

ALGORITHM FOR DETERMINATION OF S-N CURVES OF THE STRUCTURAL ELEMENTS SUBJECTED TO CYCLIC LOADING

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Abstract. *Fatigue life prediction of structural elements subjected to cyclic loading is usually performed using S-N curves, obtained from the experimental data from fatigue tests. However, in some cases the samples do not exhibit failure, due to reaching the predetermined number of cycles, failure of a non-relevant segment or terminating the test because of some other reason. These samples are usually referred to as runouts, and the data obtained from them could be used for determination of S-N curves as well. In this paper, the algorithm based on Maximum Likelihood method is proposed for the determination of S-N curves from experimental data that contain runouts. Following the algorithm, a MATLAB code was written and the verification was performed using the experimental data from the literature. The results showed that it could be successfully used for taking into account the runouts in the process of determination of S-N model parameters. It was concluded that the inclusion of runouts could significantly influence the predicted fatigue life, especially at the lower stress levels.*

Key words: *Fatigue, S-N curve, Algorithm, Maximum Likelihood method.*

1. INTRODUCTION

Most of the engineering structures in practice are subjected to changing load. This type of loading is usually represented as a cyclic loading, defined by the load amplitude, mean value and frequency. It causes the structures to undergo fatigue, which is defined as progressive and localized structural damage that occurs when a material is subjected to cyclic loading. The fatigue failure is usually different from the failures of structures subjected to static loading. They occur after certain time, i.e. the number of cycles, under loads lower than the ultimate load obtained from the static tests (sometimes significantly lower).

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The prediction of number of cycles to failure, referred to as fatigue life, is divided into two categories: high-cycle fatigue (HCF) and low-cycle fatigue (LCF). The difference between these two categories is that the first is associated with purely elastic behavior while the second involves plastic behavior. Because of this, the methods for fatigue life prediction associated with HCF and LCF are known as stress-life and strain-life, respectively. In order to select appropriate method for prediction of fatigue life it is very important to define the limit between these two fields. Although it is considered in the case of metals that the structures are subjected to HCF load when failures occur after more than 10^4 cycles, a clear limit could not be strictly defined. The difficulties in distinction between HCF and LCF are elaborated in [1].

Fatigue life prediction of structural elements is based on the experimental data obtained from the repeated fatigue tests. Identical specimens are subjected to cyclic loading at different stress levels and constant load ratio R , i.e. the ratio between the minimum and the maximum load in the cycle. The endured number of cycles at the moment of failure is recorded. If the specimen at a certain stress level does not reach the failure until predetermined number of cycles (usually 10^6 - 10^7), it is considered to have infinite fatigue life. On the other hand, sometimes the tests could be terminated due to some other reasons, such as failure of secondary element that is not of interest, technical issues etc. These specimens are referred to as “runouts”. They are usually censored and in such cases fatigue life prediction is based solely on the failed samples data. The discrete experimental data are approximated by S-N curves, often called Wöhler curves. These curves are mathematically represented by S-N models which contain parameters that need to be determined by fitting the equations to experimental data. There are several methods for determination of model parameters, such as regression analysis [2], maximum likelihood method [3], as well as some specific methods [4,5]. For obtaining the model parameters using regression analysis only the data from the failed samples could be used. Other methods could take into account runouts as well. However, besides the regression analysis, which is available in commercial software, other methods require high computational skills and complicated algorithms. In this paper the algorithm was developed for the determination parameters of S-N model using the maximum likelihood method, based on which a MATLAB code is written. It is verified using the experimental data from the literature. The results showed that it could be successfully used for taking into account the runouts in the process of determination of S-N model parameters.

