COMPARATIVE ANALYSIS OF USING DEGREE OF RIGIDITY AND ROTATIONAL STIFFNESS OF CONNECTIONS IN STRUCTURAL DESIGN

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Abstract. Using classical formulation of stiffness method, impact of semi-rigid connections on the stresses and strains can be analyzed by the degree of rigidity or by rotational stiffness of connections. In this paper, functional dependence between the degree of rigidity and rotational stiffness of connections is formulated and the comparative analysis of these two approaches in the analysis of semi-rigid connections behavior of members in real structures, is implemented.

Key words: semi-rigid connections, rotational stiffness of connection, degree of rigidity of connection

1. INTRODUCTION

1.1. Mathematical model of a semi-rigid connection

Generally, the semi-rigid connection of two members allows a certain degree of additional relative displacement in the plane of the connection, in the direction of all generalized displacements, which, for the linear element in plane means a relative horizontal and vertical displacement and rotation of the cross-section at the joint location (Figure 1). As for the majority of frame structures, vertical and horizontal displacements (slipping) of the joints are negligible in comparison to the relative rotation of the cross section at the joint location, the influence of semi-rigid connections can be modeled with an elastic spring with rotational stiffness $S_{kr}$. This coefficient includes the impact of rotational stiffness of the connection on the change of static and deformation parameters in the structure. The numerical value of rotational stiffness of the connections of the member „i,k“ at i, and k, nodes is determined by the expressions:

\[
S_{kr} = \frac{k_{ir}}{r_{ri}} + \frac{k_{kr}}{r_{rk}}
\]

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where:

$M_i, M_k$ – is the bending moment at the location of the connection in the node $i$, that is, $k$,

$\Phi_i, \Phi_k$ – relative rotation of the connection in node $i$, that is, $k$.

Rotational stiffness of the connection, in the geometrical terms, represents the angle which is constituted by the tangent and abscissa on the moment-rotation curve of the connection that is the gradient of the $M - \Phi$ connection curve (Figure 2).

1.2. Basic strain parameters of a straight member according to the first order theory

Fig. 3 Straight member with semi-rigid connections at the ends $i$ and $k$, before and after deformation
Let the member be decomposed from the given structural system (Figure 3), which is connected to the rest of the structure on its nodes. The isolated member, under the arbitrary load $p$ and unknown intersecting forces in the node $i$: $N_{ik}$, $T_{ik}$ and $M_{ik}$, and in the node $k$: $N_{ki}$, $T_{ki}$ and $M_{ki}$, is in equilibrium.

Due to the deformability of connections, the rotation of the end cross sections of the member $\varphi_{ik}$, that is $\varphi_{ki}$, is not equal to the rotation of nodes $\varphi_i$, that is $\varphi_k$, but it is:

$$\varphi_i - \varphi_{ik} = \bar{\tau}_{ik} - \tau_{ik} = \Phi_{ik}, \quad \text{and} \quad \varphi_k - \varphi_{ki} = \bar{\tau}_{ki} - \tau_{ki} = \Phi_{ki} \quad \text{(2)}$$

If the angles of tangent rotations are designated with $\alpha_{ik}^{(o)}$ and $\alpha_{ki}^{(o)}$ on the nodes $i$ and $k$ of a simple beam, due to the applied load $p$; with $\varepsilon_{ik}^{(\Delta t)}$ and $\varepsilon_{ki}^{(\Delta t)}$ the same rotation angles due to temperature differences $\Delta T$; and with $\alpha_{ik}$ and $\beta_{ik}$, that is $\alpha_{ki}$ and $\beta_{ki}$, the slope angles due to unit bending moments $M_{ik}$ = 1 and $M_{ki}$ = 1 (Figure 4), then on the basis of the equations (1) and (2), and the principle of superposition, the expressions for the moments on the ends $i$ and $k$, of the member „ik” can be obtained:

$$M_{ik} = \bar{a}_{ik} \cdot \varphi_i + \bar{b}_{ik} \cdot \varphi_k - \bar{c}_{ik} \cdot \psi_{ik} + \bar{m}_{ik}^{(o)} + \bar{m}_{ik}^{(\Delta t)}, \quad M_{ki} = \bar{b}_{ki} \cdot \varphi_i + \bar{a}_{ki} \cdot \varphi_k - \bar{c}_{ki} \cdot \psi_{ik} + \bar{m}_{ki}^{(o)} + \bar{m}_{ki}^{(\Delta t)}, \quad \text{(3)}$$

