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STRESS RECOVERY PROCEDURE BASED ON THE KNOWN DISPLACEMENTS

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Abstract. *Finite element method provides only the numerical simulation of a real behaviour of the observed physical problem. Hence, it is particularly important to have a reliable stress recovery technique and also the tools for error estimates of calculated stresses. In this paper a stress recovery procedure, based not on the conventionally calculated stresses but on the displacements, is studied in detail. Another useful property of the proposed approach is that all expressions are coordinate independent. These features have the following implications. First, a global continuous stress field is obtained directly from displacements and second, there is a possibility to use different coordinate systems at each global node.*

Along with a basic demand for reliability of a mathematical model, there is also an important question of a solution time needed for the accuracy required. It has been shown by the numerical evidence that proposed procedure evidently increases the quality of finite element (FE) solution and, for finer meshes, can even reduce the solution time, at least compared with the simplest form of stress averaging, i.e. the arithmetic mean of nodal stresses from neighbouring elements.

1. INTRODUCTION

In the continuum mechanics problems, along with the determination of displacements and strains, stresses occur as variables of primary interest. Usually, the stress prediction is of crucial importance in the structural analysis. In the classical finite element method displacements are state variables and they are continuous over a model. It is known that strains and stresses are related by constitutive equation and hence, at least in elasticity, the stresses are proportional to the gradients of displacements. In the usual case when C^0

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continuity for displacements is satisfied, raw finite element (FE) stresses are discontinuous at the interelement boundaries. Discontinuity is a main problem in the stress results interpretation for an average user, expecting continuous stress distribution throughout the model. Fortunately, the theory of finite elements has been expanded to overcome this ambiguous situation and different stress smoothing procedures and stress recovery techniques have been introduced.

It is well known that the accuracy of the analysis of a well–posed FE model depends on the number of elements in the finite element mesh, and that quality of a solution can be improved by the mesh refinement. However, increase of the mesh density drastically raises the solution time. Obviously, an acceptable stress recovery procedure should deliver accurate enough continuous stresses over the model in a reasonable time. In addition, derivative recovery techniques are used for error estimates in adaptive finite element procedures.

There are two general classes of the stress smoothing procedures [1,2,3]. If carried out over a whole finite element domain the procedure is known as a global smoothing. Local smoothing is performed at each node or small group of nodes. One of the simplest local smoothing procedures is the averaging of stresses from neighbouring elements at a particular common node. For the purpose of the present paper the procedure is named FEAavrg.

However, all these approaches including also innovative [4] ones, are based on the conventionally calculated stresses. Hence, all known approaches are scalar, which means that they approximate only one stress component at a time. It has been shown in [5] that to maintain invariance of the finite element approximations under the coordinate transformations, tensorial character of these approximations should be strictly respected.

A significant novelty of the proposed procedure should be pointed out – it is both global and tensorial. These two features, along with high accuracy and computational efficiency, are combined in the present FEDSS method.

2. FEDSS METHOD

As it has been already noted, at variance with the prevailing number of usual approaches, proposed FEDSS (Finite Element Displacement type Stress Smoothing) procedure is based not on the conventionally calculated stresses, but on the displacements. This technique is inspired with some results in [6], related to the two-field finite element models. Nevertheless, it has been shown that similar ideas can be applied in the stress recovery problems.

Proposed method has its basic application as a stress smoothing procedure. Since its smoothing scheme is expected to be more efficient than these of more popular local procedures, at least than that of FEAavrg, it can be used also for error estimation. We also emphasize that tensorial character of approximations allows a priori introduction of arbitrary coordinate systems, at all global nodes of a model, for which the continuous stresses will be interpreted. Also, one can use a different set of nodal coordinate systems for the interpretation of displacements. The proposed approach is particularly useful for the precise introduction of boundary conditions, either for displacement and stresses.

2.1 Tensoral formulation

The aim of the present formulation is to obtain a system of linear equations having values of known displacements at the right side and unknown values of nodal stress components at the left side. One can start from linear constitutive equations in a form:

$$\mathbf{e} = \boldsymbol{A} : \mathbf{t}. \tag{1}$$

Let us remind that linear strain-displacement relationships are given by:

$$\mathbf{e} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \tag{2}$$

So, similar as in [6], the stress–displacement relationships are given by:

$$\boldsymbol{A} : \mathbf{t} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \tag{3}$$

where $\mathbf{t}$ is the stress tensor, $\mathbf{u}$ the displacement vector, and $\boldsymbol{A}$ the elastic compliance tensor. A weak solution of (3) one can get by the Galerkin procedure [7].

