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EXTENSION OF THE BERNOULLI'S CASE OF A BRACHISTOCHRONIC MOTION TO THE MULTIBODY SYSTEM IN THE FORM OF A CLOSED KINEMATIC CHAIN

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Abstract. *Considering the brachistochronic motion in a homogeneous field of gravity of the multibody system given in the form of a closed kinematic chain, free from external constraints, we prove that the trajectory of the system in the part of the configuration space (this part being the one which includes all the generalized coordinates of the system except the Cartesian coordinate y_C of the system's centre of inertia referring to the vertical axis) is a geodesic. Having in mind that this geodesic is completely determined by the known initial and terminal conditions given for the brachistochronic motion considered, as well as by the nature of the mentioned part of the configuration space, and using the fact that the configuration of our multibody system in this part of the space is determined by the position of a representative point on the geodesic, we further define the position of the whole system using two coordinates only. One of them is the arc-lenght σ, which determine the position of a representative point on the geodesic, and another is the coordinate y_C, determining the one-dimensional subspace of the configuration space. This subspace, together with the subspace containing the geodesic, constitute the complete configuration space of our system. Considering the motion of the system in such a way, we obtain the result referring to the trajectory of the system which is completely analogous to the famous Bernoulli's result, found in the case of a single particle. Obtained result is illustrated by the numerical example.*

Key words: *Brachistochronic motion, Closed kinematic chain, Analogy with the Bernoulli's case of a brachistochronic motion*

1. INTRODUCTION

The problem of the motion of a mechanical system subject to the action of

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scleronomic holonomic ideal constraints, from the configuration defined by the initial time $t = t_0$ to the configuration specified by $t = t_1$, demanding that the interval $\tau = t_1 - t_0$ takes its minimal value (brachistocronic motion of the system) was considered from the moment when John Bernoulli, in 1696, formulated and solved the brachistocrone problem in the case of a free particle moving in a homogeneous field of gravity.

Extension of the Bernoulli's case to the system of particles was performed only in view of deducing the differential equations of the brachistochronic motion of the system, as well as of determining the generalized control forces which make possible such a motion. Generalizations of such a kind can be found, for instance, in [2, 8] (in the case of a conservative mechanical system), [6, 3] (where a non-conservative system was considered), [1] (the case of a dissipative system), and [4, 9, 10] (where the non-holonomic systems were treated). However, even in the case of a very simple mechanical system of particles which moves in a homogeneous field of gravity, the results of the Bernoulli's case referring to the trajectory along which the particle moves (this trajectory being cycloid in a vertical plane - see, e.g., [7]), were not generalized, up to now.

In this paper we obtain such a generalization for a rather large class of mechanical systems. Namely, we demonstrate that in the case of a system of rigid bodies forming a closed kinematic chain, moving in a homogeneous field of gravity and free from external constraints, there exists, concerning the trajectory of the system, a complete analogy with the Bernoulli's case.

2. Formulation of the Problem

We consider the mechanical system consisting of r rigid bodies forming a closed kinematic chain (Fig.1) which moves in a homogeneous field of gravity. Each pair of the two adjoining bodies in this chain represents a kinematic pair of the fifth class. Arbitrary body (S_i), where $i = 2, 3, \ldots, r$, can move having either rectilinear translation with respect to the adjoining body ($S_{i\text{-}1}$), or rotation about an axis fixed in ($S_{i\text{-}1}$). There are no external constraints imposed to the system, while the internal constraints are ideal.

In order to determine the configuration of the system of rigid bodies (further: the multibody system) considered, we introduce the following frames of reference (Fig. 1):

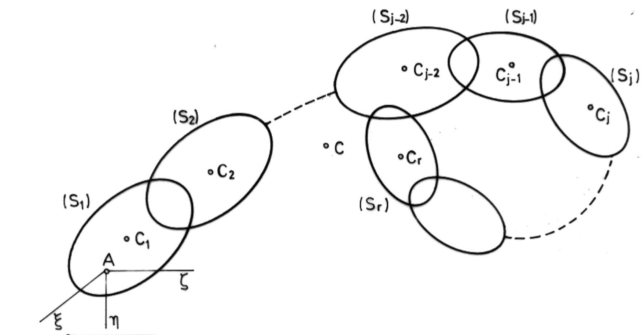

Fig. 1

- Inertial rectangular frame $Oxyz$, with the axis Oy directed vertically downwards;
- Translationally moving rectangular frame $A\xi\eta\zeta$, attached to an arbitrary point A of the body (S_1), with $A\xi \parallel Ox$, $A\eta \parallel Oy$.