2. DETERMINATION OF PARAMETERS OF S-N MODEL

S-N curves are the most widely used tool for fatigue life prediction of structures. Although the fatigue life is a dependent variable of S-N models, experimental data are usually given on the graphs where X axis is fatigue life, and Y axis is maximum stress or stress amplitude. Depending on the nature of experimental results, the relationship between stress level and fatigue life could be represented by normal or logarithmic scale. Linear dependence of logarithm of fatigue life and stress level is given by the following expression:

$$\log N = a + bS, \quad (1)$$

where N is the fatigue life, S is the stress level and a and b are parameters to be defined by fitting S-N model to the experimental data. If the data show linear dependence between logarithms of fatigue life and stress level, the S-N curve is represented with the following equation:

$$\log N = a + b \log S, \quad (2)$$

or, rewritten in the power law form:

$$N = cS^b, \quad (3)$$

where $c = 10^a$. S-N model given by the Equation (3) is known in the literature as Basquin's model [6].

2.1. Maximum Likelihood Method

General and, in most cases, satisfactory assumption of the statistical analysis of fatigue data is that fatigue lives could be represented by normal distribution. If only the data of failed specimens are available, it is simple to calculate the mean value of the fatigue life at a certain stress level. However, if the dataset contains runouts, it could not be analyzed using simple procedures. In such cases, maximum likelihood method is often utilized.

Maximum likelihood method is well known for a long time, and its wide use started in the early twenties of the 20th century. The basic principle of this method is that from the number of possible parameter values, the most likely value is the one that gives the highest probability of obtaining the observed results. The disadvantage of this method is that it requires relatively complicated calculations. For the purpose of simplifying the calculations, it is convenient to use logarithms of the probabilities ($\log p_i$) instead of the probabilities themselves (p_i).

Starting from the assumption of linear dependence of the logarithm of stress level and logarithm of fatigue life (Eq. 2 and 3), the probability of certain fatigue data could be calculated as a function of the variable t , given by the following expression:

$$t = \frac{x - \mu}{\sigma}, \quad (4)$$

where μ and σ are assumed to be mean and standard deviation values of the observed results $x = \log N$ or $\log S$. For the assumed parameters of S-N model, these values are simple to be determined. The most probable set of parameters is the one which maximizes the value of likelihood function that is equal to the sum of the probabilities of each observed experimental data [3]:

$$L(\theta) = \sum_{i=1}^n L(\theta)_i = \sum_{i=1}^n \log p_i, \quad (4)$$

where $L(\theta)$ is total likelihood of the assumed set of parameters θ , $L(\theta)_i$ is the contribution of each experimental data, and $n = n_f + n_r$, the total number of data which is equal to the sum of number of failed specimens and runouts.

There is the difference between the contributions to the likelihood of failed specimens and runouts. The explanation lies in the fact that the failed specimens are represented by the probability of failure occurrence at the observed number of cycles, given by the probability density function (Eq. 6), while the runouts are represented by the probability

that the specimen will survive more than observed endured number of cycles, which is given by the cumulative distribution function (Eq. 7).

$$p^{fail} = (2\pi)^{0.5} \cdot \exp(-t^2 / 2) dt \quad (6)$$

$$p^{runout} = \int_t^{\infty} (2\pi)^{0.5} \cdot \exp(-u^2 / 2) du \quad (7)$$

The likelihood contributions, expressed from the Eq. (6) and (7) are given by the following equations:

$$L(\theta)_i^{fail} = -t^2 / 2 - \log \sigma \quad (8)$$

$$L(\theta)_i^{runout} = \log_e \left[\int_t^{\infty} (2\pi)^{0.5} \cdot \exp(-u^2 / 2) du \right] \quad (9)$$

2.2. Algorithm for determination of the parameters of S-N curves

The first step in the process of determination of S-N curves is the selection of a mathematical model. In this paper, the model that represents linear dependence between logarithm of stress level and logarithm of fatigue life, most frequently given by Eq. (2) and (3) was used. By making simple mathematical modifications it could be rewritten in the following form:

$$N = N_d (S / S_d)^k, \quad (10)$$

where k is the slope of S-N curve, S_d is endurance limit and N_d is number of cycles for which the endurance limit was defined, which in this case is equal to 2 million cycles. The parameters from the Eq. (10) are illustrated in Fig. 1.

