UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Mechanics, Automatic Control and Robotics** Vol.2, No 8, 1998 pp. 635 - 639
Editor of series: Katica (Stevanovi}) Hedrih, e-mail: katica@masfak.masfak.ni.ac.yu
Address: Univerzitetski trg 2, 18000 Niš, YU, Tel: (018) 547-095, Fax: (018)-547-950
http:// ni.ac.yu/Facta

ON EXISTENCE OF PERIODICAL SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECTS

UDC:517.9

Valeriy Hr. Samoylenko [1], K. K. Yelgondyev [2]

[1] Institute of Mathematics, National Academy of Sciences of Ukraine
Tereshchenkyvs'ka Str. 3, 252601 Kyiv, Ukraine
e-mail: vsam@imath.kiev.ua
[2] Karakalpakskiy Univ., Universitetskaya Str., 1, 742012 Nukus, Uzbekistan

Abstract. *The periodic solutions for linear ordinary differential equations of second order with impulse effects at non-fixed moments of time are studied. It is proved that the problem under some kind of impulse effects could demonstrate rather interesting properties presented by lots of periodical solutions with arbitrary natural number (starting with one!) of impulse "transmissions" per period. The corresponding example is presented.*

Mathematics is successfully applied in numerous technical problems, biology, economics, control theory, etc. In general, taking into account different properties and effects intrinsic of a considered system, it gives us the possibility to study processes and phenomena by means of mathematical simulation more exactly. For instance, many areas of industry deal with technical systems being subjected to shocks, impacts or impulses. Thus, in cases when an external perturbation for the system is of a short-term (shock, impact or impulse) nature and its duration could be disregarded while formulating the corresponding mathematical model we have to study dynamical system with discontinuous trajectories.

A classical example of that problem is the model of a clock mechanism [1], which could be described by ordinary differential equation with discontinuous right side function [2] or by ordinary differential equation with additional conditions of impulse effects [3]. The problems mentioned above could be briefly called as impulse differential equations or differential equations with impulse action [3].

There are many different methods to mathematically describe the conditions of impulse effects, among them we more often meet the conditions of impulse effects at fixed and non-fixed moments of time. The condition of impulse effects mathematically

Received December 16, 1996

means that there are some rules according to which the moving point of the considering dynamical system at the moments of impulse action changes its way by shifting from a trajectory of the ordinary differential equation to another one. In fact, however, the main sense of them lies in determination of the new initial data at the moment of impulse action.

Despite formal simplicity of such a problem, the behaviour of trajectories of the dynamical system defined by ordinary differential equation and some conditions of impulse effects, even in the simplest cases and due to these conditions it may be too complicated and essentially different from trajectories behaviour of this differential equation with lack of impulse action.

In general case the conditions of impulse effects turn this dynamical system into a essential nonlinear one and may cause rather compliceted behaviour of its trajectories. It should be mentioned that here the problems with impulse effects at non-fixed moments of time are more complex than problems with impulse effects at fixed moments of time.

Differential equations with impulse effects form a wide set of different problems. During the last 25 years those problems were intensively studied. Paper [4] was of great significance for the development of differential equations systems with impulse effects as it caused the raising interest to the study of such problems. For the results on the theory of impulse differential equations, see monograph [3,5,6]. In [3] the main definitions and results of the theory of systems of ordinary differential equations with impulse effects were given for the first time; similarity and differentiality of such problems of applied mathematics with corresponding problems of ordinary differential equations (and without the conditions of impulse effects) were demonstrated; general characteristics of these systems were described; periodic and almost-periodic solutions of differential equations with impulse effects were studied; problems of stability for solutions of such systems were researched. Fundamental results concerning the application of asymptotic methods of nonlinear mechanics [7] to weakly nonlinear differential equations with impulse effects are given in papers [8,9].

Even in simple cases interesting results can be obtained due to impulse effects. Thus, behaviour of ordinary mechanical oscillator with continuous trajectories and discontinuous velocities (due to impulse effects at fixed moments of time) could be regarded as an unusual [10] one in some sense.

