}, \vec{\rho}]dm \tag{4}$$

where $\vec{\rho}$ is the vector of the rigid body points position of the elementary body mass dm with respect to the common pole O. For the vector $\vec{\mathrm{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point O for the axis oriented by the unit vector $\vec{n}$ we can write:

$$\vec{\mathrm{S}}_{\vec{n}}^{(O)} = [\vec{n}, \vec{\rho}_C]M = [\vec{n}, \vec{M}^{(O)}] \tag{5}$$

The illustration is given in the Figure No. 1a.

3* Vector $\vec{\mathrm{J}}_{\vec{n}}^{(O)}$ of he body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathrm{J}}_{\vec{n}}^{(O)} = \iiint_V [\vec{\rho},[\vec{n}, \vec{\rho}]]dm \tag{6}$$

It can also be considered the body mass square moment vector at the point O for the axis, through the pole, oriented by the unit vector $\vec{n}$. The vector $\vec{\mathrm{J}}_{\vec{n}}^{(O)}$ at the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ can be decomposed into three components: the collinear with the axis $\mathrm{J}_{\vec{n}}^{(o)}$ and the two other ones $\mathrm{D}_{nu}^{(O)}$ and $\mathrm{D}_{nv}^{(O)}$ in the directions, $\vec{u}$ and $\vec{v}$, normal to the orientation axis $\vec{n}$. The collinear component represents the axial moment of the body mass inertia for the axis oriented by the unit vector $\vec{n}$ through the pole O. The other two components represent the deviational moments of the body mass for a couple of normal axes oriented by unit vectors $\vec{n}$ and $\vec{u}$, that is, $\vec{n}$ and $\vec{v}$:

$$\vec{\mathrm{J}}_{\vec{n}}^{(O)} = \mathrm{J}_{n}^{(O)}\, \vec{n} + \mathrm{D}_{nu}^{(O)}\, \vec{u} + \mathrm{D}_{nv}^{(O)}\, \vec{v} \tag{7}$$

The definition-expression for the body mass inertia moment vector $\vec{\mathrm{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ can be obtained starting from the expression for the axial body mass inertia moment $\mathrm{J}_{\vec{n}}^{(o)}$ for the axis oriented by unit vector $\vec{n}$ drawn through the point O and for the deviational body mass moments for the couples of the orthogonal axes oriented by unit vectors $(\vec{n},\vec{u})$ and $(\vec{n},\vec{v})$, $\mathrm{D}_{nu}^{(O)}$ and $\mathrm{D}_{nv}^{(O)}$, according to the Ref. [21], [34]. By means of them we form the vector $\vec{\mathrm{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathrm{J}}_{\vec{n}}^{(O)} = \vec{n}\iiint_V [\vec{n}, \vec{\rho}]^2 dm + \vec{u}\iiint_V ([\vec{n}, \vec{\rho}],[\vec{u}, \vec{\rho}])dm + \vec{v}\iiint_V ([\vec{n}, \vec{\rho}],[\vec{v}, \vec{\rho}])\, dm \tag{8}$$

The rigid body axial mass inertia moment is:

$$\mathrm{J}_{\vec{n}}^{(O)} = \iiint_V [\vec{n}, \vec{\rho}]^2 dm \tag{8*}$$

The rigid body mass deviation moment vector $\vec{D}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ is in the following form:

$$\vec{D}_{\vec{n}}^{(O)} = \vec{u}\iiint_V([\vec{n},\vec{\rho}],[\vec{u},\vec{\rho}])dm + \vec{v}\iiint_V([\vec{n},\vec{\rho}],[\vec{v},\vec{\rho}])dm = \vec{T}\iiint_V([\vec{T},\vec{\rho}],[\vec{n},\vec{\rho}])dm$$

$$\vec{D}_{\vec{n}}^{(O)} = \iiint_V[\vec{n},[[\vec{\rho},[\vec{n},\vec{\rho}]],\vec{n}]]dm = [\vec{n},[\vec{J}_{\vec{n}}^{(O)},\vec{n}]] \qquad (9)$$

By means of the previous expressions (8) for the vector $\vec{J}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point *O* for the axis oriented by the unit vector $\vec{n}$ we can write the expression identical to the expression (6) which has been set as a definition.

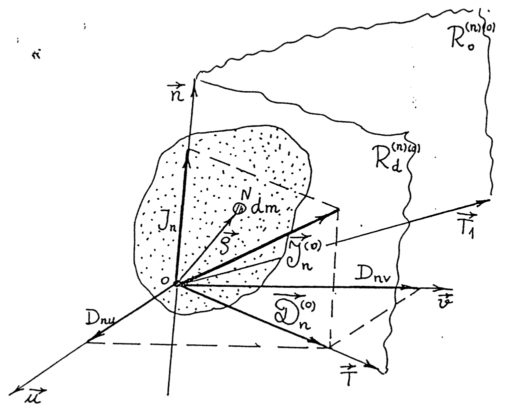

Fig. No.1a

Fig. No. 1a shows the vector $\vec{J}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point *O* for the axis oriented by the unit vector $\vec{n}$, the rigid body mass deviation moment vector $\vec{D}_{\vec{n}}^{(O)}$ at the point *O* for the axis oriented by the unit vector $\vec{n}$, the axial moment of the body mass inertia $J_{\vec{n}}^{(o)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole *O*, and the other two components, $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$, the deviational moments of the body mass for a couple of normal axes oriented by unit vectors $\vec{n}$ and $\vec{u}$, that is, $\vec{n}$ and $\vec{v}$, through the pole *O*.

Fig. No. 1b shows the vector $\vec{J}_{\vec{n}}^{(O)}$ of the material particle mass inertia moment at the point *O* for the axis oriented by the unit vector $\vec{n}$, the material particle mass deviation moment vector $\vec{D}_{\vec{n}}^{(O)}$ at the point *O* for the axis oriented by the unit vector $\vec{n}$, the axial moment of the material particle mass inertia $J_{\vec{n}}^{(o)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole *O*.

Fig. No. 1 c shows an eccentrically skewly positioned discus respect to the axis of the shaft, as well as the vector $\vec{J}_{\vec{n}}^{(O)}$ of the discus mass inertia moment at the point *O* for the axis oriented by the unit vector $\vec{n}$, the discus mass deviation moment vector $\vec{D}_{\vec{n}}^{(O)}$ at the point *O* for the axis oriented by the unit vector $\vec{n}$, the axial moment of the discus mass inertia $J_{\vec{n}}^{(o)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole *O*.

