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EXPONENTIAL SPLITTING OF THE LINEAR PARABOLIC EQUATION

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Abstract. *The linear parabolic equation with the fast changing coefficients is considered. The problem of construction of the transformation which realizes the splitting of the initial equation is solved. The important role of integral manifolds in solving this problem is established.*

Key words. *Coercive operator, parabolic equation, fast-changing coefficients, linear manifold, exponential splitting.*

1. INTRODUCTION

Let B be a basic separable Banach space, Hom(B,B) is the Banach space of linear operators $B\to B$ with the operator norm. Consider the differential operator

$$L_\omega = \frac{d}{dt} + A(\omega t)\,,$$

where $A(t)$ is continuous and bounded on real axis R function in Hom(B,B) with uniform average value

$$\overline{A} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T+\alpha}^{T+\alpha} A(s)ds\,,\quad \alpha \in R.\,,$$

N. N. Bogoljubov [1, 2] proved that the operator L_ω was close by its properties to the averaged operator

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$$\overline{L} = \frac{d}{dt} + \overline{A} \ .$$

Application of the classic methods leads to some difficulties which are stipulated by the problem on considering unbounded operators. V. V. Zhykov [3,4] showed that the operator L_ω was not close to the averaged operator even in the simplest case when $A(t) - \overline{A}$ is a bounded operator. Zhykov studied in detail the problems on regularity, correctness, exponential dichotomy of the linear parabolic operators and the connection of these problems with the problem of averaging on the real axis.

Conception of the exponential splitting is close to the conception of the exponential dichotomy. M. G. Krein and Yu. L. Daletsky [5] established, that operator

$$L = \frac{d}{dt} + A \ ,$$

with the bounded operator - function $A(t)$ in Hom(B,B) which admitted the exponential splitting was stable with respect to perturbations small by norm.

In this paper we show that the problem on constructing of this splitting transformation is connected with the problem on existing some linear integral manifolds of the initial equation.

Let H be a Hilbert space with the scalar product (,). Reflexive space E with conjugate space E^* is called an enclosed space, if there is continuous and dense enclosure $E \subset H \subset E^*$, and a bilinear form $y(x)$, $y \in E^*$, $x \in E$ coincides with scalar product on H, if $x, y \in H$.

Linear operator A: $E \to E^*$ is called the coercive operator or the strongly elliptic operator, if the inequality $\mathrm{Re}(Ax, x) \geq c_1 \| x \|_E^2 - c_2 \| x \|_H^2$, where constants $c_1 > 0$, $c_2 > 0$, holds. We can regard A as unbounded operator in H, if we represent the domain of its definition by $D = \{x \in H: Ax \in H\}$. It is true, when we consider the spectrum of the operator A [3]

The operator is called a parabolic operator, if $A(t)$ is a continuous function in Hom(E,E^*) and the coercive condition is fulfilled uniformly with respect to $t \in R$. For the parabolic operator the condition of solvability is fulfilled on the right, if the Hilbert space H is a basis space. Let

$$H = H_1 + H_2 , \tag{1}$$

and H_1 is the m-dimensional space. The decomposition $E = E_1 + E_2$, $E^* = E_1^* + E_2^*$ corresponds to (1).

Definition 1. Let $\Phi(t)$ be continuous and bounded function with values in Hom(E_1,E_2). We call a set $\sigma_m \in$ R $\times$ E of the form

$$\sigma_\mathrm{m} = \{(t,u): u = x + \Phi(t)x\}$$

the m-dimensional linear integral manifold of the equation $Lu = 0$, if for each point $(t_0,u_0) \in \sigma_m$ this equation has a solution $u(t)$ defined on R and satisfying the conditions $u(\mathrm{t}_0) = \mathrm{u}_0$, $(t,u(t)) \in \sigma_m$ for all $t \in R$.