In this paper we consider a nonlinear dynamical system, the movement in which is described by second order linear differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0\,, \quad \omega \in \mathbf{R}_+\,, \tag{1}$$

and by conditions of impulse effects at non-fixed moments of time

$$\Delta \frac{dx}{dt}\Big|_{x=x_0} = I(\dot{x})\,. \tag{2}$$

Impulse effects in such dynamical system occur at moments when the moving point passes some fixed position $x = x_0$. If $t = t_0$ is the meaning of time, when the moving point of considered dynamical system reached the point $x = x_0$, then at this moment of time the velocity of given point instantly changes according to the formula (2), i.e.

$$\frac{dx(t_0+0)}{dt} = \frac{dx(t_0-0)}{dt} + I\left(\frac{dx(t_0-0)}{dt}\right). \tag{3}$$

The solution of the problem (1), (2) is a function $x(t)$, which is continuous with respect to $t \in \mathbf{R}$ and continuously differentiable for all $t \in \mathbf{R}$ except the moments of impulse effects, when its derivative $\dot{x}(t)$ is continuouos from the right and has discontinuities of first kind.

The case of constant impulse effects $I(x) \equiv const \in \mathbf{R}$ was fundamentally studied in [3], where implicit formulas of solutions of the problem (1), (2) were obtained and the behaviour of its phase trajectories was described. In general case, when $I(x) \neq const$ the trajectories of dynamical system (1), (2), the initial data of which belong to domain $D = \{(x, \dot{x},: \omega^2 x^2 + \dot{x}^2 \le \omega^2 x_0^2\} \subset \mathbf{R}^2$, are periodic ones with the period $T = 2\pi/\omega$ and they are not influenced by impulse effects.

If initial data for the solution $x(t)$ do not belong to domain D, then at some moment of time moving point of dynamical system (1), (2) *starts* from $(x_0, \dot{x})$, and in course of time it reaches the line $x = x_0$ at $(x_0, -\dot{x})$, after that this point is influenced by impulse forces (2), and in result it *instantly* appears in $(x_0, -\dot{x} + I(-\dot{x}))$, and then, in certain time crosses the line $x = x_0$ again, – now at $(x_0, -\dot{x} - I(-\dot{x}))$, where instant forces (2) act again, etc.

Thus, there defined some mapping of line $x = x_0$ into itself according to the rule $f: \mathbf{R} \to \mathbf{R}$, where

$$f(y) = -y + I(-y), \quad y = \dot{x}. \tag{4}$$

Stationary points of the mapping $f(y)$ are corresponded by periodic regimes of dynamical system (1), (2) and at the same time the considered system (1), (2) is influenced by impulse effect only once per period.

If the mapping (4) has periodic point y_0 of period $n \in \mathbf{N}$, i.e. $f^n(y_0) = f(f(\ldots f(y_0))) = y_0$, then the point y_0 is corresponded by some T-periodic regime of dynamical system (1), (2), at the same time the considered system (1), (2) is influenced by impulse effects exactly n times per period.

In the case when the mapping (4) is continuous and has a periodic point of period 3, then there exist [11] periodic points of arbitrary period n, which are corresponded by $T(n)$ - periodic regimes of dynamical system (1), (2), at the same time the considered system (1), (2) is influenced by impulse effects exactly n times per period $T(n)$.

Thus, if dynamical system (1), (2) has such a periodic regime when for given periodic regime the system is influenced by impulse effects exactly three times per period, then there exist $T(n)$ - periodic regimes of the system (1), (2) at the same time the considered system (1), (2) is influenced by impulse effects exactly n times per period, where n is arbitrary natural number. It should be noted that in this case several periodic regimes may exist in system (1), (2) when the considered system (1), (2) is influenced by impulse effects exactly n times per period and at the same time the periods of such regimes may not coincide.

The stability of described above $T(n)$ - periodic solutions for problem (1), (2) is adequate and determined by stability of corresponding stationary points of mapping $f^n(y)$.

Thus, the following statement is valid.

Theorem: Let the mapping $\mathbf{R} \ni y \to I(y) \in \mathbf{R}$ is continuous, a dynamical system (1), (2) has periodic trajectory, while moving along it the phase point is influenced by impulse effects exactly three times per period.

Then linear differential equation (1) with impulse effects at non-fixed momemts of time (2) has $T(n)$ - periodic solution with exactly n impulse effects per period, where n is arbitrary natural number.