where the following designations are introduced:

$$\bar{a}_{ik} = \frac{\alpha_{ik} + \frac{1}{S_{ik}}}{D}, \quad \bar{a}_{ki} = \frac{\alpha_{ki} + \frac{1}{S_{ki}}}{D}, \quad \bar{b}_{ik} = \frac{\beta_{ik}}{D} = \bar{b}_{ki}, \quad D = (\alpha_{ik} + \frac{1}{S_{ik}}) \cdot (\alpha_{ki} + \frac{1}{S_{ki}}) - \beta_{ik}^2 \quad \varepsilon_{ik} = \bar{a}_{ik} + \bar{b}_{ik}, \quad \varepsilon_{ki} = \bar{a}_{ki} + \bar{b}_{ki}, \quad \bar{m}_{ik}^{(o)} = -(\bar{a}_{ik} \cdot \alpha_{ik}^{(o)} - \bar{b}_{ik} \cdot \alpha_{ki}^{(o)}), \quad \bar{m}_{ki}^{(o)} = -(\bar{a}_{ki} \cdot \alpha_{ki}^{(o)} - \bar{b}_{ki} \cdot \alpha_{ik}^{(o)}) \quad \bar{m}_{ik}^{(\Delta t)} = -\bar{a}_{ik} \cdot \alpha_{ik}^{(\Delta t)} - \bar{b}_{ik} \cdot \alpha_{ki}^{(\Delta t)} \quad \bar{m}_{ki}^{(\Delta t)} = \bar{a}_{ki} \cdot \alpha_{ki}^{(\Delta t)} - \bar{b}_{ki} \cdot \alpha_{ik}^{(\Delta t)}.$$

In the general case, the values in the equation (3), for the member with $EI_{ik} = const$ are $\bar{a}_{ik} = \bar{a}_{ki}$, $\bar{b}_{ik} = \bar{b}_{ki}$ and $\varepsilon_{ik} = \varepsilon_{ki}$. Since the parameters $\bar{a}_{ik}, \bar{b}_{ik}, \varepsilon_{ik}, \bar{a}_{ki}, \bar{b}_{ki}, \varepsilon_{ki}$, do not depend only on the geometrical characteristics of the cross-sections and mechanical properties of the basic material of the member, but also on the characteristics of the connections on the nodes of the member, and they are therefore named the constants of the member with semi-rigid connections.

On the basis of the conducted analysis, it can be concluded that, in the further analysis all the members of any type can be treated as a single type of members with semi-rigid connections at their nodes, with their appropriately calculated rotational stiffness of connections, which simplifies and standardizes the calculation, and is particularly important for application of structural design software.
Fig. 4 Physical meaning of beam constants and the initial bending moments at the ends of a beam with semi-rigid connections

2. MEMBER STIFFNESS MATRIX IN THE CASE OF USING THE ROTATIONAL STIFFNESS OF CONNECTIONS

A large number of general and specialized software were developed for analysis and calculation of the structures, practically covering all the areas of analysis of stress-strain state of structures. The stiffness method presented in the matrix form is, almost regularly, the basis for production of contemporary computer software for linear system structural analysis, due to its general character, simplicity and significant automatism in calculation.

Regarding that the contemporary stress-strain analysis of complex engineering structures, cannot be imagined without the matrix formulation and application of electronic computers. The corrected member stiffness matrix will be used for the members semi-rigidly connected at the nodes, starting from the previously defined classic stiffness method, for the calculation of the system with semi-rigid connections of members.
Axial strain and bending of elements, represent two independent theories within the first order theory, so they can be considered independently. This can be concluded based on the structure of the stiffness matrix for the „κ“ type member, from which it can be seen that the members characterizing the axial rigidity, depend exclusively on the displacement in the direction of the element axis. The correction of stiffness matrix to take into account the deformability of the member connections, are performed only on the part of matrix, which refers to the bending issue. The terms of the stiffness matrix, referring to the axial load, remain unchanged, for the case of the members with deformable connections.