After the selection of appropriate mathematical model, initial values of model parameters k and S_d , as well as standard deviation σ , have to be assumed. Although the recommendation for the calculation of these initial values is given in [3], in this paper they are set to be equal to the values obtained by performing regression analysis using only the failed specimen data. For this, a separate subroutine was written according to the procedure given in [2]. In the next step, using the assumed parameters, the fatigue lives are being calculated for each stress level from experimental dataset. Calculated values represent the assumed mean values of the fatigue lives at each stress level (μ_i). After obtaining σ and μ_i using Eq. (4-10), it is relatively simple to calculate the value of likelihood $L(\theta)$. The values of likelihood are being calculated for a set of values of parameters k and S_d , that are in the range of predetermined minimum and maximum values. These values are chosen to be on the distance of $\pm 25\%$ from the initial values of parameters. Thereafter, the values of parameters that correspond to the maximum value of likelihood are used to calculate a new set of likelihood values using a number of standard deviation values, defined in the same way as in the case of parameters k and S_d . The maximum of the new set of likelihood values, and corresponding parameters and standard deviation represent the results of the first iteration. In each subsequent iteration, the minimum and maximum values of model parameters and the standard deviation are taken to be equal to the values of the array elements that were distanced ± 2 index positions from

the values that correspond to the output values of the previous iteration. The iterative procedure is terminated when the relative difference of the maximum likelihood values of two consecutive iterations is less than or equal to 0,1%. The algorithm of the described procedure is given in Fig. 2.

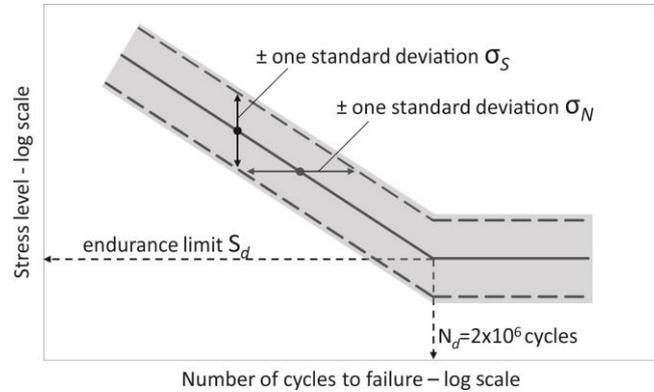


Fig. 1 Typical S-N curve given by Eq. (10) [3]

3. VERIFICATION AND VALIDATION OF THE PROPOSED ALGORITHM

As it was already pointed out, the aim of this paper was to develop an algorithm for the determination of the parameters of S-N model that could take into consideration both failed specimens and runouts. The functioning of the algorithm is verified by using the experimental data published in [7] (taken from [8]). The selected data represent the results of the experimental investigation of the fatigue life of the nickel based superalloy obtained by subjecting specimens to the cyclic load in the load controlled regime. The dataset consists of 22 failures and 4 runouts which is relatively large number of data for fatigue investigation. Thus, it is considered to be appropriate for the verification of the proposed algorithm. The results are shown in Table 1. In the case of runouts, i.e. the specimens 3,6,9 and 12, the results in the table represent the endured number of cycles. All other numbers of cycles represent the fatigue lives of corresponding specimens.

In order to verify the functioning of the MATLAB code, six sets of data were created by making some alterations of the original dataset. These alterations are summarized in Table 2. The first dataset, consisting only of failed specimens, was used to obtain S-N curve which will serve as the basis for alterations of the data. After analyzing the first set, four other sets were created by adding one data point, marked with diamonds in Fig. 3 and 4. The data points are added to the stress level of 80MPa, with the numbers of cycles that are equal to the predicted fatigue life of the first dataset, changed by 2σ , -1σ , $+1\sigma$ and $+2\sigma$ in cases 2, 3, 4 and 5, respectively. The values obtained in this way are shown in Table 3. The sixth set of data is equal to original experimental data.