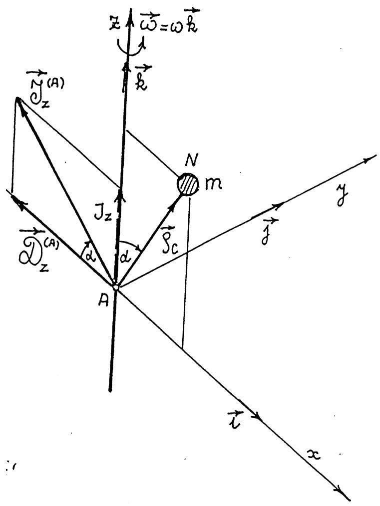

Fig. No.1 b

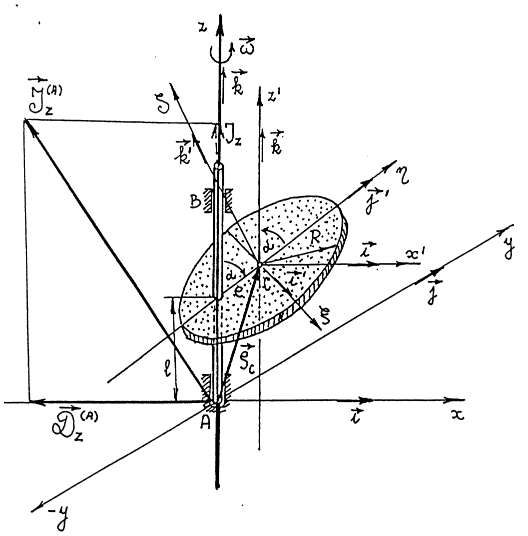

Fig. No.1c

3. THE HUYGENS-STEINER THEOREM GENERALIZED TO THE MATERIAL BODY MASS INERTIA MOMENT VECTOR FOR THE AXIS THROUGH REFERENTIAL POINT

The Figure No. 2a shows the material body and two referential points-poles O and O_1 and two parallel axes through them oriented by unit vector $\vec{n}$. The same Figure also shows the denoted elementary mass dm at the point N of the rigid body and $\vec{\rho}$ and $\vec{r}$, the position vector of that point with respect to the pole O, that is, pole O_1, as well as the position vector $\vec{\rho}_O$ of the pole O_1 with respect to pole O.

Now it is necessary to determine the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ and its relation to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O_1)}$ of the body mass inertia moment at the point O for the axis oriented by the same unit vector $\vec{n}$.

This means we are interested in the change of the body mass inertia moment vector a certain axis which moves from one point to another retaining its orientation. By using the expression (6) defining the mass inertia moment vector for a certain point and axis as well as the expression $\vec{\rho} = \vec{\rho}_O + \vec{r}$, we can write the following:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint_V [\vec{\rho}_O + \vec{r}, [\vec{n}, \vec{\rho}_O + \vec{r}]] dm = \vec{\mathfrak{J}}_{\vec{n}}^{(O_1)} + [\vec{\rho}_O, \vec{\mathfrak{S}}_{\vec{n}}^{(O_1)}] + [\vec{\mathfrak{M}}_C^{(O_1)}, [\vec{n}, \vec{\rho}_O]] + [\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]M \quad (10)$$

We see that all the members in the last expression have the same structures.

In the case when the pole O_1 is the center C of the body mass the vector $\vec{r}_C$ (the position vector of the masses center with respect to the pole O_1) is equal to zero, whereas the vector $\vec{\rho}_O$ turns into $\vec{\rho}_C$ so that the last expression (10) can be written in the following form (See Figure No. 2 b):

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C,[\vec{n},\vec{\rho}_C]]\mathrm{M} \tag{11}$$

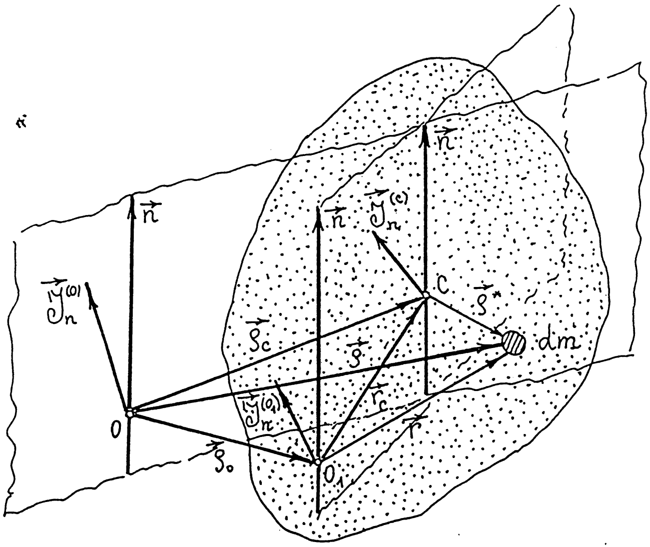

Fig. No. 2a

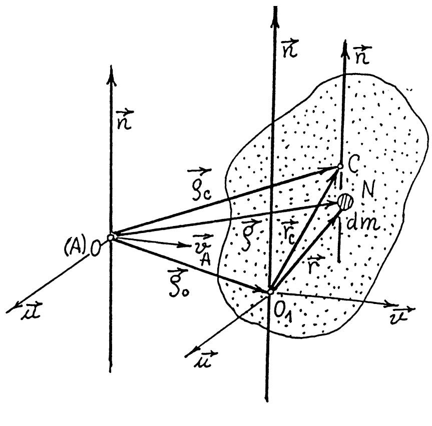

Fig. No. 2b

This expression represents the generalized Huygens-Steiner theorem with respect to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point *O* for the axis oriented by the same unit vector $\vec{n}$ passing trough the mass center *C* and any other point *O*.

The vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the body mass inertia moment for the body mass center *C* as well as for the axis oriented by unit vector $\vec{n}$ passing trough the mass center *C* we are going to call the central or proper (eigen, personal) vector of the body mass inertia moment for the axis oriented by unit vector $\vec{n}$.

The part $\vec{\mathfrak{J}}^{(O)}_{\vec{n},position} = [\vec{\rho}_C,[\vec{n},\vec{\rho}_C]]\mathrm{M}$ from the expression (11) represents the position part of the body mass inertia moment vector and we going to call it *the body mass inertia position moment vector for the point O and the axis oriented by unit vector $\vec{n}$ in relation to the body mass center C*. We can see that the body mass inertia moment vector for the axis trough the mass center C is the "smallest" vector since for all the other parallel axes the position part $\vec{\mathfrak{J}}^{(O)}_{\vec{n},position} = [\vec{\rho}_C,[\vec{n},\vec{\rho}_C]]\mathrm{M}$ has to be taken into consideration. This can be expressed by means of the vector $\vec{\mathfrak{S}}^{(O)}_{\vec{n}}$ of the body mass linear moment for the point O and the axis oriented by unit vector $\vec{n}$ in the form $[\vec{\rho}_C, \vec{\mathfrak{S}}^{(O)}_{\vec{n}}]$.