The stiffness matrix of the member with semi-rigid connections can be obtained starting from the basic stiffness matrix. For this purpose, it is necessary to establish relation between the basic deformation parameters and displacement parameters of the member, and the relation between the general forces and basic static parameters of the member.

In figure 3, a straight member with the semi-rigid connections on its nodes i and k, is presented, before and after the deformation. Assuming that those are small displacements and holding that \( \alpha_{ik} = 0 \), in the figure 3 the following interdependencies between the basic deformation and displacement parameters are obvious:

\[
\bar{\bar{\varphi}}_{ik} = \varphi_i - \psi_{ik} = \varphi_i - \frac{1}{2} (v_k - v_i), \\
\bar{\varphi}_{ki} = \varphi_k - \psi_{ik} = \varphi_k - \frac{1}{2} (v_k - v_i),
\]

(4)

which can be presented in the following matrix form:

\[
\begin{bmatrix}
\bar{\bar{\varphi}}_{ik} \\
\bar{\varphi}_{ki}
\end{bmatrix} = 
\begin{bmatrix}
\frac{1}{l} & 1 & -\frac{1}{l} & 0 \\
\frac{1}{l} & 0 & -\frac{1}{l} & 1
\end{bmatrix}
\begin{bmatrix}
\varphi_i \\
\varphi_k \\
v_i \\
v_k
\end{bmatrix}
\]

(5)

where:

\[
\begin{bmatrix}
\frac{1}{l} & 1 & -\frac{1}{l} & 0 \\
\frac{1}{l} & 0 & -\frac{1}{l} & 1
\end{bmatrix} = [C]
\]

(6)

is the matrix defining the relation between the basic deformation parameters of the „ik“ member and generalized displacements.

The relation between the basic stiffness matrix and stiffness matrix [4], can be written in the following way:

\[
[K] = [C]^T[K_0][C]
\]

(7)

where:

\([K_0]\) – is the basic stiffness matrix,

\([C]^T\) – transposed matrix, of the matrix \([C]\), which defines the relation between the basic static parameters and generalized forces, and

\([C]\) – the matrix defining the relation between the basic deformation parameters of the member and generalized displacements.

The basic stiffness matrix is equal to the inverse flexibility matrix - \([D_0]\), i.e.:

\[
[K_0] = [D_0]^{-1} = \frac{1}{\det[D_0]} \cdot adf[D_0]
\]

(8)
The flexibility matrix \([D_0]\), represents the relation between the basic deformation and statically independent parameters of the member:

\[
\{\delta\} = [D_0]\{X\}
\]  

(9)

where:

\(\{\delta\}\) – vector of deformation parameters of the member,

\([D_0]\) – flexibility matrix, and

\(\{X\}\) – vector of statically independent parameters of the member.

From the equation (8) [5], the basic matrix or stiffness is:

\[
[K_0] = [D_0]^{-1} = \begin{bmatrix}
\tilde{a}_{ik} \cdot \eta_1 & \tilde{b}_{ik} \cdot \eta_2 \\
\tilde{b}_{ik} \cdot \eta_2 & \tilde{a}_{ki} \cdot \eta_3
\end{bmatrix} = \begin{bmatrix}
\tilde{a}_{ik} & \tilde{b}_{ik} \\
\tilde{b}_{ik} & \tilde{a}_{ki}
\end{bmatrix}
\]  

(10)

The member stiffness matrix, with partially rigid connections, according to the expression (7) [5], has the following form:

\[
[K] = \begin{bmatrix}
\frac{\tilde{e}_{ik} + \tilde{e}_{ki}}{I^2} & \frac{\tilde{e}_{ik}}{I} & -\frac{\tilde{e}_{ik} + \tilde{e}_{ki}}{I} \\
\frac{\tilde{e}_{ik}}{I} & \frac{\tilde{e}_{ik} + \tilde{e}_{ki}}{I^2} & \frac{\tilde{e}_{ki}}{I} \\
-\frac{\tilde{e}_{ki}}{I} & \frac{\tilde{e}_{ki}}{I} & \frac{\tilde{e}_{ki} + \tilde{e}_{ik}}{I^2}
\end{bmatrix}
\]  