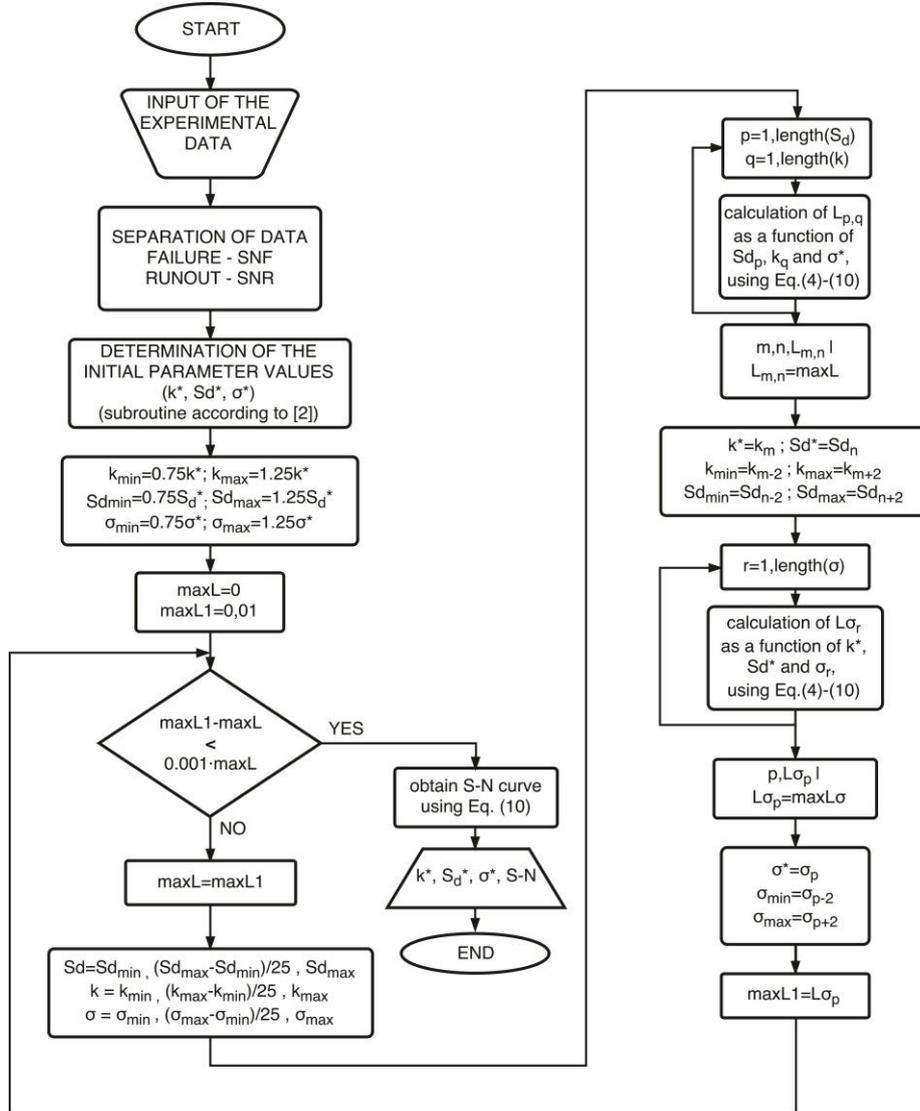


Fig. 2 Simplified algorithm for determination of the parameters of S-N model

Table 1 Fatigue test results from [7]

No.	S[MPa]	N [cyc.]	Status
1.	80.3	211629	failed
2.	80.6	200027	failed
3.	80.8	57923	<u>runout</u>
4.	84.3	155000	failed
5.	85.2	13949	failed
6.	85.6	112968	<u>runout</u>
7.	85.8	152680	failed
8.	86.4	156725	failed
9.	86.7	138114	<u>runout</u>
10.	87.2	56723	failed
11.	87.3	121075	failed
12.	89.7	122372	<u>runout</u>
13.	91.3	112002	failed
14.	99.8	43331	failed
15.	100.1	12076	failed
16.	100.5	13181	failed
17.	113.0	18067	failed
18.	114.8	21300	failed
19.	116.4	15616	failed
20.	118.0	13030	failed
21.	118.4	8489	failed
22.	118.6	12434	failed
23.	120.4	9750	failed
24.	142.5	11865	failed
25.	144.5	6705	failed
26.	145.9	5733	failed