Having in mind that the configuration of the body (S_1) relative to $A\xi\eta\zeta$ is defined by means of Euler's angles ψ, θ, φ, it is evident that this body can be replaced by the kinematic chain composed of three bodies (denoted by (V_1), (V_2), (V_3), Fig.2), with kinematic pairs of the fifth class. The first body (V_1) from this chain has rotation ψ about the axis $A\zeta$, the second body (V_2) has rotation θ about the axis attached to the body (V_1) and chosen so that it coincides with $A\xi$ when $\psi = \theta = \varphi = 0$, and finally the third body (V_3) has rotation φ about the axis attached to (V_2), chosen so that it coincides with $A\zeta$ when $\psi = \theta = \varphi = 0$. Masses of the bodies (V_1), (V_2), and consequently the elements of their tensors of inertia, are equal to zero, while the third body (V_3) is equivalent to the body (S_1) from a standpoint of the distribution of masses. In such a way we obtain the new kinematic chain, equivalent to our original chain, with $n = r + 2$ bodies (V_1), (V_2), ..., (V_n), for which we have

$$(V_{i+2}) = (S_i), \quad i = 1,2,\ldots,n.$$

It is evident that we can consider the motion of the system as to be compounded of a translation, together with the pole A, and its motion relative to a frame $A\xi\eta\zeta$. To describe this relative motion, we shall choose the generalized coordinates q^1, q^2,... , q^n, denoting by q^1, q^2, q^3 Euler's angles ψ, θ, φ, and by q^k ($k = 4, 5, \ldots, n$) either the angle of rotation of (V_k) relative to (V_{k-1}) about the axis defined by the unit vector $\vec{e}_k$, fixed in (V_{k-1}), or rectilinear translation of (V_k) with respect to (V_{k-1}), in the direction defined by $\vec{e}_k$. Motion of the pole A with respect to the system $Oxyz$ is determined by its coordinates x_A, y_A, z_A. We note that coordinates q^1, q^2,..., q^n are not mutually independent since our kinematic chain is closed, but for the present we shall not eliminate redundant coordinates. The position of the centre of inertia of the multibody system, C, is determined by the relation

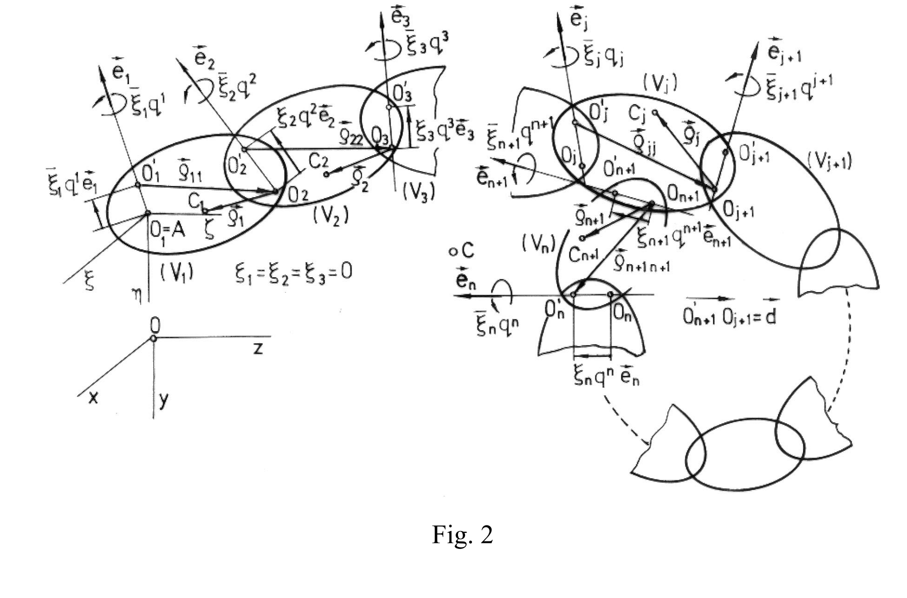

Fig. 2

$$\overrightarrow{OC} = \overrightarrow{OA} + \sum_{i=1}^{n} \frac{m_i}{m} \overrightarrow{AC_i}, \tag{1}$$

