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AGAIN ON THE ABSOLUTE INTEGRAL *

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Abstract. *The fact that the still contested idea of introducing an absolute integral as the operator inverse to the operation of covariant differentiation has its origins in papers dating back to the first third of our century – is pointed out. Some stipulations are done.*

Keywords: *absolute integral*

1. INTRODUCTION

Attending lectures at the Department of Mechanics (Faculty of Sciences, University of Belgrade), I had occasion to wonder more than once whether the various derivations of equations in three-dimensional Euclidian space, connected to the procedure of integration, unavoidably had to be carried out *in Cartesian coordinates*. This was usually justified by "formal difficulties" arising in an attempt to derive these same equations in curvilinear cordinates – hence, the equations derived in Cartesian coordinates were proclaimed, on the basis of their tensorial form[1] , to be valid in the case of arbitrary coordinates.

In 1976 I obtained an answer to these questions for the first time, from communications and papers [7] and [9] of Professor Veljko Vujičić, in which he postulated the absolute integral of a tensor as an integral operator "... *by which it is possible to obtain the initial tensor from its absolute differential*." (s. [9], p. 375). The doubt with which the audience responded to these communications, concerning the sense of the introducing such a notion, could in my opinion be resolved only by proving that this idea – introduced in an affine *n*-dimensional space – follows in a natural way from

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[1] This way of concluding is frequently encountered in literature: "*The tensor equation... having been established in a special co-ordinate system, is valid in all co-ordinate systems*." ([2], p. 543) or: "*Since this formula is constructed in full tensorial form, it is true not only in Cartesian coordinates... but also in any coordinate system*." ([5], p. 172).

the usual notion of a curvilinear integral after the introduction of arbitrary generalized coordinates, at least in three-dimensional Euclidean spece.

This was done in paper [12]: taking the example of an absolute integral of an absolute differential of a sufficiently smooth vectorial functions ***V***, from point P_o to point P on an *arbitrary* curve in Euclidean space, the following formula was quoted

$$\overset{\nabla}{\int\limits_{P_oP}} g_{\beta}^{\cdot\alpha}(M,P)DV^{\beta}(M) = V^{\alpha}(P) - V^{\beta}(P_o)g_{\beta}^{\cdot\alpha}(P_o,P) = V^{\alpha}(P) - A^{\alpha}(P_o,P)\,. \qquad (1)$$

It should be noted that a slightly different symbol from the one in [7] or [9] has been used for the absolute integral

$$\overset{\nabla}{\int\limits_{P_oP}} DV^{\alpha} \equiv \overset{\nabla}{\int\limits_{P_oP}} g_{\beta}^{\cdot\alpha}(M,P)DV^{\beta}(M) \qquad (2)$$

in order to emphasize that this operation in Euclidean space reduces to an integration in accordance with Ericksen's concept of integration of vector (tensor) fields in curvilinear coordinates (s.p. 808 in [3]). Here M is the "current" point of integration, and $g_{\beta}^{\cdot\alpha}$ are the shifting operators ("Euclidean shifters"; [3], p. 806); Einstein's summation convention for diagonally repeated indices is used, and Greek indices have the range {1,2,..., n}, where n is the number of the dimensions of the space; the vector **A**, i.e. A^{α}, having been obtained by the parallel transport of the vector $V^{\beta}(P_o)$, represents a covariantly constant vector field. As an illustration of the use of the notion of the absdolute integral, an integration of Killing's equations in generalized coordinates was done in the above mentioned paper. It is worth mentioning that, after the presentation of this paper at the Congress [11], a question concerning the possibility of applying an absolute integral to determine the displacement field from the strain field (so-called Cesàro's formula), but in curvilinear coordinates, was raised; my answer was affirmative, and this was realized[2] in [16].

In the meantime, from the discussions following new communications of Professor Vujičić, an impression could be gained that, instead of the first a'priori resistance to the notion of an absolute integral, an opinion prevailed in the audience that this notion in Euclidean space "*does not represent anything new*", that it has been "*known for a long time already*" or is even "*superfluous*"[3] , but has no sense in non-Euclidean spaces!