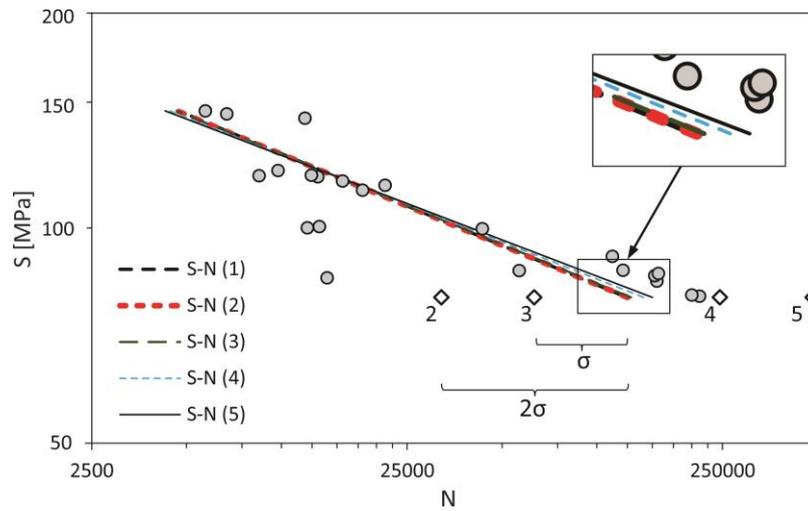
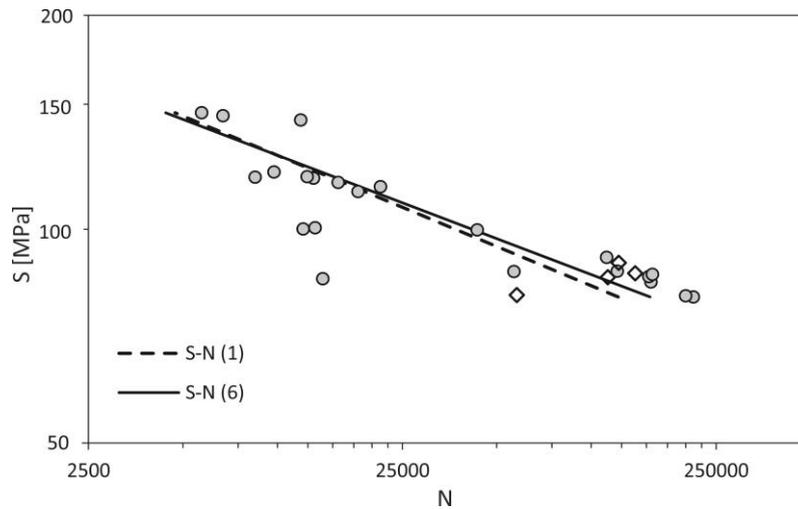
Table 2 The analyzed datasets

Case	S[MPa]
1.	Only the data of failed specimens.
2.	The same as the first case with added data point at 80MPa and number of cycles equal to $N(1) - 2\sigma$
3.	The same as the first case with added data point at 80MPa and number of cycles equal to $N(1) - \sigma$
4.	The same as the first case with added data point at 80MPa and number of cycles equal to $N(1) + \sigma$
5.	The same as the first case with added data point at 80MPa and number of cycles equal to $N(1) + 2\sigma$
6.	Entire dataset given in Table 1.

Table 3 Calculated additional data points

Case.	S [MPa]	N [cyc.]
2.	80	32129
3.	80	63280
4.	80	245486
5.	80	483511

The parameters of S-N curves, obtained from datasets 1-6, as well as standard deviations, are given in Table 4. The S-N curves are plotted and compared in Fig. 3 and 4.