(11)

Expression reduction yields to the member stiffness matrix which is semi-rigidly connected at the node 1, and hinged at the other end, having the following form [5]:

\[
[K] = \begin{bmatrix}
3 \frac{E I}{I^2} \tilde{\eta}_1 & 3 \frac{E I}{I^2} \tilde{\eta}_1 & -3 \frac{E I}{I^2} \tilde{\eta}_1 \\
3 \frac{E I}{I^2} \tilde{\eta}_1 & \frac{E I}{I^2} \tilde{\eta}_1 & -\frac{3 E I}{I^2} \tilde{\eta}_1 \\
-3 \frac{E I}{I^2} \tilde{\eta}_1 & -\frac{3 E I}{I^2} \tilde{\eta}_1 & \frac{3 E I}{I^2} \tilde{\eta}_1
\end{bmatrix}
\]  

(12)

It should be noticed that, as opposed to the member stiffness matrix for the members with semi-rigid connections on both nodes, where the influence of semi-rigid connections on the ends is covered by three different parameters, this influence is covered by one parameter only, when the matrix of the member with semi-rigid connection at one node, and hinge of the other, is concerned.

The stiffness matrices, determined by the basic stiffness matrix, excluding the differences in marking are completely equivalent to the stiffness matrices derived in the papers by D. Stojić, D. Bašić and E. Mešić [5], [6], [7], as well as with the stiffness matrices given in ECCS, „Analysis and Design of Steel Frames with Semi-Rigid Joints”, published in London in 1992.
3. MEMBER STIFFNESS MATRIX IN THE CASE OF USING THE DEGREE OF RIGIDITY OF CONNECTIONS

In the papers [1] [2] [3], using the classical formulation of the stiffness method, the influences of semi-rigid connections at the nodes of the members, are introduced through the degree of rigidity of connection:

\[ \mu_{ik} = \frac{\varphi_{ik}}{\varphi_{li}}, \quad \mu_{kl} = \frac{\varphi_{kl}}{\varphi_{ki}}; \]

where:
\( \varphi_{li} \) and \( \varphi_{ki} \) – are the rotation angles of node „i””, that is „k””, and
\( \varphi_{ik} \) and \( \varphi_{kl} \) – are the rotation angles of end cross sections of the member „ik”.

By introducing the designations, according to the equation (2) and the figure 3, the degree of rigidity of connection can be expressed in the function of rotation of end cross sections of the member „ik” and relative rotation of the connections at the nodes of the member ik:

\[ \mu_{ik} = \frac{\varphi_{ik}}{\varphi_{li}} - \frac{\varphi_{ik}}{\varphi_{li} + \varphi_{ik}}, \quad \mu_{kl} = \frac{\varphi_{kl}}{\varphi_{ki}} - \frac{\varphi_{kl}}{\varphi_{ki} + \varphi_{kl}}; \]

or, in the function of rotation of node „i”, i.e., „k” and relative rotation of the connection at the nodes of the member „ik”:

\[ \mu_{ik} = \frac{\varphi_{ik}}{\varphi_{li}} - \frac{\varphi_{ik}}{\varphi_{li} + \varphi_{ik}}, \quad \mu_{kl} = \frac{\varphi_{kl}}{\varphi_{ki}} - \frac{\varphi_{kl}}{\varphi_{ki} + \varphi_{kl}}; \]

That is:

\[ \mu_{ik} = 1 - \frac{\varphi_{ik}}{\varphi_{li}}, \quad \mu_{kl} = 1 - \frac{\varphi_{kl}}{\varphi_{ki}}; \]

The same authors derived the basic stiffness matrix, whose general form is, [3]:

\[ [K_0] = \begin{bmatrix} \tilde{a}_{ik} & \tilde{b}_{ik} \\ \tilde{b}_{ik}^* & \tilde{a}_{ik}^* \end{bmatrix} \]

For the member with \( EI = const. \) written in a derived form, it is:

\[ [K_0] = \begin{bmatrix} \mu_{ik}(3 + \mu_{kl}) \frac{E_l}{l} & 2\mu_{ik}\mu_{kl} \frac{E_l}{l} \\ 2\mu_{ik}\mu_{kl} \frac{E_l}{l} & \mu_{kl}(3 + \mu_{ik}) \frac{E_l}{l} \end{bmatrix} \]