After recasting of all quantities in the coordinate form, one obtains the relationships:

$$A_{\Lambda st \Gamma uv} t^{\Gamma uv} = B^{\Gamma q}_{\Lambda st} u_{\Gamma q}, \tag{4}$$

where, for the domain Ω_e of an element e:

$$\begin{aligned} A^{(e)}_{\Lambda st \Gamma uv} = \Omega^{(e)L}_{\Lambda} \Omega^{(e)M}_{\Gamma} \frac{\partial z_i}{\partial y^{(\Lambda)s}} \frac{\partial z_j}{\partial y^{(\Lambda)t}} \frac{\partial z_k}{\partial y^{(\Gamma)u}} \frac{\partial z_l}{\partial y^{(\Gamma)v}} \\ \int_{\Omega_e} P_L P_M A^{\alpha\beta\gamma\delta} \frac{\partial z^i}{\partial \xi^\alpha} \frac{\partial z^j}{\partial \xi^\beta} \frac{\partial z^k}{\partial \xi^\gamma} \frac{\partial z^l}{\partial \xi^\delta} d\Omega_e, \end{aligned} \tag{5}$$

$$B^{(e)\Pi q}_{\Lambda st} = \Omega^{(e)L}_{\Lambda} \Omega^{(e)\Pi}_{K} \frac{\partial z_i}{\partial y^{(\Lambda)s}} \frac{\partial z_k}{\partial y^{(\Lambda)t}} \frac{\partial z^k}{\partial x^{(\Pi)p}} \int_{\Omega_e} P_L P^K_\alpha \frac{\partial z^i}{\partial \xi^\beta} g^{\alpha\beta} g^{(\Pi)pq} d\Omega_e. \tag{6}$$

In these expressions $t^{\Gamma uv}$ and $u_{\Pi q}$ are the stress component $t^{\Gamma uv}$ at node Γ and displacement component u_q at node Π respectively; $\Omega^{(e)\Lambda}_K$ is the incidence matrix which maps global nodes Λ onto the local nodes K of an element; P_K is a shape function connected with the node K; symbols $g^{(\Pi)pq}$ are the components of the contravariant metric tensor at node Π; indices *p, q* refer to the nodal coordinate system $x^{(\Pi)p}$ used for the displacements, while indices *s, t, u, v* refer to the nodal coordinate system $y^{(\Lambda)s}$ used for stresses; *i, j, k, l* are the indices of global Cartesian coordinate system z^i of a model; symbols $g^{\alpha\beta}$ are the components of the contravariant metric tensor in the interior of an element and $\alpha, \beta, \gamma, \delta$ are the indices of a local, element coordinate system ξ^α. Finally, $A^{\alpha\beta\gamma\delta}$ are the contravariant components of the elastic compliance tensor. Omission of indices (e) in (5), (6), etc., means global values of the quantities under consideration, i.e., simple summation of these.

For flat two-dimensional and for three–dimensional configurations, one can use Cartesian coordinates for the elastic compliance coefficients, and the expression (5) has the following interpretation:

$$A^{(e)}_{\Lambda st\Gamma uv} = \Omega^{(e)}_{\Lambda L}\,\Omega^{(e)}_{\Gamma M}\,\frac{\partial z^i}{\partial y^{(\Lambda)s}}\frac{\partial z^j}{\partial y^{(\Lambda)t}}\frac{\partial z^k}{\partial y^{(\Gamma)u}}\frac{\partial z^l}{\partial y^{(\Gamma)v}}\int_{\Omega_e} P_L A_{ijkl} P_M \, d\Omega_e . \tag{7}$$

2.2 Matrix representation of the proposed technique

Once the appropriate types of basis functions are selected and the correspondence between local and global nodes evoked, coordinate systems at each global node chosen (not necessarily different from the global coordinate system of a model) as input values from FE preprocessing part, along with the given displacements, next steps are the assembly of the expressions (5) and (6) for all elements and all global nodes in the system matrices, and after that the solution of a system (4). Naturally, this should be done automatically by the program code and the input values required are only the coordinate systems data at global nodes, if different from global coordinate system of a model. Matrix formulation will be made in the same manner as it has been done in [6]. Global system of linear equations (4) has the following matrix representation:

$$[A_{\Lambda st\Gamma uv}]\{t^{\Gamma uv}\} = \{B^{\Pi q}_{\Lambda st} u_{\Pi q}\}\,. \tag{8}$$

Matrix $[A_{\Lambda st\Gamma uv}]$ is positive definite, symmetric and sparse. Coefficients (5) or (7) of this matrix can be integrated analytically, but usually full Gaussian, or Lobatto type [1] formulae are applied.