4. THE CHANGE OF THE BODY MASS INERTIA MOMENT VECTOR FOR THE POINT AND AXIS ORIENTATION CHANGE THROUGH THE REFERENTIAL POINT

Let's now define the vectors $\vec{\mathfrak{J}}^{(O)}_x$, $\vec{\mathfrak{J}}^{(O)}_y$ and $\vec{\mathfrak{J}}^{(O)}_z$ of the body mass inertia moments at the point O and for the coordinate axes O_x, O_y and O_z. These vectors can be expressed in the form:

$$\vec{\mathfrak{J}}^{(O)}_x = \iiint\limits_V [\vec{\rho},[\vec{i},\vec{\rho}]]\,dm \quad \vec{\mathfrak{J}}^{(O)}_y = \iiint\limits_V [\vec{\rho},[\vec{j},\vec{\rho}]]\,dm \quad \vec{\mathfrak{J}}^{(O)}_z = \iiint\limits_V [\vec{\rho},[\vec{k},\vec{\rho}]]dm \tag{12}$$

If we denote the senses cosine of the unit vector $\vec{n}$ with $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ when the unit vector defines the orientation of the axis passing though the point O, then we can successively multiply the expressions (12) and we obtain them added:

$$\vec{\mathfrak{J}}^{(O)}_x \cos\alpha + \vec{\mathfrak{J}}^{(O)}_y \cos\beta + \vec{\mathfrak{J}}^{(O)}_z \cos\gamma = \iiint\limits_V [\vec{\rho},[\vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma,\vec{\rho}]\,dm = \iiint\limits_V [\vec{\rho},[\vec{n},\vec{\rho}]]\,dm$$

From the previous expression we conclude that the body mass inertia moment vector $\vec{\mathfrak{J}}^{(O)}_{\vec{n}}$ at the point O for the axis oriented by the unit vector $\vec{n}$ is equal to:

$$\vec{\mathfrak{J}}^{(O)}_{\vec{n}} = \vec{\mathfrak{J}}^{(O)}_x \cos\alpha + \vec{\mathfrak{J}}^{(O)}_y \cos\beta + \vec{\mathfrak{J}}^{(O)}_z \cos\gamma \tag{13}$$

The last expression is analogous to the equation for determining the total stress vector $\vec{p}^{(O)}_{\vec{n}}$ at the point O of the stressed body for the plane with normal unit vector $\vec{n}$ which is known as the Cauchy equation in the elasticity theory. There fore we are going to call it the Cauchy equation giving the relation of the body mass inertia moment vector $\vec{\mathfrak{J}}^{(O)}_{\vec{n}}$ at the point O for the axis oriented by the unit vector $\vec{n}$ and the vectors $\vec{\mathfrak{J}}^{(O)}_x$, $\vec{\mathfrak{J}}^{(O)}_y$ and $\vec{\mathfrak{J}}^{(O)}_z$ of the body mass inertia moments at the point O and for the coordinate axes O_x, O_y and O_z.

5. CAUCHY EQUATIONS IN THE MATRIX FORM

Now by means of the mass inertia moment tensor matrix $\mathbf{J}^{(O)}$ the Cauchy vector equation (13) can be written in the matrix form:

$$\{J_{\vec{n}}^{(O)}\} = (\{J_x^{(O)}\}\{J_y^{(O)}\}\{J_z^{(O)}\})\{n\} = \mathbf{J}^{(O)}\{n\} \tag{14}$$

Now for the body mass axial inertia moment $\mathrm{J}_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$, as well as for the body mass deviation moment we can write by means of the mass inertia moment $\mathrm{D}_{\vec{n}\vec{v}}^{(O)}$ for the orthogonal axes $\vec{n}$ and $\vec{v}$ we can write the following expressions:

$$\mathrm{J}_{\vec{n}}^{(O)} = (n)\{J_{\vec{n}}^{(O)}\} = (n)\mathbf{J}^{(O)}\{n\} \quad , \quad \mathrm{D}_{\vec{n}\vec{v}}^{(O)} = (v)\{J_{\vec{n}}^{(O)}\} = (v)\mathbf{J}^{(O)}\{n\} \tag{15}$$

The invariants of the body mass inertia moment state at a certain point can be determined as the first $\mathrm{J}_1^{(O)}$, second $\mathrm{J}_1^{(O)}$ and third $\mathrm{J}_1^{(O)}$ scalar of the body mass inertia moment tensor matrix.

The rigid body mass inertia moment tensor matrix $\mathrm{J}^{(O)}$ for a certain pole can be separated into two matrices corresponding to the spherical $\mathrm{J}^{(O)sph}$ and deviational $\mathbf{J}^{(O)dev} = \mathbf{D}^{(O)dev}$ part of the rigid body mass inertia moment tensor (which is analogous to the stress tensor matrix and strain (relative deformation) tensor matrix in the elasticity theory):

$$\mathbf{J}^{(O)sph} = \frac{1}{3}\mathrm{J}_1^{(O)}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\mathrm{J}_1^{(O)} & 0 & 0 \\ 0 & \frac{1}{3}\mathrm{J}_1^{(O)} & 0 \\ 0 & 0 & \frac{1}{3}\mathrm{J}_1^{(O)} \end{pmatrix} \tag{16}$$

$$\mathbf{J}^{(O)dev} = \mathbf{J}^{(O)} - \mathbf{J}^{(O)sph} = (\{J_{\vec{n}}^{(O)}\}\{J_{\vec{u}}^{(O)}\}\{J_{\vec{v}}^{(O)}\}) - \frac{1}{3}\mathrm{J}_1^{(O)}\mathrm{I} \tag{17}$$

6. AXIAL AND DEVIATIONAL PART OF THE RIGID BODY MASS INERTIA MOMENT VECTOR

The body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ can be written in the transformed form in which we separate the part $\vec{\mathfrak{J}}_{\vec{n}}^{(O)aks}$ collinear with axis oriented by unit vector $\vec{n}$ and the part $\vec{D}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)dev}$ normal to the axis oriented by unit vector $\vec{n}$ as it is shown in the Figure No. 1a and No. 3.

Now the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ can be transformed to the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)aks} + \vec{\mathfrak{J}}_{\vec{n}}^{(O)dev} = \vec{n}(\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}) + [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] = \vec{\mathfrak{J}}_{\vec{n}}^{(O)aks} + \vec{D}_{\vec{n}}^{(O)} \tag{18}$$

with components:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)aks} = \vec{n}(\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}) = \vec{n}\,\mathrm{J}_{\vec{n}}^{(O)} \tag{19}$$

$$\vec{D}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)dev} = [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] \tag{20}$$

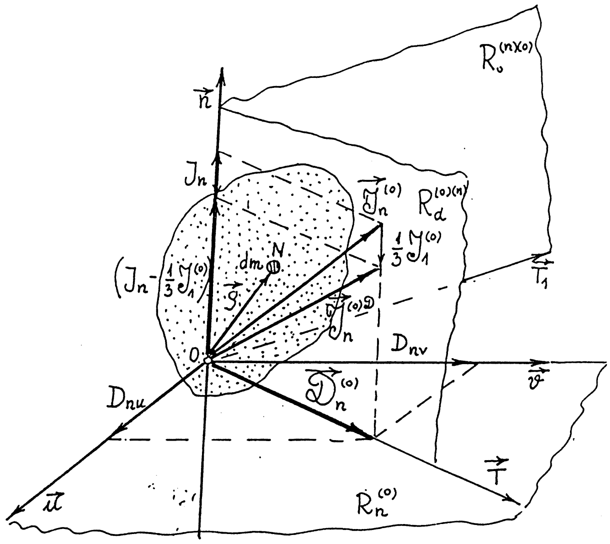

Fig. No. 3

The first part $\vec{\mathfrak{J}}_{\vec{n}}^{(O)aks}$ collinear with axis oriented by unit vector $\vec{n}$ given by formula (19) represents body mass axial inertia moment vector at the point and for the axis oriented by unit vector $\vec{n}$, and it does not depend on the pole position on the axis.