4. FUNCTIONAL DEPENDENCE BETWEEN DEGREE OF RIGIDITY AND ROTATIONAL STIFFNESS OF CONNECTIONS

Analyzing under which conditions the basic stiffness matrix (18) will be identical to the basic stiffness matrix derived in this paper (10), a system of three independent non-linear equations is obtained, which in the matrix form are:

\[ \begin{bmatrix} \mu_{ik}(3 + \mu_{kl}) \frac{E_l}{l} & 2\mu_{ik}\mu_{kl} \frac{E_l}{l} \\ 2\mu_{ik}\mu_{kl} \frac{E_l}{l} & \mu_{kl}(3 + \mu_{ik}) \frac{E_l}{l} \end{bmatrix} = \begin{bmatrix} 4\frac{E_l}{l} \cdot \eta_1 & 2\frac{E_l}{l} \cdot \eta_2 \\ 2\frac{E_l}{l} \cdot \eta_1 & 4\frac{E_l}{l} \cdot \eta_2 \end{bmatrix} \]
4.1. The case when the member is semi-rigidly connected at one node, and connected with ideal hinge at the other node, $\mu_{ik} \neq 0$ and $\mu_{kl} = 0$, i.e. $\Psi_{ik} \neq 0$ and $\Psi_{ki} \rightarrow \infty$

![Semi rigid connection at left node and connection with ideal hinge at right node](image)

The system of equations (19) represents a system of three independent equations where only one unknown parameter $\mu_{ik}$ is present. By solving this system of equations according to the unknown parameter $\mu_{ik}$, the following is obtained:

$$\mu_{ik} = \frac{4}{3} \cdot \eta_1 = \frac{4}{3} \cdot \lim_{\Psi_{ki} \to \infty} \frac{1 + 3\Psi_{ki}}{1 + 4(\Psi_{ik} + \Psi_{ki}) + 12\Psi_{ik}\Psi_{ki}} = \frac{4}{3} \cdot \frac{3}{4 + 12\Psi_{ik}}$$

That is:

$$\mu_{ik} = \frac{1}{1 + 3\Psi_{ik}}, \quad \mu_{kl} = 0; \quad \Psi_{ik} = \frac{1 - \mu_{ik}}{3\mu_{ik}}, \quad \Psi_{ki} \rightarrow \infty \quad (S_{ki} = 0);$$

![Functional dependence between $\mu_{ik}$ and $\Psi_{ik}$ for the case shown in figure 5](image)
4.2. The case when the member is semi-rigidly connected at one node, and ideally rigidly connected at the other node, $\mu_{ik} \neq 0$ and $\mu_{kl} = 1$, i.e. $\psi_{ik} \neq 0$ and $\psi_{kl} = 0$

\[ \psi_{ik} = 0 \quad \psi_{kl} = 0 \]

Fig. 7 Semi rigid connection at left node and ideally rigid connection at right node

The system of equations (19) represents a system of three independent equations where only one unknown parameter $\mu_{ik}$ is present. By solving this system of equations according to the unknown parameter $\mu_{ik}$, the following is obtained:

\[ \mu_{ik} = \frac{1}{1 + 4\psi_{ik}}, \quad \mu_{kl} = 0; \]
\[ \psi_{ik} = \frac{1 - \mu_{ik}}{4\mu_{ik}}, \quad \psi_{kl} \to \infty \quad (S_{kl} = 0); \]  

(22)

Fig. 8 Functional dependence between $\mu_{ik}$ and $\psi_{ik}$ for the case shown in figure 7

It should be emphasized that these dependencies could also be obtained through the geometrical-static interpretation.

For these two special cases of member connections, there is a direct functional dependence between the rotational stiffness and the degree of rigidity of connection, meaning that one can easily calculate their rotational stiffness, if the degree of rigidity of connection of such members is known, or vice versa.

