UNIVERSITY OF NIŠ
The scientific journal **FACTA UNIVERSITATIS**
Series: **Mechanics, Automatic Control and Robotics** Vol.2, No 8, 1998 pp. 709 - 713
Editor of series: Katica (Stevanovi}) Hedrih, e-mail: katica@masfak.masfak.ni.ac.yu
Address: Univerzitetski trg 2, 18000 Niš, YU, Tel: (018) 547-095, Fax: (018)-547-950
http:// ni.ac.yu/Facta

A HYBRID WKB - GALERKIN METHOD AND ITS USING TO APPLIED MECHANICS PROBLEMS

UDC: 531:539.3.

V. Z. Gristchak, E. M. Dimitrijeva

Applied Mathematical Department, Zaporozhye State University
66 Zhukovskogo str., UA - 330600, Zaporozhye, Ukraine

Abstract. *In this paper, a hybrid WKB-Galerkin method is presented, as well as the possibilities of its applications on applied mechanic problems.*

1. INTRODUCTION

Solution of many mechanical problems that deals with the differential equations with "small" or "large" parameters with using of numerical methods in some cases can give the results that are far away from exact solutions because the accumulation of errors takes place when the parameters strive to be equal zero or infinite, respectively. Therefore, in these cases some approximate asymptotic methods are applied, though their direct using have limit area of application.

Geer and Andersen [2], [3] have discussed a two-step hybrid perturbation - Galerkin method for the solution of some types of differential equations and applied mechanical problems that involve a parameter.

A hybrid WKB-Galerkin method [4], [5] is one of the new approaches to the solution of boundary problems of mathematical physics that are based on the differential equations with parameter. Proposed hybrid approach consists from the two step method: WKB-Galerkin method gives possibility to build approximate solutions as well for "small" as for "large" values of parameter using only few first terms in asymptotic expansion. Using the criteria of Galerkin orthogonality, the effect of higher terms of asymptotic solution is included into the lowest terms. The hybrid WKB-Galerkin method was used for different classes of differential equations and for some applied problems of mechanics of deformable bodies.

Received June 6, 1997

2. DESCRIPTION OF THE METHOD

Suppose we are seeking an approximate solution $U(x,\varepsilon)$ to the boundary problem.

$$L(U,x,\varepsilon)=0\,, \tag{1}$$

where L is some linear differential operator of n-th order (in the general case with variable coefficients); ε is a parameter near the highest derivative; x is located in some interval (a,b); and $U(x,\varepsilon)$ is satisfied by the given boundary conditions.

In the first step we present the problem solution $U(x,\varepsilon)$ in correspondence with the WKB - procedure in the form

$$U(x,\varepsilon)=\exp\left(\int_a^x\sum_{i=0}^{\infty}\gamma_i(\varepsilon)\psi_i(x)dx\right). \tag{2}$$

where $\gamma_i(\varepsilon)$ is an appropriate asymptotic sequence and each $\psi_i(x)$ are chosen as coordinate functions for the Galerkin technique and an approximation $U_H(x,\varepsilon)$ for $U(x,\varepsilon)$ is sought in the form

$$U_H(x,\varepsilon)=\exp\left(\int_a^x\sum_{i=0}^{\infty}\delta_i(\varepsilon)\psi_i(x)dx\right). \tag{3}$$

where the unknown parameters $\delta_i(\varepsilon)$ are complex functions of ε and all $\psi_i(x)$ are approximate coordinate functions that were found in the previous step. To determine the unknown coefficients $\delta_i(\varepsilon)$ ($i=0,\ldots,N$) we substitute (3) into (1). Thus we obtain a product of the right hand side of an expression (3) and some expression in which a leading derivative of functions $\psi_i(x)$ is one order less than the leading derivative in equation (1)

$$L(U_H,x,\varepsilon)=\exp\left(\int_a^x\sum_{i=0}^{\infty}\delta_i\psi_i dx\right)R(\psi_0,\ldots\psi_N,\frac{d\psi_0}{dx},\ldots\frac{d^{n-1}\psi_N}{dx^{n-1}},\delta_0,\ldots\delta_N,x,\varepsilon)\,. \tag{4}$$

It is necessary to satisfy the right hand side of the governing equation (1), that leads to

$$R(\psi_0,\ldots\psi_N,\frac{d\psi_0}{dx},\ldots\frac{d^{n-1}\psi_N}{dx^{n-1}},\delta_0,\ldots\delta_N,x,\varepsilon)=0\,. \tag{5}$$

Therefore we demand that the residual R be orthogonal to the $N+1$ coordinate functions $\psi_i(x)$ over the interval (a, b), i.e.

$$\int_a^b R(\psi_0,\ldots\psi_N,\frac{d\psi_0}{dx},\ldots\frac{d^{n-1}\psi_N}{dx^{n-1}},\delta_0,\ldots\delta_N,x,\varepsilon)\psi_i dx=0\,,\ i=0,\ldots N\,. \tag{6}$$

Equations (6) represent a set of $N+1$ unknown coefficients δ_i. While equations (6) must generally be solved numerically, their solutions are simpler than a direct numerical solution of equation (1).