where m_i is the mass of the body (V_i), C_i - its centre of inertia, and where m denotes mass of the whole multibody system. Since obviously

$$\overrightarrow{AC} = \overrightarrow{AC}(q^1, q^2, ..., q^n),$$

from (1) we have

$$\begin{aligned} x_C &= x_A + F_x(q^1,...,q^n) \\ y_C &= y_A + F_y(q^1,...,q^n) \\ z_C &= z_A + F_z(q^1,...,q^n) \end{aligned} \tag{1'}$$

where F_x, F_y, F_z are the projections of the vector $\sum_{i=1}^{n} \frac{m_i}{m} \overrightarrow{AC_i}$ on the corresponding axes of the frame $Axyz$. Now, as the relations (1') determine x_A, y_A, z_A in terms of x_C, y_C, z_C and q^1,..., q^n, it is evident that the configuration of the multibody system considered at time t can be described by the set of generalized coordinates

$$x_C, y_C, z_C, q^1, q^2, ... q^n.$$

The conditions of closing the kinematic chain impose to the coordinates q^1, ..., q^n following equations of constraint (ref. [11])

$$\begin{aligned} f^1(q^1,...,q^n) &= (\vec{e}_{n+1})[A_{j,n}]\{\vec{e}_{n+1}\} - 1 = 0, \\ f^2(q^1,...,q^n) &= (\vec{q})[A_{j,n}]\{\vec{e}_{n+1}\} = 0, \\ f^3(q^1,...,q^n) &= (\vec{q}) \sum_{k=j+1}^{n} [A_{j,k}]\{\vec{\rho}_{kk} + \xi_k q^k \vec{e}_k\} + (\vec{q})\{\vec{d}\} = 0, \\ f^4(q^1,...,q^n) &= (\vec{r}) \sum_{k=j+1}^{n} [A_{j,k}]\{\vec{\rho}_{kk} + \xi_k q^k \vec{e}_k\} + (\vec{r})\{\vec{d}\} = 0, \\ f^5(q^1,...,q^n) &= \xi_{n+1}((\vec{q})[A_{j,n}]\{\vec{q}\} - \bar{\xi}_{n+1}\cos q^{n+1}) + \\ &+ \bar{\xi}_{n+1}\left((\vec{e}_{n+1}) \sum_{k=j+1}^{n} [A_{j,k}]\{\vec{\rho}_{kk} + \xi_k q^k \vec{e}_k\} + (\vec{e}_{n+1})\{\vec{d}\} + \xi_{n+1} q^{n+1} \right) = 0, \end{aligned} \tag{2}$$

where $[A_{j,k}]$ denotes the matrix of orthogonal transformations of a vector coordinates from a local frame of reference $C_k\xi_k\eta_k\zeta_k$, fixed in (V_k), to a local frame $C_j\xi_j\eta_j\zeta_j$, fixed in (V_j), and where $\xi_k = 1$ if the body (V_{k+1}) has translational motion with respect to the body (V_k), $\xi_k = 0$ in the case of a rotational motion of (V_{k+1}) relative to (V_k), and $\bar{\xi}_k = 1 - \xi_k$. Vector $\vec{q}$, appearing in (2), denotes the unit vector attached to the body (V_j) and orthogonal to $\vec{e}_{n+1}$, vector $\vec{r}$ is given by

$$\vec{r} = \vec{e}_{n+1} \times \vec{q}$$

while the meaning of the vectors $\vec{e}_{n+1}$, $\vec{\rho}_{kk}$ and $\vec{d}$ is evident from the Fig. 2.