Definition 2. We call a set $\sigma^m \in R \times E$ of the form

$$\sigma^m = \{(t,u): u = F(t)y + y \ , F \in C(\mathrm{Hom}(E_2,E_1))\}$$

the m-codimensional linear integral manifold of the equation $Lu = 0$ if for each point $(t_0,u_0) \in \sigma^m$ this equation has a solution $u(t)$ defined for $t \geq t_0$ and satisfying the conditions $u(t_0) = u_0$, $(t,u(t)) \in \sigma^m$ for all $t \geq t_0$.

2. Existence of integral manifolds

Consider the parabolic equation

$$L_\omega u = \left(\frac{d}{dt} + A(\omega t)\right)u = 0 \ ,$$

for $\omega >> 1$.

Assume the following conditions:

1. Enclosure $E \subset H$ is compact.
2. The space E^* has an approximate property: there is sequence of the linear operators $P_m : E^* \to E^*$ such that $P_m E^* \subset E$, $P_m y \xrightarrow{E^*} y (y \in E^*)$, $m \to \infty$.
3. $A(t)$ is compact function on R in $Hom(E,E^*)$ with a uniform average value.
4. There is μ such that $A(t)(\overline{A} + \mu I)^{-1} \in C(\mathrm{Hom}\, E, E)$.
5. Operator

$$\overline{L} + \mu I = \frac{d}{dt} + \overline{A} + \mu I \ ,$$

where μ is determined in 4, is regular.

6. $\mathrm{Re}\,\sigma(\overline{A}) \geq 0$ and the proper subspace of operator $\overline{A}$ corresponding to the spectral set $\sigma_0 = \sigma(\overline{A}) \cap \{\lambda : \mathrm{Re}\,\lambda = 0\}$ is m-dimensional

Remark. By virtue of the condition 6 the equation $\overline{L}u = 0$ has a trivial m-dimensional linear integral manifold and a trivial m-codimensional linear integral manifold. By Theorem 1 these integral manifolds are stable with respect to perturbations which are integrally small.

Theorem 1. Assume conditions 1 - 6. Then there exists ω_0 such that for any $\omega > \omega_0$ the following statements are valid.

1) Equation (1) has the m-dimensional linear integral manifold

$$\sigma_m : u = x + \Phi(t,\omega)x \ ,$$

for which $\Phi(t,\omega)$ satisfies relation

$$\lim \| \Phi(t,\omega) \|_{C(\mathrm{Hom}(E_1,H_2))} = 0 \ .$$

Moreover the function $\Phi(t,\omega)$ has a derivative which satisfies $\Phi(t,\omega) \in C(\mathrm{Hom} E_1, E_2^*)$.

2) Equation (1) has the m-codimensional linear integral manifold

$$\sigma^m : u = F(t,\omega)y + y \ ,$$

for which $F(t,\omega)y$ satisfies relation

$$\lim_{\omega\to\infty} \| F(t,\omega) \|_{C(\mathrm{Hom} E_2, E_1)} = 0 .$$

Moreover the function $F(t,\omega)$ has derivative locally integrable with square on the whole real axis.

Our techniques for proving Theorem I are much influenced by [7]. So, Theorem 1 follows from the statement which we formulate as Theorem 2.

Theorem 2. Assume the conditions of Theorem 1 and ω_0 is defined as in Theorem 1. Then for any $\omega > \omega_0$ there exists $\mu > 0$ such that for any point $(\tau,\xi) \in \mathrm{R} \times \mathrm{E}_1$ the equation (1) has a solution unique on the whole real axis

$$u(t,\tau,\xi,\omega) = x(t,\tau,\xi,\omega) + y(t,\tau,\xi,\omega),$$

for which $x(\tau,\tau,\xi,\omega) = \xi$, $e^{-\mu|t-\tau|}y(t,\tau,\xi,\omega)$ is continuous and bounded function with the value in E_2. The derivative $y_t(t,\tau,\xi,\omega)$ is continuous function in E_2^*. The mapping $\tau\to(x(\cdot,\tau,\xi,\omega), y(\cdot,\tau,\xi,\omega))$ is continuously differentiated on the whole real line. Moreover the function $y(t,t,\xi,\omega)$ satisfies the condition