The second part $\vec{D}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)dev}$ normal to the axis oriented by unit vector $\vec{n}$ given by formula (20) lies in the plane formed by the axis oriented by unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment. This plane is determined by the axis selection and by the body mass distribution with respect to the axis and the pole.

The vector $\vec{D}_{\vec{n}}^{(O)}$ is the deviation load by the rigid body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ and it can be defined as the rigid body mass inertia moment vector component normal to the axis and in the plane which is formed by the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment. This can be seen in the figure No. 1a and No. 3. We conclude that the vector magnitude is equal to the deviation moment of the body mass for the axis oriented by the unit vector $\vec{n}$ and the axis oriented by the unit vector $\vec{T}$ normal to the axis oriented by the unit vector $\vec{n}$, in the direction of the cutting line of the plane normal to the axis trough the pole O and of the plane formed by the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the pole and for axis oriented by the unit vector $\vec{n}$. The unit vector of this cutting line is denoted with $\vec{T}$. The unit vector normal to the unit vectors $\vec{n}$ and $\vec{T}$ is denoted with $\vec{T}_1$. We conclude that the body mass deviation moment for the axes $\vec{n}$ and $\vec{T}_1$ passing through the pole O is equal to zero. This means that for an arbitrary axis at the observed point O there can always be found at least one axis normal to it oriented by $\vec{T}_1$ for which, together with the axis oriented by the unit vector $\vec{n}$, the body mass deviation moment is equal to zero. This axis is normal to the axis oriented by the unit vector $\vec{n}$ and to *the deviation plane* formed by the unite vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$. The deviaton plane we denote by R_d. Only for the mass inertia moment main axis through a retain point-pole the deviation plane is not defined nor it can be said it exists since if the axis oriented by the unit vector $\vec{n}$ through a certain point is the main axis of the body mass inertia moment then for this axis the deviation load to the axis is equal to zero. In this case the body mass inertia moment vector has only one component collinear with the axis. That is, if a certain axis through a certain point-pole is the main mass inertia moment than the vector of its deviation load by the body mass inertia moment is equal to zero.

7. Spherical and deviatorial part of the rigid body mass moment vector

If we now follow the idea of the formation of matrices of the spherical and deviatorial part of the inertia tensor according to the analogy (See Ref. [21], [20] and [34]) with the spherical and deviatorial part of the stress tensor, that is, of the relative deformation (strain) tensor we can define two vectors (See Figure No.3):

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)sph}$ the vector spherical part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)sph} = \frac{1}{3}\mathrm{J}_1^{(O)}\vec{n} = \frac{1}{3}\mathrm{J}_O^{(O)}\vec{n} \tag{21}$$

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)D}$ the vector deviatoriall part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)D} = \vec{n}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}) - \frac{1}{3}\mathrm{J}_1^{(O)}\vec{n} + [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] = \vec{n}\left\langle(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}) - \frac{1}{3}\mathrm{J}_1^{(O)}\right\rangle + \vec{D}_{\vec{n}}^{(O)} \tag{22}$$

Let's now consider the modification of the Huygens-Steiner theorem in its application to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)dev} = \vec{D}_{\vec{n}}^{(O)}$ the deviation part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$, as well as the vector of the deviation load by the rigid body mass inertia moment on the axis oriented by the unit vector $\vec{n}$ in the transition from the mass center C to the pole O (See Figure No. 2 b). We use the definition of the vector $\vec{D}_{\vec{n}}^{(O)}$ of the deviation load by the mass inertia moment (18) and the formula (11) derived in the paragraph for the Huygens-Steiner formula of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$ so that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)dev} = \vec{D}_{\vec{n}}^{(O)} = [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] = \vec{D}_{\vec{n}}^{(C)} - (\vec{n},\vec{\rho}_C)[\vec{n},[\vec{\rho}_C,\vec{n}]]\mathrm{M} \tag{23}$$

The expression (23) represents the Huygens-Steiner Theorem modified to the vector $\vec{D}_{\vec{n}}^{(O)}$ of the deviation load by the mass inertia moment of the axis oriented by the vector

$\vec{n}$ connected to the pole O. From this expression we conclude that the vector $\vec{D}_{\vec{n}}^{(O)}$ of the axis deviational load through an arbitrary point O oriented by the unit vector $\vec{n}$ equal to the sum of the vector $\vec{D}_{\vec{n}}^{(C)}$ of the axis deviation load through the center C of the body mass for the parallel axis and the position deviation load in the transition of the axis from the pole C - mass center to the pole- arbitrary point O determined from the expression:

$$\vec{D}_{\vec{n}}^{(C\to O)} = [\vec{n},[[\vec{\rho}_C,[\vec{n},\vec{\rho}_C]]\vec{n}]]\mathrm{M} = -(\vec{n},\vec{\rho}_C)[\vec{n},[\vec{\rho}_C,\vec{n}]]\mathrm{M} \tag{24}$$

If the pole O and the center C of the body mass are located on the same normal to the axis oriented by the unit vector $\vec{n}$ then the position part of the deviation load in the transition from the axis through the mass center C to the parallel axis through the pole O is equal to zero. This means that the deviation load vectors of the axis by the body mass inertia moment for the central plane points corresponding to the given axis are equal to the deviation load belonging to the central axis $\vec{D}_{\vec{n}}^{(C)}$.

8. MAIN MASS INERTIAL MOMENT DIRECTIONS, MAIN MASS INERTIA MOMENT VECTORS

By means of the vector $\vec{\mathrm{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$ we can introduce a new definition of the main inertia axes. Through one pole O we can draw an infinite number of axes of orientations. Among them we are looking for the axis for which the vector $\vec{\mathrm{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment had only one component, collinear with the axis, that is, the one for which the vector $\vec{D}_{\vec{n}}^{(O)}$ of the deviation load of the axis by the body mass inertia moment is equal to zero. Using the analogy given in the papers [21] and [34] as well as the analogy with the matrix interpretation from books [11], [10] , [26] and [20] as more appropriate for this case and by denoting the unit vector of the main inertia axis orientation with $\vec{n}_s$ which is in accordance with the Fig. No. 3, we can write:

$$\{\mathrm{J}_{\vec{n}_s}^{(O)}\} = \mathbf{J}^{(O)}\{n_s\} = \mathrm{J}_{\mathrm{s}}^{(O)}\{n_s\} \quad \Rightarrow \quad (\mathbf{J}^{(O)} - \mathrm{J}_{\mathrm{s}}^{(O)}\,\mathrm{I})\{n_s\} = \{0\} \tag{25}$$

so that the Hamilton equation for determining the main mass inertia moments is:

$$f(\mathrm{J}_{\mathrm{s}}^{(O)}) = \left|\mathbf{J}^{(O)} - \mathrm{J}_{\mathrm{s}}^{(O)}\,\mathrm{I}\right| = 0 \tag{26}$$

while for the senses cosines of the main mass inertia moment axes the following relations are obtained:

$$\frac{\cos\alpha_s}{\mathbf{K}_{31}^{(s)}} = \frac{\cos\beta_s}{\mathbf{K}_{32}^{(s)}} = \frac{\cos\gamma_s}{\mathbf{K}_{33}^{(s)}} = C_s \qquad \cos^2\alpha_s + \cos^2\beta_s + \cos^2\gamma_s = 1 \tag{27}$$

where $\mathbf{K}_{3k}^{(s)}, k = 1,2,3$ are co-factors of the third kind elements and the corresponding matrix column, successively for the roots $\mathbf{J}_s^{(O)}$, $s = 1,2,3$ of the Hamilton equation (26), which are the main mass inertia moments and which represent the axial mass inertia moments for the main mas inertia moments axes. There are three roots and three orthogonal main axes at every point with respect to which the rigid mass inertia moment vectors are determined. The Hamilton equation coefficients are the first, second and third invariants of the mass inertia moment state at referent point, and they are the first, second and third scalar of the body mass inertia moment tensor matrix at referent point (see Ref. [21] or [20]).