If

$$rank\left[\frac{\partial f^{\nu}}{\partial q^{a}}\right]<5 \tag{3}$$

the relations (2) are not mutually independent[1]. In this case it is necessary to eliminate the dependent equations of constraint. Without reducing the generality of our considerations, we can further suppose that l independent equations of constraints are given by

$$f^{\nu}(q^{1},...,q^{n})=0\,,\quad \nu=1,2,...,l \tag{4}$$

where

$$l=rank\left[\frac{\partial f^{\nu}}{\partial q^{a}}\right]\leq 5\,.$$

The kinetic energy of the multibody system considered has the form

$$T=\frac{1}{2}m(\dot{x}_C^2+\dot{y}_C^2+\dot{z}_C^2)+\frac{1}{2}a_{ab}\dot{q}^{a}\dot{q}^{b}\,, \tag{5}$$

where $a_{ab}(q^1,..., q^n)$ denotes the fundamental tensor in the space of coordinates $q^1,..., q^n$. This tensor,as we can prove, has the form

$$a_{ab}=\sum_{i=1}^{n}(m_i(\vec{t}_{a(i)})\{\vec{t}_{b(i)}\}+(\vec{e}_a)[J_i]\{\vec{e}_b\}) \tag{6}$$

where $\vec{t}_{a(i)}$ are quasibasic vectors, given by

$$\vec{t}_{a(i)}=\vec{T}_{a(i)}-\sum_{k=1}^{n}\frac{m_k}{m}\vec{T}_{a(k)}\,,$$

with $\vec{T}_{a(i)}=\dfrac{\partial\vec{r}_{ci}}{\partial q^{a}}$, and where

$$[J_i]=\begin{bmatrix} J_{\xi_i} & -J_{\xi_i\eta_i} & -J_{\xi_i\varsigma_i} \\ -J_{\eta_i\xi_i} & J_{\eta_i} & -J_{\eta_i\varsigma_i} \\ -J_{\varsigma_i\xi_i} & -J_{\varsigma_i\eta_i} & J_{\varsigma_i} \end{bmatrix}$$

is a tensor of inertia of the rigid body (V_i),whose coordinates are given with respect to the local frame of reference $C_i\xi_i\eta_i\zeta_i$.

Introducing further the denotations

$$x_C=q^{n+1},\, z_C=q^{n+2},$$

we can write (5) in the form

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[1] In (3), as well as further throughout the paper, the index a takes the values 1,2,...,n, white ν runs from 1 to l. In the sequel the following indices will also be used: b = 1,2,...,n; ρ = $1,2,...,l$; α,β,γ = 1,2,..., n, n+1, $n+2$; π,θ = 1,2,..., $m = n - l$; ν', ρ' = m+1, m+2, ..., $m+l = n$. The repeated index will denote summation.

$$T = \frac{1}{2} a_{\alpha\beta} \dot{q}^{\alpha} \dot{q}^{\beta} + \frac{1}{2} m \dot{y}_C^2 , \tag{5'}$$

where we put

$$a_{\alpha n+1} = a_{\alpha n+2} = a_{n+1 n+2} = 0$$

$$a_{n+1 n+1} = a_{n+2 n+2} = m.$$

The potential energy of the multibody system is

$$V = -mgy_C . \tag{7}$$

Remembering now that as a consequence of (4) the coordinates $q^1, ..., q^n$ are not mutually independent,we further take, without loss in generality, the first m = n – l of them as the independent ones, so that last l of them are dependent coordinates, for which, in virtue of the theorem of the implicit function, we have

$$q^{\nu'} = \psi^{\nu'}(q^{\pi}) , \tag{8}$$

where, as we already said, $\nu' = m + 1,..., m + l$, and $\pi = 1,...,m$.

The time our multibody system needs to move from a known initial position (P_0), given by $q^{\pi} = q_0^{\pi}$, $f^{\nu}(q_0^1,...,q_0^n) = 0$, $q^{n+1} = x_0$, $q^{n+2} = z_0$ and $y_c = 0$ at time $t = t_0$, where t_0 is fixed, to a final configuration (P_1), specified by $q^{\pi} = q_1^{\pi}$, $f^{\nu}(q_1^1,...,q_1^n) = 0$, $q^{n+1} = x_1$, $q^{n+2} = z_1$ and $y_c = y_1$ at time $t = t_1$, where t_1 is to be determined, is given by

$$\tau = \int_{t_0}^{t_1} dt.$$

In the case of a brachistochronic motion from (P_0) to (P_1) of our multibody system, analogous to the brachistochronic motion of a particle in the famous Bernoulli's case, we require that