2.3 Comparison of the fedss and some more common stress recovery methods

Numerical results of the present procedure are, up to the order of the rounding error, the same as in global stress smoothing methods proposed earlier by Oden and Brauchly [2], Hinton and Campbell [3] and Zienkiewicz and Zhu [1] despite the fact that in the kernel of the system matrices (5) or (7) we have elastic coefficients, while in the aforementioned methods there are only the shape functions, $P_L P_M$, in the kernel. Of course, the right hand side is also different. It should be however pointed out that, to get identical results, the same set of the points of numerical integration should be taken for both methods. The advantage of the proposed procedure is a fact that it can be directly used in the case when only the displacements are available, either from theoretical considerations, numerical analysis, or an experiment. In addition, at least in principle, formulation (4) allows introduction of the stress boundary conditions as the essential ones.

Also, special attention is paid to the tensorial invariance of the of proposed procedure. Accordingly, the present formulation allows use of arbitrary coordinate systems, possibly different at each observed node. Obviously, tensorial invariance can be kept independently on the kind of the stress smoothing approach [8].

3. DETERMINATION OF THE STRAIN ENERGY AND APPROPRIATE NORMS

Obtaining of a reliable solution of an appropriate mathematical model is one of

crucial demands posed in front of the FE end users in the engineering practice. Hence, a new procedure intended to be introduced in a FE software must have a sound mathematical basis and pass a serious numerical examination. When FE analysis (FEA) is used it is known that in a typical situation only an approximate solution of the model problem is computed. The main users' question is how accurate are such results. There are several ways of investigation of the FEA error. Commonly used way of determination of an approximation error is by the use of an appropriate norm. If the discretization error estimate for a given mesh is needed, it is practical to introduce energy (error) norm:

$$\|e\| = \left(\int_{\Omega} (\mathbf{t} - \mathbf{t}_h) : \boldsymbol{A} : (\mathbf{t} - \mathbf{t}_h) d\Omega \right)^{1/2}. \tag{9}$$

In this expression $\mathbf{t}$ is an exact stress solution, $\mathbf{t}_h$ finite element stress, $\boldsymbol{A}$ the elastic compliance tensor and Ω the domain of the model. It is usually taken as granted that smoothed (continuous) stress field is "more accurate" than finite element discontinuous stress pattern. Also, the *strain energy*:

$$U = \frac{1}{2} \int_{\Omega} \mathbf{t} : \boldsymbol{A} : \mathbf{t} \, d\Omega, \tag{10}$$

has been traditionally used [9] for estimation of accuracy and convergence of the finite element solution. The popularity of above measure is partially due to a fact that it is, at a system level, because of the First Law of Thermodynamics, equal to the work of the external forces (at least for hyperelastic materials) which can be easily calculated [11]. The energy of a finite element solution is obviously:

$$U_h = \frac{1}{2} \int_{\Omega} \mathbf{t}_h : \boldsymbol{A} : \mathbf{t}_h \, d\Omega, \tag{11}$$

Note once more that $\mathbf{t}_h$ is a finite element stress, *dependent on the stress recovery procedure*. Now, one can introduce the total energy norm:

$$\|U\| = \sqrt{2U}, \tag{12}$$

or, more specifically:

$$\|U_{\mathrm{FEA}}\| = \left(\int_{\Omega_e} \mathbf{t}_{\mathrm{FEA}} : \boldsymbol{A} : \mathbf{t}_{\mathrm{FEA}} d\Omega \right)^{1/2}, \tag{13}$$

where index FEA denotes the value obtained by the straightforward application of the classical FEA. It has been proven in [11] that the energy of the error of the finite element solution is equal to the error of the energy, i.e.