9. Extreme values of the mass deviation moments

The Ref. [21] gives an analogy between the stress state model, the strain state model and the mas inertia moment state of the body at the observed body point. For determining the mass deviation moments extreme values we shall use this analogy which exists between the stress tensor, the strain tensor and the body mass inertia tensor, as well as between the vector $\vec{p}_{\vec{n}}^{(O)}$ of the total stress at a certain body point for the plane with the normal oriented by unit vector $\vec{n}$, the vector $\vec{\delta}_{\vec{n}}^{(O)}$ of the total strain (relative deformation) of the line element drawn from the observed point in the direction of the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the observed pole for the axis oriented by unit vector $\vec{n}$.

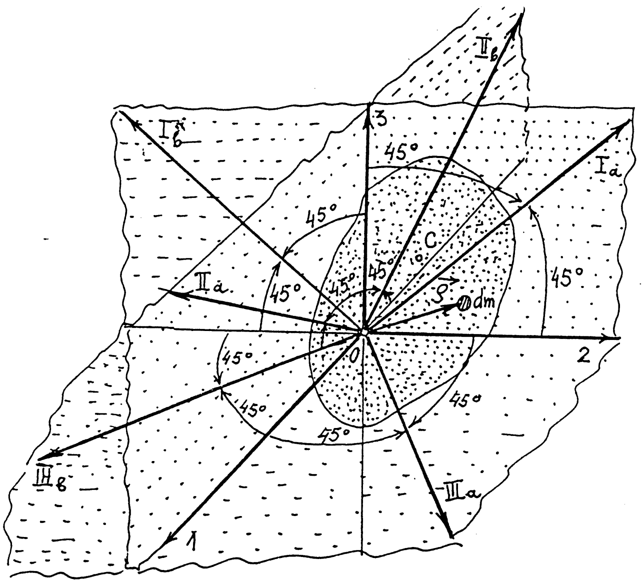

Fig. No. 4 a

On the basis of the given analogy in the quoted References [21] and [20], the following conclusions are drawn, though without proofs: on the basis of the analogy between the mass deviation moments extreme values for a couple of orthogonal axes (that is, of the mass centrifugal moments) and yield stress extreme values in the orthogonal planes that pass in pair through one main stress direction and form an angle of 45^0 with the other two main stress direction, we conclude that the mass deviation moments extreme values appear for the axes pairs I_a and I_b, II_a and II_b, III_a and III_b that pass in pairs through the main body mass inertia moment axis trough the given point and form angles of 45^0 with the other two main mass inertia moment axes (see Figure No. 4 a and No. 4 b). For these pairs of the defined axes the mass deviation moments (the mass centrifugal moments) are equal to the semi-difference between the two main (axial)body mass inertia moment and for each axis in the corresponding pair the axial inertia moments are equal to the semi-sum of the two corresponding main moments of the body mass inertia for the given point.

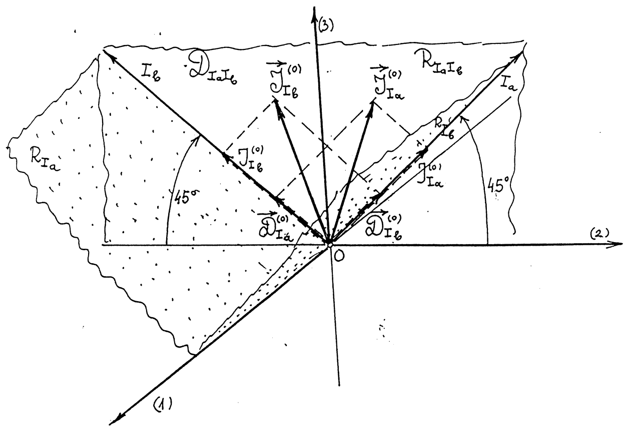

Figure No.4b

The pairs of these coupled axes are the body mass inertia moments asymmetry axes since for them the mass centrifugal moments are extreme values and the axial mass inertia moments for both the axes in pair are mutually equal. The concept of "asymmetry" can be accepted since for symmetry axes the body mass centrifugal moment is equal to zero and for these axes the body mass centrifugal moment is of extreme value so that this leads to the conclusion about the asymmetry of the material body mass inertia moment properties. On the basis of the given analogy we cam write the values of the mass deviation moments and the body mass axial inertia moments of these axes (see Figure No. 4 a and No. 4 b):

$$\begin{aligned} &\mathrm{D}^{(O)}_{I_aI_b} = \pm\frac{1}{2}(\mathrm{J}^{(O)}_2 - \mathrm{J}^{(O)}_3) \qquad \mathrm{J}^{(O)}_{I_a} = \mathrm{J}^{(O)}_{I_b} = \frac{1}{2}(\mathrm{J}^{(O)}_2 + \mathrm{J}^{(O)}_3) \\ &\mathrm{D}^{(O)}_{II_aII_b} = \pm\frac{1}{2}(\mathrm{J}^{(O)}_2 - \mathrm{J}^{(O)}_3) \qquad \mathrm{J}^{(O)}_{II_a} = \mathrm{J}^{(O)}_{II_b} = \frac{1}{2}(\mathrm{J}^{(O)}_1 + \mathrm{J}^{(O)}_3) \\ &\mathrm{D}^{(O)}_{III_aIII_b} = \pm\frac{1}{2}(\mathrm{J}^{(O)}_1 - \mathrm{J}^{(O)}_2) \qquad \mathrm{J}^{(O)}_{III_a} = \mathrm{J}^{(O)}_{III_b} = \frac{1}{2}(\mathrm{J}^{(O)}_1 + \mathrm{J}^{(O)}_2) \end{aligned} \tag{28}$$

In the coordinate system of the main body mass inertia directions $\vec{n}_s$, $s = 1,2,3$ the vectors $\vec{\mathfrak{J}}^{(O)}_{\vec{n}_s}$, $s = 1,2,3$ for the referential point as the pole are the body mass inertia moment vectors for the main mass inertia moment axes and we see that they have only the components collinear with the corresponding main mass inertia moment axes $\vec{\mathfrak{J}}_{\vec{n}_s}^{(O)} = \mathrm{J}_s^{(O)}\vec{n}_s$, $s = 1,2,3$.