In the case of member with elastic connections at its both ends, it is possible to find only approximate solution that gives dependence between the degree of rigidity and rotational stiffness as a real parameter of connection. One of such solutions, giving the approximate value in a particular domain of the accuracy (1), can be determined by solving the system of
equations (19) using the method of elimination. The solutions of the system of equations are given in the following form:

\[
\mu_{ik} \approx \frac{1+4\psi_{ki}}{1+4(\psi_{ik}+\psi_{ki})+12\psi_{ik}\psi_{ki}} \\
\nu_{ik} \approx \frac{1+4\psi_{ki}}{1+4(\psi_{ik}+\psi_{ki})+12\psi_{ik}\psi_{ki}}
\]

These solutions meet the first and third equation in the system, while the second equation is just approximately satisfied.

It is obvious that prior obtained exact solutions for two special cases, are derived from this approximate solution.

5. CONCLUSION

In the papers [1] [2] [3], using the classical formulation of deformation method, the influence of semi-rigid connections of members is introduced through the degree of rigidity of connections \( \mu_{ik} \) and \( \mu_{ki} \). In this paper is also applied the classic formulation of the deformation method for the analysis of stress and strain fields of structures with semi-rigid connections of members, but the influence of semi-rigid connections of members is included by the rotation stiffness of connections \( S_{ik} \) and \( S_{ki} \).

Based on the comparative analysis of these two approaches, which in different ways introduce into the calculus the influence of connection deformation (\( \mu \) that is \( S \)) on the variation of the stress and strain field of the structure, the following conclusions can be drawn:

- For determination of the degree of rigidity of the semi-rigidly connected member, using its definition (13), (14) or (15), it is necessary to know two parameters, those being:
  - Node „i”, that is „k”, rotation angles \( \varphi_i \) and \( \varphi_k \), and rotation angles of end cross sections of the member „ik” - \( \varphi_{ik} \) and \( \varphi_{ki} \), or
  - rotation angles of end cross sections of the member „ik” \( \varphi_{ik} \) and \( \varphi_{ki} \), and relative nodal rotation angles of the member „ik” due to connection deformation \( \Phi_{ik} \) and \( \Phi_{ki} \), or
  - rotation angles of node „i”, that is „k”, - \( \varphi_i \) and \( \varphi_k \), and relative nodal rotation angles of the member „ik” due to the connection deformation \( \Phi_{ik} \) and \( \Phi_{ki} \).

The degree of rigidity of connection depends not only on deformation of semi-rigid connections at the nodes of the member, but also on deformation parameters which relate to the deformation of nodal cross-sections of the member or deformation of nodes where the member is connected to the other structural members by semi-rigid connections. For one and the same designed connection, the degree rigidity of connection will not have the same numerical value, so it cannot be adopted as a parameter characterizing the behavior of the very connection, as opposed to the rotational stiffness of the connection characterizing the connection itself.

- The value of the degree of rigidity of connection depends on type of stress and strain analysis of the structure. Namely, the value of the degree of rigidity of connection for structural analysis, according to the First order theory, does not have the same numerical value as when such structure with the same, designed connections, would be calculated according to the Second order theory, which
additionally complicates and limits its application. As opposed to the degree of rigidity, the rotational stiffness of the connection, for the stress and strain field in the elastic domain, has the constant value, irrespective of whether the calculation is done according to First or Second order theory.

- Rotational stiffness of the connection changes depending on the intensity of the connection bending moment, and once calculated for one and same connection, it can be applied for the analysis of the stress and strain field in the elastic, elasto-plastic or plastic domain.

REFERENCES


KOMPARATIVNA ANALIZA KORIŠĆENJA STEPENA UKLJEŠTENJA I ROTACIONE KRUTOSTI VEZA U PRORAČUNU KONSTRUKCIJA

Primenom klasične formulacije metode deformacija, uticaj polukrutih veza štapova na polje napona i deformacijama može se analizirati korišćenjem stepena uklještenja ili rotacione krutosti veza. U ovom radu formulisana je funkcionalna zavisnost između stepena uklještenja i rotacione krutosti veza i izvršena komparativna analiza ova dva pristupa pri analizi ponašanja polukrutih veza štapova u realnim konstrukcijama.

Ključne reči: polukrute veze, rotaciona krutost veze, stepen uklještenja