$$\|e\|^2 = |2U - 2U_{\mathrm{FEA}}|. \tag{14}$$

As simple differences between two scalar quantities, expressions of the type (14) can be used for the determination of the energy difference between any two solutions of the problem under consideration. At variance, the expressions of the type (9) calculate energy error correctly only if the stresses under consideration are based on some global

projection procedure. The use of (9) in the case when $\mathbf{t}_h$ is *not* an orthogonal projection of $\mathbf{t}$, can seriously underestimate the energy error norm. To be perfectly clear, note that use of (9) for the calculation of the energy error is justified if and only if $\mathbf{t}_h = \mathbf{t}_{\mathrm{FEA}}$. Hence, the popular use of (9) for the determination of the error of a smoothed solution ($\mathbf{t}_h = \mathbf{t}_{\mathrm{SMOOTH}}$) is overoptimistic and misleading. To avoid any ambiguity, for the energy error determination in this paper we will use only the expressions of the type:

$$\left\|e_{\mathrm{SMOOTH}}\right\|^2 = \left|2U - 2U_{\mathrm{SMOOTH}}\right|, \tag{15}$$

or specifically:

$$\left\|e_{\mathrm{FEDSS}}\right\|^2 = \left|2U - 2U_{\mathrm{FEDSS}}\right|, \tag{16}$$

$$\left\|e_{\mathrm{FEAavrg}}\right\|^2 = \left|2U - 2U_{\mathrm{FEAavrg}}\right|. \tag{17}$$

It is easy now to define the *relative percentage error* [1] or *precision* [12] as:

$$\eta = 100\,\frac{\left\|e\right\|}{\left\|U\right\|}, \tag{18}$$

or specifically:

$$\eta_{\mathrm{FEDSS}} = 100\,\frac{\left\|e_{\mathrm{FEDSS}}\right\|}{\left\|U\right\|}, \tag{19}$$

$$\eta_{\mathrm{FEAavrg}} = 100\,\frac{\left\|e_{\mathrm{FEAavrg}}\right\|}{\left\|U\right\|}. \tag{20}$$

When a largest local error is required, the supreme norm is usually applied. Per instance,

$$\left\|e_{\mathrm{SMOOTH}}\right\|_{\infty} = \max_{L}\left|t_L^{jk} - t_{L\,\mathrm{SMOOTH}}^{jk}\right| \tag{21}$$

is a largest stress error of the smoothed stress component t^{jk}, obtained at node L.

4. NUMERICAL EXAMPLE

In this section the quality of the proposed FEDSS stress recovery procedure is illustrated by means of a plane stress numerical example. The problem of a square plate with a circular hole [13] is shown on Figure 1. This problem is particularly interesting due to high stress concentrations occurring in the regions where the coordinate axes intersect the contour of a hole. Due to the symmetry, only the quarter of a plate is analysed. Isotropic, homogeneous material properties and the plane stress behaviour are assumed. Modulus of elasticity is 1 and Poisson's ratio 0.3. Plain isoparametric four–noded quadrilateral elements and 2×2 Gaussian quadrature are used. Estimated converged value of the strain energy is $U = 3.582$, where at least three significant digits are considered to be accurate.

Numerical studies were made for the sequence of meshes from 1×1 to 38×38. One of

these (5×5) is shown on Figure 2.

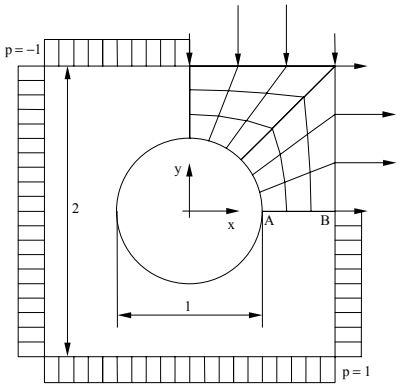

Fig.1. Plate with a circular hole

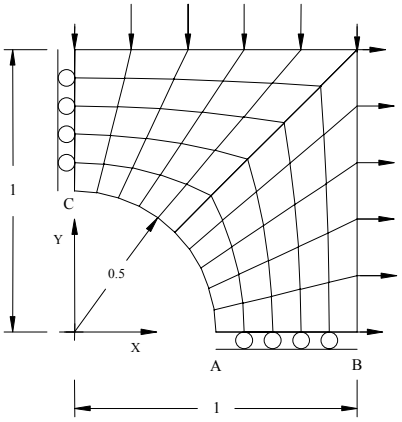

Fig.2. Finite element mesh

One possible user defined set of coordinate systems, useful for the determination of hoop stresses at global nodes is shown on Figure 3.