Let's now define the vectors $\vec{\mathfrak{J}}_{I_a}^{(O)}$, $\vec{\mathfrak{J}}_{II_a}^{(O)}$ and $\vec{\mathfrak{J}}_{III_a}^{(O)}$ of the body mass inertia moment at the observed point for the axis oriented by the unit vector $\vec{n}_{I_a}$, or $\vec{n}_{II_a}$ or $\vec{n}_{III_a}$ of the mass inertia moment asymmetry axis I_a or II_a or III_a by using the definition of this vector so that we have (see Figure No. 4 b):

$$\vec{\mathfrak{J}}_{I_a}^{(O)} = \frac{\sqrt{2}}{2}\left\langle \vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)} \right\rangle;\quad \vec{\mathfrak{J}}_{II_a}^{(O)} = \frac{\sqrt{2}}{2}\left\langle \vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)} \right\rangle;\quad \vec{\mathfrak{J}}_{III_a}^{(O)} = \frac{\sqrt{2}}{2}\left\langle \vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} \right\rangle \quad (29)$$

Let's now define the vectors $\vec{\mathfrak{J}}_{I_b}^{(O)}$, $\vec{\mathfrak{J}}_{II_b}^{(O)}$ and $\vec{\mathfrak{J}}_{III_b}^{(O)}$ of the body mass inertia moment at the observed point for the axis oriented by the unit vector $\vec{n}_{I_b}$ or $\vec{n}_{II_b}$ or $\vec{n}_{III_b}$ of the mass inertia moment asymmetry axis I_b or II_b, or III_b by using the definition of this vector so that we have:

$$\vec{\mathfrak{J}}_{I_b}^{(O)} = \frac{\sqrt{2}}{2}\left\langle -\vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)} \right\rangle;\quad \vec{\mathfrak{J}}_{II_b}^{(O)} = \frac{\sqrt{2}}{2}\left\langle -\vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)} \right\rangle;\quad \vec{\mathfrak{J}}_{III_b}^{(O)} = \frac{\sqrt{2}}{2}\left\langle -\vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} \right\rangle \quad (30)$$

Now we define the components of the vector $\vec{\mathfrak{J}}_{I_b}^{(O)}$. The collinear one with the body mass inertia moments symmetry axis $\boldsymbol{I_a}$:

$$(\vec{\mathfrak{J}}_{I_a}^{(O)}, \vec{n}_{I_a}) = \frac{\mathrm{J}_2^{(O)} + \mathrm{J}_3^{(O)}}{2} = \mathrm{J}_{I_a}^{(O)} = \mathrm{J}_{I_b}^{(O)} \quad (31)$$

The component normal to the body mass inertia moment asymmetry axis lying in the deviation plane representing the vector $\vec{D}_{I_a}^{(O)}$ of the deviation load by the body mass inertia moment of the mass inertia moment asymmetry axis according to the previously given definition in the form:

$$\vec{D}_{I_a}^{(O)} = [\vec{n}_{I_a}, [\vec{\mathfrak{J}}_{I_a}^{(O)}, \vec{n}_{I_a}]] = \frac{\mathrm{J}_2^{(O)} - \mathrm{J}_3^{(O)}}{2}\vec{n}_{I_b} = \mathrm{D}_{I_a I_b}^{(O)}\,\vec{n}_{I_b} \quad (32)$$

Analysis the expressions from (28) to (32) we conclude the following:

1* The expressions given in (28) on the analogy basis are correct;

2* Both the vectors $\vec{\mathfrak{J}}_{I_a}^{(O)}$ and $\vec{\mathfrak{J}}_{I_b}^{(O)}$ of the rigid body mass inertia moment for the pole O and the axis of the pair I of the mass inertia moment asymmetry, I_a and I_b are normal to the main mass inertia moment axis (1) and they lie in the plane $R_{I_aI_b}$ which is their mutual deviation plane. This plane is normal to the main mass inertia moment axis (1) and contains the other two main mass inertia moment directions (2) and (3);

3* The vector $\vec{D}_{I_a}^{(O)}$ of the deviation load by the body mass inertia moment of the mass inertia moment asymmetry axis oriented by unit vector $\vec{n}_{I_a}$ at given point lie in the direction of the second mass inertia moment asymmetry axis oriented by unit vector $\vec{n}_{I_b}$

of the pair I which is normal to the main mass inertia moment direction (1) and to the axis the mass inertia moment asymmetry I_a and vice versa. These two vectors, that is, $\vec{\mathfrak{D}}_{I_a}^{(O)}$ and $\vec{\mathfrak{D}}_{I_b}^{(O)}$, are the same magnitude and of the same components, of axial and deviational, and they have the same axial mass inertia moments. In a similar way the calculation can be applied to the other two pairs of the mass inertia moment asymmetry axes and the corresponding conclusions can be drawn in accordance with the expressions (28) and the previous conclusions.

10. MASS INERTIA MOMENT VECTORS FOR THE OCTAHEDRON DIRECTIONS IN THE REFERENTIAL POINT

In analogy with defining the octahedron directions a certain point of the stressed and strained body as it is done in the elasticity or plasticity theory we shall define the octahedron directions at a certain point of the rigid body form the viewpoint of the body mass inertia moment state with respect to this pole as the direction that forms the same angles with the main inertia axes, that is, with the main inertia directions. There are eight such octahedron directions.

The vector $\vec{\mathfrak{J}}_{oct}^{(O)}$ of the mass inertia moment at the point O for the octahedron direction by using the basic definition is calculated as:

$$\vec{\mathfrak{J}}_{oct}^{(O)} = \iiint\limits_V [\vec{\rho},[\vec{n}_{oct},\vec{\rho}]]dm = \frac{\sqrt{3}}{3}(\vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)}) \tag{33}$$

and we can decompose it into two components.

1* The axial component in the octahedron direction:

$$\mathrm{J}_{n_{oct}}^{(O)} = (\vec{n}_{oct}, \vec{\mathfrak{J}}_{oct}^{(O)}) = \frac{1}{3}\mathrm{J}_1^{(O)} = \frac{2}{3}\mathrm{J}_O^{(O)} \tag{34}$$

which represents the axial moment of the rigid body mass inertia moment for the octahedron direction axis for the given pole and it is equal to one third of the first mass inertia moment invariant or one third of the first scalar of the mass inertia polar moment for the pole O.

2* Normal component to the octahedron direction which is equal to the vector $\vec{\mathfrak{D}}_{oct}^{(O)}$ of the octahedron axis deviation load, by the body mass inertia moment and has the form:

$$\vec{\mathfrak{D}}_{oct}^{(O)} = -\frac{2\sqrt{6}}{9}(\vec{\mathfrak{D}}_{I_a}^{(O)} + \vec{\mathfrak{D}}_{II_a}^{(O)} + \vec{\mathfrak{D}}_{III_a}^{(O)}) \tag{35}$$

The vector $\vec{\mathfrak{D}}_{oct}^{(O)}$ of the deviation load by the body mass inertia moment of the octahedron axis can be expressed as the linear combination of the vectors $\vec{\mathfrak{D}}_{I_a}^{(O)}, \vec{\mathfrak{D}}_{II_a}^{(O)}, \vec{\mathfrak{D}}_{III_a}^{(O)}$ of the deviation load of the mass inertia moments asymmetry axes when it is related to one of the pair.