$$\lim_{\omega\to\infty} \| y(t,t,\xi,\omega) \|_{C(H_2)} = 0 .$$

There exists $\bar{\mu} > 0$ such that for any point $(\tau,\eta) \in \mathrm{R} \times \mathrm{E}_2$ the equation (1) has the solution unique on semiaxis $t \geq \tau$

$$u(t,\tau,\xi,\omega) = x(t,\tau,\xi,\omega) + y(t,\tau,\xi,\omega)$$

for which $y(t,\tau,\eta,\omega) = \eta$, $x(t,\tau,\eta,\omega)e^{\mu(t-\tau)} = \eta$, is bounded on $[\tau,\infty)$ function and

$$\int_t^{t+1} \| e^{\mu(s-\tau)} \dot{x}(s,\tau,\eta,\omega) \|^2 \, ds < \infty \quad \text{for all} \quad t \geq \tau .$$

The mapping $\tau\to(x(\cdot,\tau,\eta,\omega),y(\cdot,\tau,\eta,\omega))$ is continuously differentiated.
Moreover the function $x(\tau,\tau,\eta,\omega)$ satisfies the condition

$$\lim_{\omega\to\infty} \| x(\tau,\tau,\eta,\omega) \|_{C(E_1)} = 0 .$$

From Theorem 2 follows:

Corollary. The function $\Phi(t,\omega)$ defining the m-dimensional linear integral manifold of the equation (1) is determined by

$$\Phi(t,\omega)\xi = y(\tau,\tau,\xi,\omega).$$

The function $\mathrm{F}(t,\omega)$ defining the m-codimensional linear integral manifold of the equation (1) is determined by

$$F(t,\omega)\eta = x(\tau,\tau,\eta,\omega).$$

3. Exponential splitting and reduction principle

Now consider the problem of exponential splitting. Let P, Q be projectors E^* on E_1^*, E_2^* respectively. Suppose that P, Q commutate with the operator $\overline{A}$. Denoting $\overline{A}_{|H_1} = B$,

$\overline{A}_{|H_2} = C$ we write the equation (1) in the form

$$\left(\frac{d}{dt} + B\right)x + P\tilde{A}(\omega t)(x+y) = 0,$$
$$\left(\frac{d}{dt} + C\right)y + Q\tilde{A}(\omega t)(x+y) = 0. \tag{2}$$

where $\tilde{A}(t) = A(t) - \overline{A}$.

It is not difficult to see, that the differentiable function $\Phi(t,\omega)$, defining the linear integral manifold of the system (2) bounded on the whole real axis is solution of the equation

$$\frac{d}{dt}\Phi + C\Phi + Q\tilde{A}(\omega t)(I+\Phi) - \Phi(B + P\tilde{A}(\omega t))(I+\Phi) = 0. \tag{3}$$

By the formula $y = \Phi(t,\omega)x + \eta$ we transform system (2) to the form

$$\frac{dx}{dt} + Bx + P\tilde{A}(\omega t)(x + \Phi(t,\omega)x + \eta) = 0,$$
$$\frac{d\eta}{dt} + C\eta + Q\tilde{A}(\omega t) + \Phi(t)P\tilde{A}(\omega t)\eta = 0. \tag{4}$$

The system (4) satisfies the conditions of Theorem 1 and has the m-codimensional linear integral manifold.

By the formula $x = \xi + F(t,\omega)\xi$ we transform the system (4) to diagonal form

$$\frac{d\xi}{dt} + B\xi + P\tilde{A}(\omega t)(\xi + \Phi(t)\xi) = 0,$$
$$\frac{d\eta}{dt} + C\eta + (Q\tilde{A}(\omega t) + \Phi(t)P\tilde{A}(\omega t))\eta = 0.$$

The transformation $(x,y)\to(\xi,\eta)$ realizes the exponential splitting of the equation (2). Simple analysis implies that the condition 6 in Theorem 1 can be exchanged by condition on boundedness specter of the operator $B = \overline{A}_{|H_1}$ from a specter of the operator $C = \overline{A}_{|H_2}$. This makes possible to generalize Theorem 5.2 [5, p. 266].