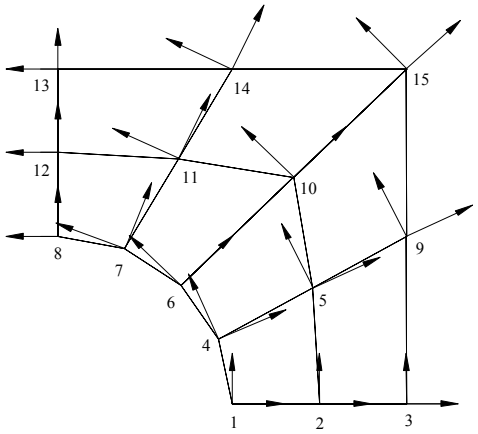

Fig.3. User defined coordinate systems

Recovered FEDSS stresses are compared with stresses simply averaged at nodes, FEAavrg. To study the effectiveness of FEDSS method as a stress recovery and smoothing procedure two types of the error evolution were considered. First, relative percentage error versus number of elements, and second, the same error as a function of the execution time is examined.

Figure 4 shows a comparison of the relative percentage error versus the number of degrees of freedom. As it can be seen, FEDSS formulation presented here is, especially for reasonably dense (more than 5×5), meshes significantly more accurate in the energy error norm than conventional smoothing, i.e. simple averaging of the stresses at nodes,

FEAavrg.

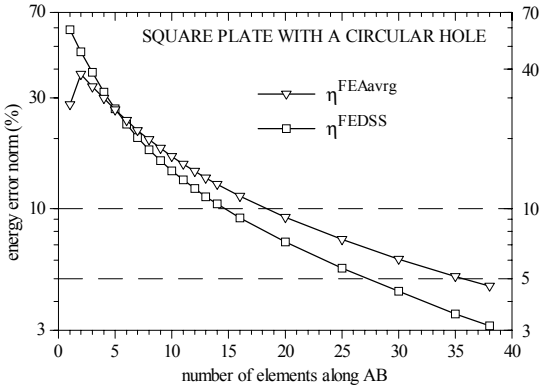

Fig.4. Energy error norm versus mesh density

It is also interesting to study the convergence rates for both procedures. From the Figure 5 it is evident that the convergence rate of FEAavrg is somewhat higher than 1 (for dense meshes) while for FEDSS approaches 1.5.

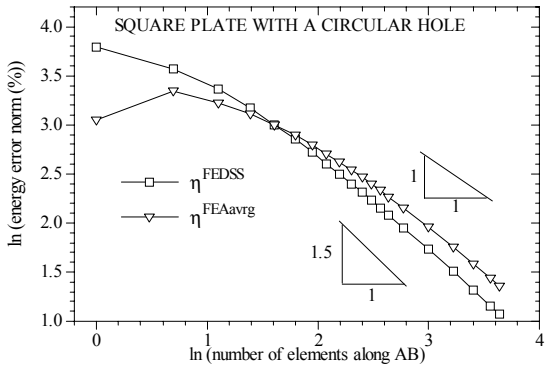

Fig.5. Energy convergence rates

Even more important is a quantitative measure, i.e. the execution time for the accuracy required. In Figure 6 relative errors related to the time of execution are compared. It can be said that FEDSS stress recovery procedure, for finer meshes even reduces the total solution time, despite the additional time necessary for the solution of (5) for stresses. In fact, if an error in the energy norm smaller than 6% is required, the proposed FEDSS procedure is more efficient than simple averaging.

Obviously, for the practical stress estimates it is not enough to check just a global energy convergence, but also the local convergence in $\|e\|_\infty$ norm, especially in the regions of high stress concentrations. It can be seen on Figure 7 that hoop stress at A, obtained either by the present FEDSS procedure (8) or by FEAavrg via Hookean law (3), as the mesh is refined overcomes the theoretical value of 10.39, gradually approaching it from above, for very fine meshes. For coarser meshes directly calculated stresses

converge (Figs 7–9) seemingly faster than FEDSS stresses. However, for meshes finer than 10×10 (Figs 8 and 9) the supremum norm error tends to be almost an order of magnitude smaller when the stresses are recalculated by the use of the FEDSS procedure.