Time passed, and other preoccupations followed... Even so, in the meantime I encountered the assertation asserting that, in a space equipped with linear connection, "*The problem of integrating a field of tensor quantities along a given curve ... reduces to one of integrating a system of ordinary linear differential equations of the first order*." ([10], p. 286), but without mentioning an (integral) operator able to realize this assertation. It should be noted that a field defined *along a curve* was discussed here, while we face quite a different situation when the tensor field is given throughout the

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[2] Even back in 1978, I did not doubt the feasibility of carrying out all these procedures in curvilinear coordinates, but the opportunity to do this arrised only recently, in connection with considerations in the shell theory. Naturally, the effective use of this formula is reduced to evaluating the usual curvilinear integrals (s. (1)), but the proposed approach to formula's derivation and its ensuing form are quite new, judging from the literature accessible to me.

[3] Nonetheless, I have not, until now, encountered the derivation of Cesàro's formula in the way proposed in [16].

whole of space or in one of its domains: "*The problem of integration, i.e. the operation inverse to covariant differentiation, then ... is difficult and has not yet been solved in its full generality*." ([10], p. 287). Hence, although "*Not much progress has been made on the problem of giving not only the integrability conditions but also the solutions. Apart from the papers of Dubnov*[4]*, Lopschitz*[5]*, and Graiff*[6] *... little has been done to date*." ([10], p. 287-288), an attempt to find an operation *inverse* to the operation of covariant differentiation has certainly not been said to be *a'priori* without sense[7] , nor have the integrability conditions been connected only with the path independence conditions.

Papers [13] and [14] appeared in the meantime, proposing the use of the idea of an absolute integral to solve some problems of analytical mechanics, but it was stated that still " *.... the problem of the covariantly constant tensor* [**A**, i.e.] $A^{\beta_1 \dots \beta_n}_{\alpha_1 \dots \alpha_m}$ *in Riemannian spaces is not solved generally*..." ([13], p. 1307). By courtesy of Professor Vujičić, I had the opportunity to look through the thesis [15], referring to tensorial integration on manifolds; there one can find quoted several authors who have been occupied with tensorial integration defined as an operation inverse to covariant differentiation,but not in the way postulated in [7] and [9]

$$\overset{\nabla}{\int\limits_{P_o P}} DV^{\alpha} = V^{\alpha}(P) - A^{\alpha}(P_o, P)\,; \tag{3}$$

cf. with (1.2) in [7] or (2.1) in [9].

However, Professor Vujičić himself just recently (during a visit to Moscow) obtained the paper [1], so that, once again thanks to him I had the opportunity to return to some of my interests now nearly two decades old. And, lo and behold − something *a'priori* declared to be *nonsens*, was the subject of a communication on one of the sessions of the French Academy od Sciences back in the distant year of 1929!

Namely, paper [1] considers the determination of a vector field ***V*** such that, along a curve *K*

$$x^{\nu} = x^{\nu}(t) \tag{4}$$

in a space equipped with linear connection, the absolute differential of this field is equal

$$\frac{DV^{\nu}}{Dt} = \upsilon^{\nu} \tag{5}$$

where υ^{ν}(t) is the field given at the points of the curve *K*; the problem reduces to solving a system of ordinary linear differential equations of the first order

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[4] Ya. S. Dubnov, *Integration covariante dans les espaces de Riemann à deux et à trois dimensions*, Trudy Sem. Vektor. Tenzor. Anal. 2-3 (1935).

[5] A. Lopschitz, *Integrazione tensoriale in una varietà riemanniana a due dimensioni*, Trudy Sem. Vektor. Tenzor. Anal. 2-3 (1935), p. 200-211.

[6] F. Graiff, *Sull'integrazione tensoriale negli spazi di Riemann a curvatura constante*, Ist Lombardo Accad. Sci. Lett. Rend. A. 84 (1951), p. 155-163.