$$\int_{t_0}^{t_1} dt \to inf.$$

$$\begin{gathered} \frac{1}{2} a_{\alpha\beta} \dot{q}^{\alpha} \dot{q}^{\beta} + \frac{1}{2} m \dot{y}_C^2 - mgy_C = 0, \\ f^{\nu}(q_1^1,...,q_1^n) = 0, \\ t = t_0 : q^{\pi} = q_0^{\pi} , \quad f^{\nu}(q_0^1,...,q_0^n) = 0 , \quad q^{n+1} = x_0 , \quad q^{n+2} = z_0 , \quad y_C = 0 \\ t = t_1 : q^{\pi} = q_1^{\pi} , \quad f^{\nu}(q_1^1,...,q_1^n) = 0 , \quad q^{n+1} = x_1 , \quad q^{n+2} = z_1 , \quad y_C = y_1 \end{gathered} \tag{9}$$

is valid.

3. The Solution of the Problem

Using (8), i.e. by eliminating the dependent coordinates, variational problem (9) gets the form

$$\int_{t_0}^{t_1} dt \to inf.$$

$$\frac{1}{2}a^*_{\pi\theta}\dot{q}^{\pi}\dot{q}^{\theta} + \frac{1}{2}m\dot{x}_C^2 + \frac{1}{2}m\dot{z}_C^2 + \frac{1}{2}m\dot{y}_C^2 - mgy_C = 0, \tag{9'}$$

$$t = t_0 : q^{\pi} = q_0^{\pi}, x_C = x_0, z_C = z_0, y_C = 0\,,$$

$$t = t_1 : q^{\pi} = q_1^{\pi}, x_C = x_1, z_C = z_1, y_C = y_1$$

where

$$a^*_{\pi\theta} = a^*_{\pi\theta}(q^1,...,q^m).$$

Having further in mind the form of the so formulated variational problem's constraint, we deduce that in the subspace of the coordinates q^1, q^2,..., q^m, x_C, z_C, the demanded trajectory of the brachistochronic motion of the system is a geodesic, which can be found as a solution of the new constrained variational problem:

$$\int_0^{\sigma_1} d\sigma \to inf.$$

$$(d\sigma)^2 - \frac{1}{m}a_{\alpha\beta}dq^{\alpha}dq^{\beta} = 0\,,\; f^{\nu}(q^1,...,q^n) = 0, \tag{10}$$

$$\sigma = 0 : q^{\pi} = q_0^{\pi}\,,\; f^{\nu}(q_0^1,...,q_0^n) = 0\,,\; q^{n+1} = x_0\,,\; q^{n+2} = z_0,$$

$$\sigma = \sigma_1 : q^{\pi} = q_1^{\pi}\,,\; f^{\nu}(q_1^1,...,q_1^n) = 0\,,\; q^{n+1} = x_1\,,\; q^{n+2} = z_1,$$

in which σ denotes the arc-length of the geodesic in the space of coordinates q^1,..., q^m, q^{n+1}, q^{n+2}.

Using a multiplier rule, the problem (10) reduces to

$$\delta\int_0^{\sigma_1}\left[1 + \lambda\left(a_{\alpha\beta}\frac{dq^{\alpha}}{d\sigma}\frac{dq^{\beta}}{d\sigma} - m\right) + \mu_{\nu}f^{\nu}\right]d\sigma = 0$$

$$\sigma = 0 : q^{\pi} = q_0^{\pi}\,,\; f^{\nu}(q_0^1,...,q_0^n) = 0\,,\; q^{n+1} = x_0\,,\; q^{n+2} = z_0 \tag{10'}$$

$$\sigma = \sigma_1 : q^{\pi} = q_1^{\pi}\,,\; f^{\nu}(q_1^1,...,q_1^n) = 0\,,\; q^{n+1} = x_1\,,\; q^{n+2} = z_1$$

where the multipliers λ and μ_ν are functions of σ to be determined. Since σ_1 is not prescribed and the integrand in (10') does not depend on σ explicitly, we easily find

$$\lambda = \frac{1}{2m}\,,$$

and the Euler's equations of the problem (10') read

$$\frac{d}{d\sigma}\left(a_{\alpha\gamma}\frac{dq^{\alpha}}{d\sigma}\right) - \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial q^{\gamma}}\frac{dq^{\alpha}}{d\sigma}\frac{dq^{\beta}}{d\sigma} = \mu_{\nu}\frac{\partial f^{\nu}}{\partial q^{\gamma}}. \tag{11}$$