As examples of the stress-strain state problem of orthotropic shell of revolution and buckling problem of conical shell are given.

3. THE STRESS-STATE PROBLEM OF ORTHOTROPIC SHELL OF REVOLUTION

We consider the stress-strain state of the orthotropic shell given by a differential equation in complex form [1]

$$\frac{d^2\sigma}{ds^2}-\frac{\sin(\vartheta)}{r}\frac{d\sigma}{ds}-\lambda\frac{\sin^2(\vartheta)}{r^2}\sigma+i\sqrt{\frac{\Omega_0}{C_{11}D_{11}}}\frac{\sigma}{R_2}=\Phi(s)\,, \tag{7}$$

where

$$\sigma=W-i\frac{C_{11}}{\Omega_0}\sqrt{\frac{\Omega_0}{C_{11}D_{11}}}V\,, \tag{8}$$

and W, V are the functions to be determined; $\Phi(s)$ is the surface load function. In this equation small parameter can be allocated

$$\varepsilon=\frac{1}{k}=\sqrt{\frac{C_{11}D_{11}}{\Omega_0}}\,. \tag{9}$$

We shall decide a given problem for the case of free boundaries. The particular solution of equation (7) is described in [1]. The general solution, have been obtained with using of hybrid WKB - Galerkin method will be recorded in the form [5]

$$\begin{aligned}\sigma_H={}&Q_1\exp\int_{s_0}^{s}\left[\left(\frac{g(s_0)-g(s_1)}{4\int_{s_0}^{s_1}g^{3/2}(s)ds}+\sqrt{\frac{1}{\varepsilon^2}+\left(\frac{g(s_0)-g(s_1)}{4\int_{s_0}^{s_1}g^{3/2}(s)ds}\right)^2}\right)\sqrt{g(s)}+\frac{1}{2(l-s)}\right]ds+\\&+Q_2\exp\int_{s_0}^{s}\left[\left(\frac{g(s_0)-g(s_1)}{4\int_{s_0}^{s_1}g^{3/2}(s)ds}-\sqrt{\frac{1}{\varepsilon^2}+\left(\frac{g(s_0)-g(s_1)}{4\int_{s_0}^{s_1}g^{3/2}(s)ds}\right)^2}\right)\sqrt{g(s)}+\frac{1}{2(l-s)}\right]ds\end{aligned} \tag{10}$$

where

$$g(s)=\frac{\left(-\frac{d\vartheta}{ds}\cos(\vartheta)r+\frac{dr}{ds}\sin(\vartheta)\right)}{2r^2k^2}+\frac{\sin^2(\vartheta)}{r^2k^2}\left(\lambda+\frac{1}{4}\right)-\frac{i}{hR_2}\,, \tag{11}$$

Q_1, Q_2 - any constantes.

For a conical shell the equation (7) allows the exact solution. However for small values of parameter ε this solution is not correct, as the determinant, made for search of free members in exact solution, has large spread of values including zero. These results are in a sharp increasing of error. The analysis of the solutions with using of a hybrid method, method WKB and Ambartsumian's method for small ε has shown reasonably good enough correspondance. In fig. 1 all three solutions are constructed for a pressure σ_3 [1] at $\varepsilon = 0.409$.

In fig. 2 the solution with using of the hybrid approach is compared at $\varepsilon = 9.09$ with the exact solution and other asymptotic methods (method WKB and Ambartsumian's method) for a normal pressure σ_3. The hybrid solution in this case is the most appropriate at all interval of coordinate s.

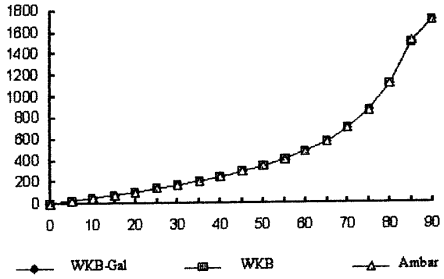

Fig 1.

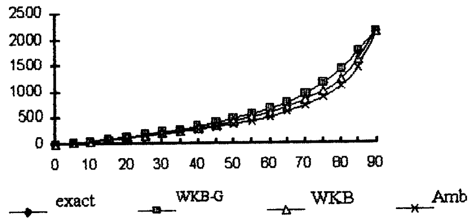

Fig. 2.