[7] As far as the problem difficulty is concerned, it could be anticipated because "*The fact that two quantities... of the same species, but attached at two different points in space, cannot be compared causes serious difficulties in tensor analysis* ..." ([10], p. 157). Namely, in order to compare, or add (like in the process of integration ...), any (physical) quantities, they must be transported to the *same point* in space, and then the question of their *parallel transport* unavoidably arises.

$$\frac{dV^{\nu}}{dt}+\Gamma^{\nu}_{\lambda\mu}V^{\lambda}\frac{dx^{\mu}}{dt}=\upsilon^{\nu}\ ; \tag{6}$$

all its solutions, as it is known, can be written in the form

$$V^{\nu}=K^{.\nu}_{\mu}\left(\int K^{\mu}_{.\lambda}\upsilon^{\lambda}dt+C^{\mu}\right)\ , \tag{7}$$

where C^{μ} are the constants, while $K^{.\nu}_{\mu}$ represent the fundamental solution of the homogeneous system corresponding to the system (6) and $K^{\mu}_{.\lambda}$ is defined by

$$K^{.\nu}_{\mu}K^{\mu}_{.\lambda}=\delta^{\nu}_{\lambda}\quad ,\quad K^{.\nu}_{\mu}K^{\lambda}_{.\nu}=\delta^{\nu}_{\mu}\ . \tag{8}$$

In the next step, by transforming the expression (7) and perceiving a "*wide analogy*" of this procedure with ordinary integration, Horàk introduced in [1] "*un symbole d'integration absolue le long d'une courbe*"(!)

$$_{-}|^{-}\ \upsilon^{\nu}dt=K^{.\nu}_{\mu}\int K^{\mu}_{.\lambda}\upsilon^{\lambda}dt \tag{9}$$

and rewrote the formula (7) in the form

$$V^{\nu}={}_{-}|^{-}\ \upsilon^{\nu}dt+K^{\nu}\ , \tag{10}$$

designating

$$K^{\nu}=K^{.\nu}_{\mu}C^{\mu}\ , \tag{11}$$

and finally defining as an "*intégrale absolue du vecteur* υ^{ν} *prise le long de* (K) *entre les limites* t_{o} *et* t" the following vector

$$_{-}|^{-t}_{t_o}\ \upsilon^{\nu}dt=K^{.\nu}_{\mu}(t)\int_{t_0}^{t}K^{\mu}_{.\lambda}\upsilon^{\lambda}K^{\mu}_{.\lambda}\upsilon^{\lambda}dt\ ! \tag{12}$$

But, let us now return to expression (7). This expression can be rewritten in a somewhat different form. Namely, if $K^{.\nu}_{\mu}$ is such a fundamental solution reducing to the Kronecker delta when $t = t_{o}$, then the solution of the nonhomogeneous system (6) can be written down as

$$V^{\nu}(t)=K^{.\nu}_{\mu}(t_o,t)\left(\int_{t_o}^{t}K^{\mu}_{.\lambda}(t_o,\tau)\upsilon^{\lambda}(\tau)d\tau+V^{\mu}_{o}\right) \tag{13}$$

where $V^{\mu}_{o}\equiv V^{\mu}(t_o)$. A similar form can be found, for example, in [6] (the expression (22) at the p. 135), but the method of presentation used here certainly points out the fact that the solution is a function of the unitial values and of the choice of the point t_{o}[8]. But, after this stipulation concerning the dependence on the variables, (13) can obviously be rewritten in the form

$$V^{\nu}(t)=\int_{t_o}^{t}K^{.\nu}_{\mu}(t_o,t)K^{\mu}_{.\lambda}(t_o,\tau)\upsilon^{\lambda}(\tau)d\tau+K^{.\mu}_{\mu}(t_o,t)V^{\nu}_{o}\ , \tag{14}$$

[8] The first index in $\mathbf{K}(t_{o},t)$, either superscript or subscript, refers to the point on curve K determined by the first argument, while the second one refers to the point determined by the second argument.

or, having in mind that the composition $K_{\mu}^{\cdot\nu}(t_o,t)K_{\cdot\lambda}^{\mu}(t_o,\tau)$ is a fundamental solution, too (s.pp. 78-79 in [8]), in the form

$$V^{\nu}(t)=\int_{t_o}^{t}K_{\lambda}^{\cdot\nu}(\tau,t)\upsilon^{\lambda}(\tau)d\tau+K_{\mu}^{\cdot\nu}(t_o,t)V_o^{\mu}\,, \tag{15}$$

where the same "kernel" is kept for this fundamental solution.