These equations, together with the relations (4), determine the geodesic demanded. Constants of integration and the arc-length σ_1 can be found from the conditions

$$\sigma = 0 : q^{\pi} = q_0^{\pi}\ , \quad f^{\nu}(q_0^1,...,q_0^n) = 0\ , \quad q^{n+1} = x_0\ , \quad q^{n+2} = z_0$$

$$\sigma = \sigma_1 : q^{\pi} = q_1^{\pi}\ , \quad f^{\nu}(q_1^1,...,q_1^n) = 0\ , \quad q^{n+1} = x_1\ , \quad q^{n+2} = z_1$$

$$a_{\alpha\gamma}\frac{dq^{\alpha}}{d\sigma}\frac{dq^{\beta}}{d\sigma} = m\ .$$

Now, having in mind that we found the trajectory of the multibody system in the subspace of the coordinates $q^1,..., q^m, x_C, z_C$, and considering the motion of the system in this subspace as the motion of a representative point along this known trajectory, we can choose the arc-length σ, measured along this trajectory, as the coordinate determining the position of the representative point, i.e. the position of our multibody system in the subspace $q^1,..., q^m, x_C, z_C$. Then it is evident that the coordinates σ and y_C may be chosen to determine the position of our multibody system, and the kinetic energy T can be written in the form

$$T = \frac{1}{2}m(\dot{y}_C^2 + \dot{\sigma}^2)\ ,$$

so that the brachistochronic motion of the system can be determined from the variational problem

$$\delta\int_{t_0}^{t_1}\left\{1 + \lambda\left[\frac{1}{2}m(\dot{y}_C^2 + \dot{\sigma}^2) - mgy_C\right]\right\}dt = 0$$

$$t = t_0 : \sigma = 0\ , \quad y_C = 0$$

$$t = t_1 : \sigma = \sigma_1\ ,\ y_C = y_1$$

The solution of the so formulated problem is a cycloid, given with respect to the orthogonal system of coordinates σ, y_C , if we take $t_0 = 0$, by the equations

$$\begin{aligned} \sigma &= \frac{g}{\omega^2}(\omega t - \sin\omega t), \\ y_C &= \frac{g}{\omega^2}(1 - \cos\omega t), \end{aligned} \tag{13}$$

where ω denotes the constant of integration. This solution coincides with the solution of the familiar Bernoulli's problem, referring to the free particle.

Finally, introducing the denotation

$$\chi = \omega t_1,$$

from (13) we easily obtain the relation

$$\chi - \sin\chi - \frac{\sigma_1}{y_1}(1 - \cos\chi) = 0 \tag{14}$$

Finding the minimal positive solution $\chi = \chi_0$ of the equation (14), we determine time of the brachistochronic motion of the multibody system considered

$$t_1 = \sqrt{\frac{y_1\chi_0^2}{g(1 - \cos\chi_0)}}\ . \tag{15}$$

4. NUMERICAL EXAMPLE

Plane mechanism ABDE (Fig. 3), consisting of the four homogeneus rods each of which having the mass m and lenght 2a, moves brachistochronically in the plane *Oxy* with the axis *Oy* directed vertically downwards. The rods are connected at their ends by pins. If the mechanisms starts moving from rest, and if its initial and terminal configurations are given by

$$\begin{aligned} &t=0: A(-a,-a),\ B(-a,a),\ D(a,a), \\ &t=t_1: A(3a,2a),\ B(3a-a\sqrt{3},3a),\ D(3a+a\sqrt{3},3a), \end{aligned} \tag{16}$$

determine the time t_1 of a brachiststochronic motion of the mechanism.