Theorem 3. Assume conditions 1 - 5 by Theorem 1. Assume also that the averaging equation $\overline{L}u = 0$ admits the exponential splitting. Then there exist ω_0 such that for any $\omega > \omega_0$ the equation (1) also admits the exponential splitting.

Theorem 3 and Theorem 3 [4] imply the following Theorem.

Theorem 4 (*Reduction principle*). Assume the conditions of Theorem 1. Then there exists ω' such that for any $\omega > \omega'$ a trivial solution of the equation (1) is stable, asymptotically stable if and only if the trivial solution of this equation on the m-dimensional linear integral manifold $\sigma_m(\omega)$ is stable, asymptotically stable, respectively.

4. ASYMPTOTIC EXPANSION OF FINITE-DIMENSIONAL LINEAR INTEGRAL MANIFOLD.

As a rule we cannot construct effectively the function $\Phi\ (t,\omega)$ defining the integral manifold $\sigma_m(\omega)$. Therefore it is important, especially while the investigating the stability problem for the equation (1), to construct the approximate linear integral manifold, corresponding to $\sigma_m(\omega)$ [7]. We can construct the approximate finitedimensional linear integral manifold of the equation (1) while considering the equation (3) and using the following theorem.

Theorem 5. Assume the conditions by Theorem 1, Assume also that the function $P(t,\omega)\in C(\mathrm{Hom}(E_1,E_2))$ such that $P(t,\omega)\to 0$, $\omega\to\infty$ in the sense of norm of space $C(\mathrm{Hom}(E_1,E_2))$ exists and satisfies the equation

$$\frac{d}{dt}\Phi + CP + Q\tilde{A}(\omega t)(I+P) - P(B + P\tilde{A}(\omega t))(I+P) = \Delta(t\omega,\omega)\,, \tag{5}$$

where $\Delta(t\omega,\omega)$ is a continuous function with respect to t with values in $\mathrm{Hom}(E_1,H_2)$ Moreover the inequality

$$\|\Delta(t,\omega)\|_{\mathrm{Hom}(E,H2)} \le k\omega^{-p}\ ,\quad k>0$$

holds for all $\omega > \omega'$ and for some natural p.

Then if the function $\Phi(t,\omega)$ defines the m-dimensional linear integral manifold of the equation (1), then there exist the constants $k > 0$, ω'' such that for $\omega > \omega''$ the inequality

$$\|\Phi(t,\omega) - P(t,\omega)\| \le k\omega^{-p}$$

holds.

Corollary. If $A(t)$ is the almost-periodic function with respect to t, then there exists a polynomial $P(\tau,\omega)$ with respect to ω^{-1} of order p with the almost-periodic coefficients, the function $\Delta(t,\omega)$ satisfying the condition (6) such that $P(t\omega,\omega)$ satisfies the equation (5).

Remark. These results are considered in details in [8].

5. EXAMPLE

Consider a problem

$$u_t = (1 + a\cos(\omega t)g(x))u_{xx} + u\ ,\quad t>0\,,\quad 0<x<\pi\ , \tag{6}$$

$$u(0,t)=0\,,\quad u(\pi,t)=0\,,$$

where a is constant,

$$|a|<1\,,\quad \omega >> 1\,,\quad g(x)=\begin{cases}0, & 0\le x\le \pi/2;\\ 1, & \pi/2<x\le\pi.\end{cases}$$