Example. Let us consider dynamical system of the form (1), (2), where

$$I(y) = \begin{cases} 1 - 3y, & \text{if } \; y \ge 0 \\ 1 + y, & \text{if } \; y < 0 \end{cases}. \tag{5}$$

The mapping

$$f(y) = -y + I(-y) = \begin{cases} 1 - 2y, & \text{if } \; y \ge 0 \\ 1 + 2y, & \text{if } \; y < 0 \end{cases}, \tag{6}$$

is continuous for all $y \in \mathbf{R}$, has one stable $y = -1$ and one unstable $y = 1/3$ stationary points, exactly two periodic points $y = -1/5$, $y = 3/5$ of period 2 and exactly six periodic points of period 3. The points of period 3 form two cycles

$$\left\{-\frac{3}{7}, \frac{1}{7}, \frac{5}{7}\right\}, \; \left\{-\frac{5}{9}, -\frac{1}{9}, \frac{7}{9}\right\}$$

for mapping (6), correspondingly.

Thus, the dynamical system (1), (2), (5) has periodic regimes, when the considered system is influenced by impulse effects exactly one, or exactly two or exactly three times per period correspodingly. For given x_0 it is not difficult to find initial data and periods of above mentioned periodic regimes. Moreover, from the result of paper [11] it follows, that mapping (6) has [11] periodic points of arbitrary period $n \in \mathbf{N}$, and consequently, the dynamical system (1), (2), (5) has such periodic regimes, when the considered dynamical system is influenced by impulse effects exactly n times per period.

All periodic points of period $n \ge 2$ for mapping (6) are unstable, so periodic solutions of equation (1) with impulse effects (2), (5) corresponding to them are also unstable.

REFERENCES

1. Bautin N. N. Dinami~eska® teori® ~asov: Stabilizaci® perioda v kolebatelÍnìh sistemah s dvum® stepen®mi svobodì. M.: Nauka, 1986. - 192 s. (In Russian)
2. MitropolÍskiy Á. A. Metod usredneni® v nelineynoy mehanike. Kiev: Naukova dumka, 1974. - 440 s. (In Russian)
3. Samoylenko A.M., PerestÓk N.A. DifferencialÍnìe uravneni® s impulsÍnìm vozdeys-tviem. - Kiev: Viça {k., 1987. - 287 s. (In Russian).
4. Mì{kis A.D. Samoylenko A.M.: Sistemì s tol~kami v zadannìe momentì vremeni. Mat. sb. - 1967. -74, N2. - C. 202-208. (In Russian).
5. Lakshmikantam V., Bainov D.D., Simeonov P.S.: *Theory of Impulsive Differential Equations*. Singapore: World Scientific, 1989. - 520 pp.
6. Samoilenko A.M., Perestyuk N.A. *Impulsive differential equations*. World Scientific Series on Nonlinear Sciences. Ser. A. Vol. 14 - Singapore-New Jersey-London-Hond-Kong, 1995. - 560 pp.
7. BogolÓbov N.N., MitropolÍskiy Á.A. Asimptoti~eskie metodì v teorii nelineynìh

kolebaniy. M.: Nauka, 1974. - 502 s. (In Russian).
8. Samoylenko A. M. Metod usredneniÔ v sistemah s tol~kami. Mat. fizika. - 1971. - VÍ{. 9. - S. 101-117. (In Russian).
9. MitropolÍskiy Á. A., Samoylenko A.M., PerestÓk N.A. Metod usredneniÔ v sistemah s impulÍsnìm vozdeystviem. Ukr. mat. `urn. - 1985. - 37, N1. - S. 56-64. (In Russian).
10. Samoylenko V.G., Elgondìev K.K. Periodi~eskie i po~ti-periodi~eskie re{eniÔ lineynìh odnorodnìh differencialÍnìh uravneniy s impulÍsnìm vozdeystviem. Mat. fizika i nelineyn. mehanika. - 1991. - Vìp. 14(49). - S. 13-20. (In Russian).
11. [arkovskiy A.N. Sosuçestvovanie ciklov neprerìvnogo preobrazovaniÔ prÔmoy v sebÔ. Ukr. mat. `urn. - 1964. - 16, N1. - S. 61-71. (In Russian).

O EGZISTENCIJI PERIODIČNIH REŠENJA ZA DIFERENCIJALNE JEDNAČINE SA IMPULSNIM EFEKTIMA

Valeriy Hr. Samoylenko, K. K. Yelgondyev

Periodična rešenja za obične linearne diferencijalne jednačine drugog reda sa impulsnim efektima u neodređenim trenucima vremena su izučavana. Dokazano je da problem pri nekoj vrsti impulsnih efekata može pokazati prilično interesantna svojstva koja se ogledaju u odsustvu periodičnih rešenja sa proizvoljnim prirodnim brojem impulsnih "transformacija" po periodu. Prikazan je odgovarajući primer.