Solution. Choosing the angles φ, θ and Cartesian coordinates x_C, y_C of mechanism's inertia centre as the indepepedent generalized coordinates determining the position of the mechanism (see Fig. 2), the kinetic energy of the mechanism can be written in the form

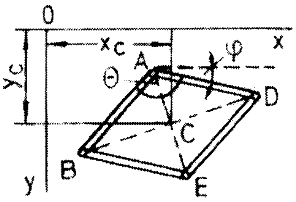

Fig. 3.

$$T=2m\dot{y}_C^2+2m\dot{x}_C^2+\frac{8}{3}ma^2\dot{\varphi}^2+\frac{4}{3}ma^2\dot{\theta}^2+\frac{8}{3}ma^2\dot{\varphi}\dot{\theta} \tag{17}$$

while the expression for the potential energy function reads

$$V=-4mgy_C\,. \tag{18}$$

The motion of the mechanism considered in the space of coordinates φ,θ,x_C can be substituted by the motion of reprensetative point along a geodesic whose element of the arc-length is given by (see (10))

$$d\sigma^2=\frac{4}{3}a^2(d\varphi)^2+\frac{2}{3}a^2(d\theta)^2+\frac{4}{3}a^2d\varphi\,d\theta+(dx_C)^2 \tag{19}$$

The differential equations of this geodesic are (see (10') and (11)) whence, taking $\sigma_{(t=0)}=0$, can be obtained

$$\varphi=k_\varphi\sigma+\varphi_0,\quad \theta=k_\theta\sigma+\theta_0,\quad x_C=k_x\sigma+x_{C0}\,, \tag{20}$$

$$\frac{d^2\varphi}{d\sigma^2}+\frac{d^2\theta}{d\sigma^2}=0,\quad 2\frac{d^2\varphi}{d\sigma^2}+\frac{d^2\theta}{d\sigma^2}=0,\quad \frac{d^2x_C}{d\sigma^2}=0,$$

where k_φ, k_θ, and k_x are constants.

Relations (19) and (20) lead to

$$\sigma^2(t_1)=\sigma_1^2=\frac{4}{3}a^2(\varphi_1-\varphi_0)^2+\frac{2}{3}a^2(\theta_1-\theta_0)^2+\frac{4}{3}a^2(\varphi_1-\varphi_0)(\theta_1-\theta_0)+(x_1-x_0)^2$$

wherefrom, as

$$\begin{aligned} &t=0:\quad \varphi=\varphi_0=0,\quad \theta=\theta_0=\frac{\pi}{2},\quad x_C=x_0=0, \\ &t=t_1:\quad \varphi=\varphi_1=\frac{\pi}{6},\quad \theta=\theta_1=\frac{2\pi}{3},\quad x_C=x_1=3a, \end{aligned}$$

it is easy to find

$$\sigma_1 = a\sqrt{\frac{4}{3}\frac{\pi^2}{36}+\frac{2}{3}\frac{\pi^2}{36}+\frac{4}{3}\frac{\pi^2}{36}+9} = 3.1486268a.$$

Finally, as in the case considered $y_c(t_1) = y_1 = 3a$, the relation (14) reads

$$\chi - \sin\chi - 1.0495423(1-\cos\chi) = 0,$$

where from $\chi_0 = 2.490437$, and (15) leads to

$$t_1 = 1.0278341\text{sec}.$$

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UOPŠTENJE BERNULIJEVOG SLUČAJA BRAHISTOHRONOG KRETANJA NA SISTEM KRUTIH TELA U OBLIKU ZATVORENOG KINEMATIČKOG LANCA

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Rešava se problem brahistohronog kretanja zatvorenog kinematičkog lanca, slobodnog od spoljašnjih veza u homogenom polju teže. Geometrizacijom problema ovo se kretanje razmatra u konfiguracionom prostoru koji se može razdvojiti na dva potprostora - jedan je jednodimenzioni i određen Dekartovom koordinatom središta masa sistema koja odgovara vertikalnoj osi, y_C*, a drugi obuhvata sve ostale generalisane koordinate sistema. Pokazuje se da je u tome drugom potprostoru trajektorija sistema geodezijska linija. Birajući dalje za koordinate koje određuju položaj sistema luk te geodezijske linije i koordinatu* y_C*, dobija se rezultat koji se potpuno poklapa sa poznatim Bernulijevim rezultatom koji se odnosi na slučaj brahistohronog kretanja jedne materijalne tačke. Rezultat rada ilustrovan je primerom.*