It is necessary to note that the analysis of a relative error of a deviation of values σ_3, found on the basis of a hybrid two terms approximation with comparison of an exact values σ_3 are given. A maximum relative error for one term approach is 2,3%, and for two terms approach is 0,13%. So, the accuracy has increased in 18 time.

4. BUCKLING OF CONICAL SHELL

The mathematical statement of a buckling problem of conical shell is reduced to boundary problem of a system of two differential equations with variable coefficients. To solve exactly this system of equations in general case is impossible. The hybrid WKB-Galerkin method for solution of this problem proposed.

We shall consider a conical shell of constant thickness loaded by uniform external pressure. By various replacements and simplifications, which are described in [6] in detail, we shall reach to a following ordinary differential equation of the fourth order, which will be recorded as follows

$$\frac{d^2}{ds^2}\left(s^3\frac{d^2\psi}{ds^2}\right)+\frac{d}{ds}\left(p^2 l_1 \eta \frac{d\psi}{ds}\right)+\left(\frac{p^4 l_1^2}{s^3}-\frac{p^3\nu}{l_1}\right)\psi = 0\,, \tag{12}$$

where s is current coordinate along a forming of a cone; $\psi(s)$ is function of normal displacement. In equation (12) we introduce a small parameter

$$\varepsilon = \frac{\sin^2(\alpha)}{n^2} \tag{13}$$

The solution constructed with using of hybrid WKB-Galerkin technique is

$$\psi_H = C_1 \exp(\delta_1 \int_{l_1}^{s} \psi_0 ds) + C_2 \exp(\delta_2 \int_{l_1}^{s} \psi_0 ds) + C_3 \exp(\delta_3 \int_{l_1}^{s} \psi_0 ds) + C_4 \exp(\delta_4 \int_{l_1}^{s} \psi_0 ds)\,, \quad (14)$$

where

$$\psi_0 = \sqrt{-\frac{\eta p^2 l_1}{2s^2\delta^3} + \sqrt{\frac{p^4 l_1^2}{\delta^4 s^6}\left(\frac{\eta^2}{4} - 1\right) + \frac{p^3\nu}{\delta^4 l_1 s^3}}}\,. \quad (15)$$

In fig. 3 dependence of external pressure on quantity of formed waves in circumferential direction is analyzed with using of the hybrid approach and WKB method [3]. As can be seen, both solutions have good conformity. The dependence $q = q(n)$ has a minimum at $n = 6$. As an exaomple from this follows that the critical value of external pressure q_{xp} is q [6]=0.821.

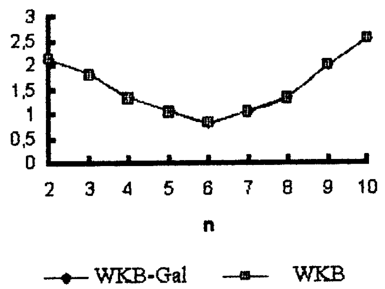

Fig. 3.

4. Concluding remarks

The results of comparison of hybrid WKB-Galerkin solution with other asymptotic methods gave good correspondence in the interval of changing of parameter where these methods take place. At large values of parameter where the tradition methods give significant error the proposed hybrid WKB-Galerkin method gives enough good correlation with exact solution, of nonhomogeneous system and etc.

References

1. Ambartsumian S.A. *General theory of anisotropic shells*. M.: Nauka. 1974. 448 p.
2. Geer J.F., Andersen C.M. *A hybrid perturbation - Galerkin method for differential equations containing a parameter* . Pan American Cngress on Appl. Mech. 1991. p. 460-463.
3. Geer J.F., Andersen C.M. *A hybrid perturbation - Galerkin technique with application to slender body theory*. SIAM J. Appl. Mech. 1989. Vol. 49, p. 344-361.
4. Gristchak V.Z., Dmitrijeva Ye.M. *A hybrid method on the basis of phase integrals' method and its application*. Mathematician, physics. Zaporozhye. 1995. p. 28-31.
5. Gristchak V.Z., Dmitrijeva Ye.M. *A hybrid WKB - Galerkin method and its application*. Technische Mechanik. 1995. 15, N^{o}4. p. 281-294.
6. Trapezin I.I. *Buckling of conical shell*. Accounts on strength. M. 1960. 5. p. 21-32.

HIBRIDNA WKB-GALERKINOVA METODA I NJENO KORIŠĆENJE NA PROBLEMIMA PRIMENJENE MEHANIKE

V. Z. Gristchak, E. M. Dimitrijeva

U ovom radu je prikazana hibridna WKB-Galerkinova metoda kao i mogućnost njene primene na probleme praktične mehanike.