In order to provide a geometrical interpretation to the previous result, we point out that the homogeneous system corresponding to (6), i.e. to (5) represents the condition of *parallel transport*, for example, of a vector **u** along the given curve *K*. However, "*From the linear homogeneous character of the differential equations*" (s.p. 59 in [4]) corresponding to (6), it follows that the vector $u_o^{\nu}\equiv u^{\nu}(t_o)$ at the point $P_o\equiv P(t_o)$ determines by parallel transport the vector $u^{\nu}\equiv u^{\nu}(t_o)$ at the point $P\equiv P(t)$ as a linear homogenous function; this, in essence, means that we may write

$$u^{\nu}=K_{\lambda}^{\cdot\nu}u_o^{\lambda} \tag{16}$$

since the linear combination at the right side in (16) is certainly a solution of the homogeneous system, and – because of the uniqueness of the solution – this combination must be equal to the vector obtained by parallel transport of the vector u_o^{λ}.

Consequently, the coefficients $K_{\mu}^{\cdot\nu}(t_o,t)$ represent *the shifting operator along a given curve*, i.e. "*the parallel propagator*" ([4], p. 59). We add that the afore mentioned composition of fundamental solutions now also receives its geometrical sense – the composition of two parallel displacements is in question. Such an operator represents a double tensor field ("*a 2-point tensor*"; [4], p. 59), but it should be noted that it depends on the chosen curve, too.

If equation (5) is satisfied along curve *K*, then (15) can be rewritten in the form

$$\int_{t_o}^{t}K_{\lambda}^{\cdot\nu}(\tau,t)\upsilon^{\lambda}(\tau)d\tau=\overset{\nabla}{\int_{P_oP}}K_{\lambda}^{\cdot\nu}(M,P)DV^{\lambda}(M)=V^{\nu}(P)-K_{\mu}^{\cdot\nu}(P_o,P)V^{\mu}(P_o)\,, \tag{17}$$

and this is, if we introduce (similarly as in (2)) the notation

$$\overset{\nabla}{\int_{P_oP}}DV^{\nu}\equiv\overset{\nabla}{\int_{P_oP}}K_{\lambda}^{\cdot\nu}(M,P)DV^{\lambda}(M)\,, \tag{18}$$

in essence the form

$$\overset{\nabla}{\int_{P_oP}}DV^{\nu}=V^{\nu}(P)-K_{\mu}^{\cdot\nu}(P_o,P)V^{\mu}(P_o)=V^{\nu}(P)-A^{\nu}(P_o,P)\,, \tag{19}$$

postulated in [13] for an absolute integral in Riemannian space; we have thus demonstrated the geometrical sense of the vector A^{ν} covariantly constant *along the curve K*, as well as how it can be evaluated. Hence this operation can be used to determine a vector field if its absolute differential is known. On the other hand, it is clear from (12) (although some inconsistency in designating the vriable of integration is noticeable there) and (13) – (15) that the notion of an absolute integral in (19) coincides with one introduced in [1].

Of course, an integral defined in this way in non-Euclidean space is not, in general, independent of the chosen curve *K*. This dependence on the path of integration may be the source of heretical ideas about the necessity for a different definition of the operations of differentiation and integration in these spaces; but, this should be the subject of future activities.

Anyhow, without hurrying to immediately proclaim the paper [1] as crucial evidence to the correctness of the idea of Professor Vujičić, I finally wich only to emphasize something that is undisputable – the least that the notion of an absolute integral deserves at this point is to be fully reconsidered once again. However, this note has no such pretension, and hence it does not offer a ffinal conclusion. A modest aim was to point out to the facts clearly showing that there is no reason to shrink from the notion of an absolute integral.

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PONOVO O APSOLUTNOM INTEGRALU

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Ukazano je na činjenicu da još uvek osporavana ideja o uvođenju apsolutnog integrala, kao operatora inverznog operaciji kovarijantnog diferenciranja, ima svoje korene i u radovima iz prve trećine ovog veka. Data su određena utanačenja.

Ključne reči: *apsolutni integral*