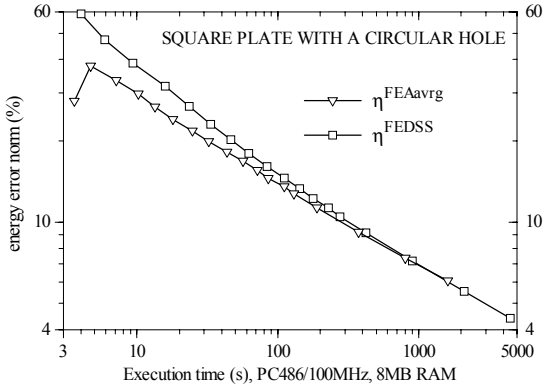

Fig.6. Energy error norm versus execution time

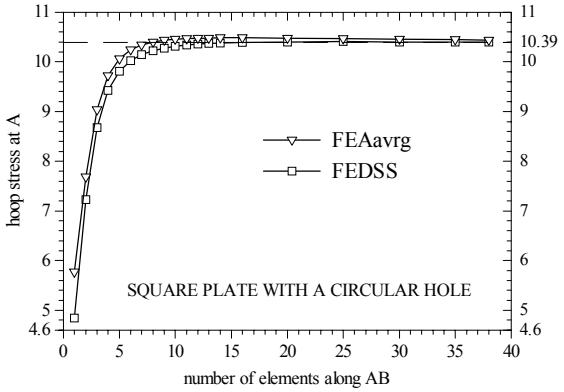

Fig.7. Stress convergence at A

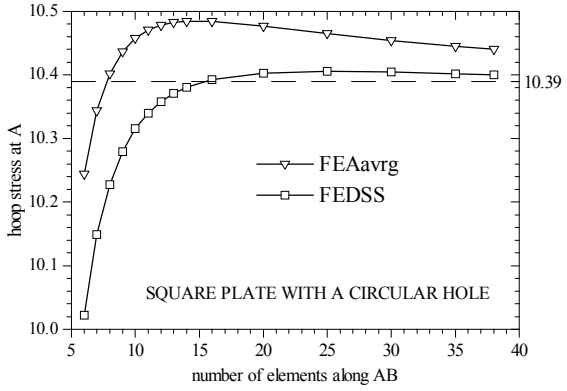

Fig.8. Stress convergence at A for finer meshes

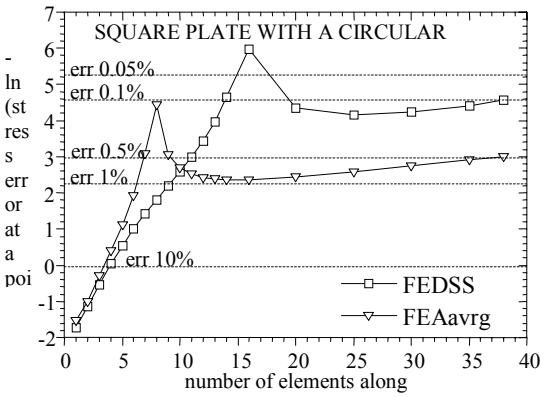

Fig.9. Stress error at A versus number of elements

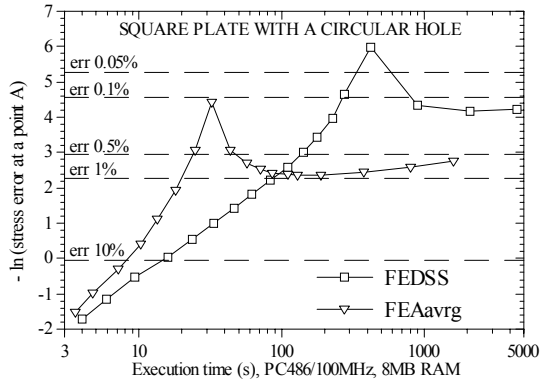

Fig.10. Stress error at A versus execution time

It is also noteworthy that maximum local stresses are much more accurate than an average stress error estimated on the basis of the energy norm (Figs 9 and 10). This behaviour of the FE solution is obviously very desirable from the practical point of view.

5. CONCLUSIONS

A global, coordinate independent projection procedure for generation of a stress continuous displacement field has been presented and applied to two-dimensional elasticity problems. Proposed procedure can be recommended if high precision of calculated stresses, both global and local, is needed. Another advantage of the present algorithm is its flexibility (i.e. possibility to use arbitrary local coordinate systems). In addition, it is implemented as an universal postprocessing routine for which the input data (displacements) can be taken from any finite element analysis package, theory, or experiment. Obviously, the proposed approach can be used not only as a postprocessing procedure, but also as a precise tool in the adaptive mesh refinement algorithms.

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POUZDANO IZRAČUNAVANJE NAPONA U METODI KONAČNIH ELEMENATA

Dubravka Mijuca, Zoran Drašković, Mladen Berković

U radu je predstavljena pouzdana, globalna, koordinatno nezavisna metoda izračunavanja kontinualne naponske slike modela opterećenog tela. Metoda se bazira na poznatim pomeranjima, koja mogu da budu određena eksperimentalno, teorijski ili numerički.