**Fig. 3** The influence of the additional data points on the slope of S-N curve (cases 1-5)**Fig. 4** Comparison of S-N curves obtained by excluding and including runouts in the determination of model parameters

From the calculation results, given in Table 4, it is clear that the inclusion of the runouts in the analysis has a greater influence on the S-N curve if the specimens endured larger number of cycles before the tests were stopped. The slope of the curve as well as the endurance limit exhibited changes with the increase of the endured number of cycles of additional data point. In the second case (Fig. 3), the data point is located relatively far away from the curve S-N (1), obtained by using only the failed specimen data. Moreover, it is located out of the 95% confidence limit (1.96σ). In accordance with this fact, the influence of the additional data point on the S-N curve is negligible. This was shown in Fig. 3, where curves S-N (1) and S-N (2) almost coincide. From the practical point of view, considering the curve S-N (1) and the corresponding standard deviation, the specimen was not expected to fail after 32 thousand cycles, which was only confirmed by the observed specimen. Similar influence on the S-N curve was seen in the case of the third dataset, though in this case there is certain difference between the curves S-N (1) and S-N (3). Here, it was also not expected that the specimen would fail after 63 thousand cycles, but the probability of failure was much greater than in the second case. In the fourth and fifth case, there is a clearly visible change in the slope of the S-N curve, which is justified by the fact that the additional samples were not expected to endure that many cycles. Finally, in Fig. 4, the influence of the real runouts on the slope of the S-N curve, as well as the predicted endurance limit, was illustrated. It could be noticed that the datasets 5 and 6 show almost the same change of the slope of the S-N curve although the runouts in the sixth dataset are much closer to the curve S-N (1). This could be attributed to the fact that there are more runouts in this dataset.

Table 4 The results of calculation using MATLAB code written according to the algorithm given in Fig. 2

Case	S_d [MPa]	k	σ^*
1.	48.042	-5.446	0.2944
2.	48.196	-5.472	0.2811
3.	48.658	-5.523	0.2792
4.	50.430	-5.756	0.2839
5.	51.667	-5.923	0.2857
6.	52.198	-5.952	0.2891

*The standard deviation values are given as $\log N$

When analyzing the influence of runouts, one must have in mind that the plots in Fig. 3 and 4 were created using logarithmic scales for both the x-axis and the y-axis. Thus, even a small change in the slope could represent a big difference in fatigue life, especially at lower stress levels. For example, at the stress level of 80MPa, the curve S-N (6) predicts the failure after about 150 thousand cycles, which is 25% more than the number of cycles predicted by the curve S-N (1) (120 thousand cycles). This fact has to be considered when deciding whether to censor runouts or not.

Since the S-N model used in reference [8] assumed a nonlinear relationship between logarithms of stress and fatigue life, the S-N curve given in it was not adequate for the validation of the results of the proposed algorithm. Another dataset that contained runouts, taken from reference [9], was used to validate the algorithm predictions. Although the size of the dataset was relatively small, due to the similarity of the adopted S-N models and methods for the determination of parameters, it was considered suitable for the purpose of validation.

The comparison of S-N curves from [9], obtained by performing three different analyses, and S-N curve obtained by using proposed algorithm is given in Fig. 5. All of them represent the linear relationship between logarithms of stress and fatigue life. The curves from reference [9] provided the fatigue life predictions in which runouts were treated by:

- not including them in analysis,
- considering them as failures,
- analyzing them using the procedure based on maximum likelihood method, different than the one proposed here.