The intensity square of the vector $\vec{\mathfrak{D}}_{oct}^{(O)}$ of the deviation load by the body mass inertia moment of the octahedron axis can be defined by the following expression:

$$\left|\vec{D}_{oct}^{(O)}\right|^2 = \frac{4}{9}\left(\left|\vec{D}_{I_a}^{(O)}\right|^2 + \left|\vec{D}_{II_a}^{(O)}\right|^2 + \left|\vec{D}_{III_a}^{(O)}\right|^2\right) \tag{36}$$

It should be noted that there are eight axes (or four axes) at each point of the rigid body for which the inertia axial moments are equal to a third of the first mass inertia moment invariant and they are the octahedron directions determined with respect to the main mass inertia moment axes. The question shousld be asked about what sort of motion the body performs while rotating around the octahedron axis and if the conclusions can be generalized to hold for the bodies with different mass inertia moment characteristics.

If this conclusion is related to the previous section we can conclude that is: one set of eight (or four) axes for which the axial inertia moments of the body mass are mutually equal and equal to a third of the first mass inertia moment invariant: Three sets of two pairs of orthogonal axes of the inertia asymmetry for the axial inertia moments are also equal to the semi-sum of two main inertia moments each. The same stand for each body and for each pole chosen within the space or outside the space of the rigid body. Only the spherical body as the pole of all fourteen axes the axial mass inertia moment is the same and the deviation load is equal to zero.

Acknowledgment: This research was supported by Science Ministry of Serbia, grant number 0402A through Mathematical Institute SANU Belgrade. The author is grateful to professor dr Veljko Vujičić for his valuable suggestions.

REFERENCES

1. Andjelić, T. i Stojanović, R., (1965), *Racionalna mehanika*, Zavod za izdavanje udžbenika, Beograd.
2. Bilimović, A., (1955), *Dinamika čvrstog tela*, SANU, Beograd.
3. Bilimović, A., (1951), *Racionalna mehanika II*, Mehanika sistema, Naučna knjiga, Beograd.89.
4. Br~i}, V., (1975), *Teorija konstrukcija*, Gra|evinska kwiga, Beograd
5. Goldstein, H., (1980), *Classical Mechanics*, Addison-Wesley Publishing Company, Massachusetts.
6. Pars, L.A., (1971), *Treatise on Analytical Dynamics*, (in russian Language), Nauka, Moscow.
7. Rašković, D., (1972), *Dinamika*, Naučna knjiga, Beograd.
8. Rašković, D., (1966), *Mehanika - Kinematika*, Zavod za izdavanje udžbenika Srbije, Beograd.
9. Rašković, D., (1974), *Analitička mehanika*, Mašinski fakultet, Kragujevac.
10. Rašković, D., (1971), *Osnovi matričnog računanja*, Naučna knjiga, Beograd.
11. Rašković, D. (1974), *Teorija elastičnosti*, Naučna knjiga, Beograd.
12. Vujičić, V.A.,(1990), Dynamics of Rheonomic Systems, Math. Inst. Beograd.
13. Vujičić, V., (1962) *Kretanje dinamički promenljivih objekata i njihova stabilnost*, doktor. dis. Beograd.
14. Vujičić, V., (1984) *O samorotaciji nebeskih tela*, Jugoslovensko društvo za mehaniku,XVI Jugoslovenski kongres Teorijske i primenjene mehanike, Bečići, A Opšta mehanika, - A1-1, 233-238.
15. Vujičić, V., (1964) *O jednom pitanju obrtanja tela promenljive mase*, Matematički vesnik,1 (16), sv. 2, 119-126.
16. Vujičić, V., (1987) *The tensotial Equations of self-rotation of celestial bodies*, Tensor, N. S., vol. 44, 96-102.
17. Wittenburg, J., (1977), *Dynamics of Systems of Rigid Bodies*, B.G. Tebner Stittgart.
18. Hedrih (Stevanović), K., (1992), *On Some interpretations of the rigid bodies kinetic parameters*, XVIII th ICTAM HAIFA 1992, Abstracts.
19. Hedrih (Stevanović), K., (1993) *Same vectorial interpretations of the kinetic parameters of solid material lines*, ZAMM. Angew.Math. Mech. 73, 4-5, T153-T156.
20. Hedrih (Stevanović), K., (1991) *Zbirka rešenih zadataka iz Teorije elastičnosti sa prilozima*, Naučna knjiga, Beograd, Prilog.
21. Hedrih (Stevanović), K., (1991), *Analogy between models of stress state, strain state and state of moment inertia mass of body*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics",

Niš, vol.1, N 1, 1991, pp. 105-120. Referisan od strane I.T.Selezova u Referativnom žurnalu, (16. Mehanika, 16V. Mehanika deformiruemogo tvrodgo tela, Vipusk svodnogo toma, N3, Moskva 1993. MethSci Disc 1993-1996, record 2 of 4, CPM 96 08, Mathematical Review)