Here $H = L^2[0,\pi]$. The averaged problem is

$$u_t = u_{xx} + u\ ,\quad t>0\,,\quad 0<x<\pi\ ,$$

$$u(0,t)=0\,,\quad u(\pi,t)=0\,.$$

According to the designations of Theorem 1

$$\overline{A}u = -u_{xx} - u\,,\quad \sigma(\overline{A}) = \{n^2 - 1,\quad n = 1,2,\dots\}\,,$$

$$H_1 = \mathrm{Span}\{\sin x\}\,,\quad H_2 = \{\varphi : \int_0^\pi \varphi(x)\sin x dx = 0\}\,.$$

The integral manifold of the parabolic problem (6) can be represented in the form

$$u(x) = s\sin x + \sum_{k=2}^{\infty} f_k s \sin kx.\,, \tag{7}$$

A flow on the integral manifold (7) is described by the equation

$$\dot{s}\sin x + \sum_{k=2}^{\infty}(\dot{f}_k s + f_k \dot{s})\sin kx =$$

$$= (1 + a\cos(\omega t)g(x))\left(-s\sin x - \sum_{k=2}^{\infty} f_k s k^2 \sin kx\right) + s\sin x + \sum_{k=2}^{\infty} f_k s \sin kx.$$

After multiplying by sin x and integrating we have

$$\dot{s} = a\cos(\omega t)\left(-\frac{1}{2} - \sum_{k=2}^{\infty} f_{2n}(2n)^2 \frac{1}{2}(-1)^n\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)\right)s.$$

After multiplying by sin $2nx$ and integrating we have

$$\dot{f}_{2n} + f_{2n} a\cos(\omega t)\left(-\frac{1}{2} - \sum_{k=2}^{\infty} f_{2n}(2n)^2 \frac{1}{2}(-1)^n\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)\right)$$

$$= (1 - (2n)^2) f_{2n} + a\cos(\omega t)\left(-\frac{1}{2}(-1)^n\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right) +\right.$$

$$\left. + \frac{1}{2}\sum_{k=2}^{n}(2k+1)^2\left(\frac{1}{2k+1-2n} - \frac{1}{2k+1+2n}\right)(-1)^{k-n} f_{2k+1}\right)$$

After multiplying by sin $(2n + 1)x$ and integrating we have

$$\dot{f}_{2n+1} + f_{2n+1} a\cos(\omega t)\left(-\frac{1}{2} - \sum_{k=2}^{\infty} f_{2n}(2n)^2 \frac{1}{2}(-1)^n\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)\right)$$

$$= (1 - (2k+1)^2) f_{2k+1} + a\cos(\omega t)\frac{1}{2}\sum_{k=2}^{n}(2k)^2\left(\frac{1}{2k+1-2n} - \frac{1}{2k+1+2n}\right)(-1)^{k-n} f_{2k+1}.$$

The integral manifold is defined in the first approximation as a periodic solution of the system

$$\dot{f}_{2n} = (1 - (2n)^2) f_{2n} + a\cos(\omega t)(-1)^n \frac{2n}{4n^2 - 1}\,,$$

$$\dot{f}_{2n+1} = (1-(2n+1)^2) f_{2n+1}.$$

We find from here

$$f_{2n} = a\left(\frac{(2n)^2 - 1}{((2n)^2-1)^2} \frac{1}{2} (-1)^{n+1} \left(\frac{1}{2n-1} + \frac{1}{2n+1} \right) \cos \omega t \right.$$

$$\left. + \frac{\omega}{((2n)^2-1)^2+\omega^2} (-1)^{n+1} \left(\frac{1}{2n-1} + \frac{1}{2n+1} \right) \sin \omega t \right),$$

$$f_{2n+1} = 0.$$

Upon substituting the obtained values in the equation on the manifold and analysing we came to the conclusion on the unstability of trivial solution of the problem (6) for the sufficiently large ω.

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EKSPONENCIJALNO DELJENJE LINEARNE PARABOLIČNE JEDNAČINE

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Razmatra se linearna parabolična jednačina sa brzo promenljivim koeficijentima. Rešen je problem konstruisanja transformacije kojom se realizuje deljenje početne jednačine. Ustanovljena je značajna uloga integralnih mnogostrukosti u rešavanju ovog problema.