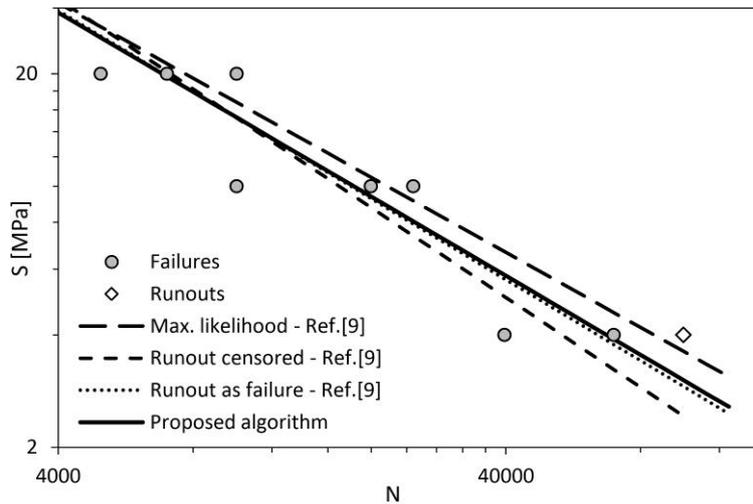


Fig. 5 Comparison of S-N predictions from Ref. [9] and predictions obtained using proposed algorithm

From the Fig. 5, it is clear that censoring of the runout gave a big difference in the fatigue life prediction, compared to the other analyses. Considering that the fatigue life of the runout is equal to or greater than the endured number of cycles, it makes sense that it should be considered at least as a failure, which is represented by the dotted curve. However, having in mind that this sample did not fail suggests that the curve should be shifted a bit to the right in this region. This was achieved by utilization of the maximum likelihood method. However, the S-N curve obtained using proposed algorithm is more logical than the one given in reference [9]. A considerable difference between the curves obtained by considering runouts as failures and analyzing them using maximum likelihood method is very non-conservative and not realistic since it is not known if the sample would fail in the next cycle. Thus, it could be concluded that the proposed algorithm gave very good fatigue life predictions.

4. CONCLUSION

In this paper, the algorithm for determination of the parameters of the S-N curve for prediction of the fatigue life of structures or elements subjected to cyclic loading is given. The algorithm is derived based on the maximum likelihood method. This method allows the

inclusion of runouts in the process of determination of model parameters. Based on the proposed algorithm, a MATLAB code is written and verified by using the experimental data from the literature. The results obtained from different sets of data showed that the MATLAB code gives the appropriate change of slope, as well as the predicted endurance limit, when runouts are included in the analysis. The results obtained by analysis of the real experimental data showed that censoring runouts could give significant difference in fatigue life prediction. The comparison of S-N curves obtained by using the proposed algorithm with the curves from the literature obtained by utilizing the maximum likelihood method showed that it gives very good predictions of the fatigue life. Having in mind all written above, in order to predict more accurate fatigue life, it is advisable to include runouts in the analysis of the experimental data, for which the proposed algorithm could serve as a good tool.

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ALGORITAM ZA ODREĐIVANE S-N KRIVIH ELEMENATA KONSTRUKCIJA IZLOŽENIH CIKLIČNOM OPTEREĆENJU

Predviđanje veka trajanja elemenata konstrukcije ili same konstrukcije pri dejstvu cikličnog opterećenja najčešće se vrši pomoću S-N krivih, dobijenih na osnovu eksperimentalnih podataka ispitivanja uzoraka cikličnim opterećenjem do loma. Međutim, prilikom ispitivanja se može desiti da uzorak izdrži unapred određeni broj ciklusa bez loma, kao i da se ono iz određenog razloga prekine. Takvi uzorci se nazivaju preživeli, a podaci dobijeni na ovaj način takođe mogu biti od značaja za određivanje S-N krive. U ovom radu je prikazan algoritam za određivanje parametara S-N krive metodom maksimalne verodostojnosti, kojim se uzimaju u obzir i podaci o preživelim uzorcima. Na osnovu algoritma je napisan MATLAB program, čija je verifikacija urađena na eksperimentalnim podacima iz literature. Rezultati su pokazali da se pri proračunu parametara S-N krive pomoću razvijenog algoritma mogu na odgovarajući način uzeti u obzir i podaci o broju ciklusa koje su nepolomljeni uzorci preživeli. Takođe, pokazano je da njihovo uključivanje u proračun može značajno uticati na predviđeni vek trajanja, pogotovo pri nižim nivoima opterećenja.

Ključne reči: *Zamor, S-N kriva, algoritam, metoda maksimalne verodostojnosti.*