22. Hedrih (Stevanović), K., (1995), *New Interpretation of the Rigid Bodies Kinetic Parameters*, Abstracts of 2-nd International Symposium of Ukrainian Mechanical Engineers in Lviv, State University "Lvivska Politechnika", Ukainian engineerr's Society in Lviv and Ukainian engineerr's Socciety of America, INC. pp. 51
23. Hedrih (Stevanović), K, (1995), *Interpretation of the Motion Equations of a variable mass object rotating around a atationary axis by means of the mass moment vector for the pole and the axis*, Procedings of the 4th Greek National Congress on Mechanics, vol. 1, Mechanics of Solids, Democritus University of Thrace, Xanthi, 1995, pp 690-696.
24. Hedrih (Stevanović), K., (1995), *Senzitivna zavisnost prinudnog oscilovanja /obrtanja/ haotičnog kretanja teškog tela, od početnih uslova, (Sensitive dpendence on initial conditions of forced vibration/rotation/stochastic-like motion of a heavy body around a stationary axis, in the field with damping)*, Proceedings of the YUCTAM NIŠ ' 95, Invited Letures, IL, YSM, Niš, pp. 239-253.
25. Hedrih (Stevanović) K., (1997) *Nonlinear dynamics of rotor with a vibrating axis*, Invited letures, Proceedings of the YUCTAM Vrnjačka Banja '97, Yugoslav Society of Mechanics, pp. 165-174.
26. Hedrih (Stevanović) Katica, (1988) *Izabrana poglavlja Teorije elastičnosti*, Mašinski fakultet u Nišu, prvo izdanje 1976, drugo izdanje, str. 424.
27. Rašković, D i Stevanović, K. (Hedrih), (1966-1967), *Ubrzanje drugog reda (trzaj) pri obrtanju tela oko nepomične tačke*, Zbornik radovaTehničkog fakulteta u Nišu, 1966-1967 godina, str. 93-100.
28. Hedrih (Stevanović), K., (1991), *On Rotation of a heavy Body Around a Stationary Axis in the Fiels with Turbulent Damping and Dynamic Pressures on Bearings*, Abstract of lectures YUCNP Niš'91, pp. 38-39.
29. Hedrih (Stevanović), K., (1992/93), *Neka razmišljanja o kinematici linijskih elemenata deformabilnog tela pri malim deformacijama, (A Contribution to the Kinematics of Line Elements of Deformable Bodies with Small Deformations)*, Zbornik radova Gra|evinskog fakulteta Univerziteta u Nišu, Broj. 13-14, 1992/1993, pp. 51-56.
30. Hedrih (Stevanović), K., (1992), *Neke vektorske interpretacije kinetike deformabilnih tela i fluida*, Zbornik radova Simpozijuma is Mehanike fluida posvećenog akademiku Konstantinu Voronjecu, Mašinski fakultet, Beograd, str. 279-286.
31. Hedrih (Stevanović), K., (1994), *Interpretation of the motion of a heavy body around a stationary axis in the field with turbulent damping and kinetic pressures on bearing by means of the mass moment vector for the pole and the axis*, "Facta Universitatis", Series "Mechanics, Automatic Control and Robotics", Vol. 1, N4, 1994, Niš. pp. 519-538.
32. Hedrih (Stevanović), K., (1994), *Vektori vezani za polove i pravu i pojmovi vektora sektorskih momenata masa i površina za pol i osu i sektorski pol*, Apstrakt, Sažeci, PRIM 94, Primenjena analiza, IX Seminar primewene matematike, Budva, 30 maj-1 jun 1994, Novi Sad, str. 44.
33. Hedrih (Stevanović), K., (1994), *Interpretation of the motion of a heavy body around a stationary axis and kinetic pressures on bearing by means of the mass moment vector for the pole and the axis*, Theoretical and Applied Mechanics, N 20, 1994, pp. 69-87. Beograd, rad po pozivu Jugoslovenskog društva za mehaniku za jubilarni broj časopisa povodom 20 godina publikovanja časopisa i ure|ivanja, posvećen uredniku prof. dr V.Vujučiću. MethSci Disc 1993-1996, reccord 2 of 4, CPM 96 08, Mathematiccal Reviews
34. Hedrih (Stevanović), K., (1991), *Analogije modela stanja napona,stanja deformacije i stanja momenata inercije mase tela*, Tehnika, prihvaćeno za štampu, 1995, N 6, str. Beograd. (1995)
35. Hedrih (Stevanović), K., (1995), *Energijska analiza kinetike konstrukcija za različite modele materijala*, Kratki prikazi radova naučnog skupa "Mehanika, materijali i konstrukcije", Srpska Akademija Nauka i Umetnosti, Odelenje tehničkih nauka, 1995, str. 106-107.
36. Hedrih (Stevanović), K., (1995), *Tenzor stanja slučajnih vibracija, (Tensor state of the random vibrations)*, Zbornik radova Mašinskog fakulteta povodom 35 godina mašinstva, Mašinski fakultet, Niš, 1995, str. 133-136.
37. Hedrih (Stevanović), K., (1995), *O jednom kinetičkom modelu rotora centrifugalne pumpe, (Some kinetic model of the double-flow castings pump rotor)*, Zbornik radova Mašinskog fakulteta povodom 35 godine mašinstva, Mašinski fakultet, Niš, 1995, str. 137-152.
38. Hedrih (Stevanović), K., (1995), *Neke vektorske interpretacije kinetike fluida - II deo, (Some vectorial interpretations of the fluid Kinetics - part II)*, Zbornik radova Mašinskog fakulteta povodom 35 godine mašinstva, Mašinski fakultet, Niš,1995, str. 153-165.
39. Hedrih (Stevanović), K., (1995), *On New Interpretations of the rigid Bodies Kkinetic Parameters and on Rotation of a Heavy body around a Stationary Axis in the Foeld with Turbulent Damping and Dynamic*

Pressures on Bearings, Conference on Differential Geometry and Applications, Masaryk University, Brno 1995, Zbornik Apstrakata.
40. Hedrih (Stevanović), K., (1995), *On one of the kinetic models of the rotor of exchangeble mass*, Facta Universitatis, Series Mechanical Engineering, vol. 1 N. 2, 1995, pp.185-208.
41. K.(Stevanović) Hedrih, (1996) *Interpretation of the Rigid Bodies Kinetics by Vectors of the Bodies Mass Moments*, Invited Lecture, Book of Abstracts, The International Conference: Stability, Control and Rigid Bodies Dynamics - ICSCD 96, Institute of Applied Mathematics & Mechanics of NAS of Ukraine, Donetsk - Mariupol, 2-6 September, pp. 35-36.(MatInst).
42. Hedrih (Stevanović), K., (1993), *The mass moment vectors at n-dimensional coordinate system*, Tensor, Vol 54 (1993), pp. 83-87.
43. Hedrih (Stevanović), K., (1996), *Neke interpretacije kinetičkih parametara krutih tela*, Tehnika, 11-12, Mašinstvo 45 (1996) 11-12, str. 8M-M13.
44. K.(Stevanović) Hedrih, (1996), *Model Rotora izmenljive mase i primena na dinamiku rotora radnog kola centrifugalne pumpe, Model of a Rotor of Exchangeable Mass and Analogous Applying to the Dynamic of Rotor of Centrifugal Pump*, II International Symposium: Contemporary Problems of Fluid Mechanics, Proceedings, 30 September - 2. October, Belgrade, YSM, Faculty of Mechanical Engineering, Belgrade, 1996, pp. 281-284. (MašFak).
45. K.(Stevanović) Hedrih, (1997), *Nelinearna dinamika rotora sa oscilujućom osom, (Nonlinear dynamics of rotor with a vibrating axis)*, Abstracts of the YUCTAM Vrnjačka Banja 97, pp.93-94.
46. Hedrih (Stevanović), K., (1997), *Nonlinear Dynamics of Rotor with a vibratin axis and sensitive dependence on initial conditions of forced vibration/rotation/stochasticlike -chaotic like motion of a hevy rotor*, Third Bogoliubov Readings: Asymptotic and Qualitative Methods of Nonlinear Mechanics, ASYM 97, Tezi dopovidey, Institut Math. NANU, Kiev, pp.73-74.

VEKTORSKA ANALIZA STANJA MOMENTA INERCIJE MASA

Katica (Stevanović) Hedrih

U ovom radu su uvedeni vektori momenata masa i tački N za osu orjentisanu jediničnim vektorom $\vec{n}$. Vektori momenata masa se mogu koristiti za interpretaciju kinetičkih parametara krutih tela i za analizu stanja momenata masa u nekoj tački tela, odnosno prostorne konfiguracije masa u odnosu na neki pol. Za promenu vektora momenata masa krutog tela pri promeni pola i zadržavanju orjentacije osa izvedena je uopštena Huygens-Steiner-ova teorema. Za promenu vektora momenata masa kasa osa kroz pol menja prvac orjentacije izvedena je vektorska jednačina koja je analogna Cauchy-jevim jednačinama iz teorije elastičnosti. Pomoću vektora momenata masa izvedena je analiza stanja prostorne konfiguracije masa krutog tela u odnosu na neki pol, koja omogućava određivanje glavnih pravaca momenata inercije, pravaca inercione asimetrije i oktaedarskih pravaca prostorne konfiguracije masa. Iz članka se da videti da uvođenje vektora momenata masa omogućava očiglednu i pristupačnu analizu stanja geometrijske konfiguracije masa tela.