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# ANALYTICAL STUDY OF THE SECTION OF THE RC BEAMS STRENGHTENED FOR FLEXURE WITH FRP MATERIALS

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**Abstract**. *Strengthening of concrete structures is applied as a solution for various deterioration problems in civil engineering practice. This paper presents an analytical study of the behaviour of cross-section of reinforced concrete (RC) beam, strengthened for flexure with fiber reinforced polymer (FRP) materials. Using the balance of internal forces in the cross section through all phases of stress through which the section passes, a program was written in the MATLAB software, the execution of which produced a curve of dependence between bending moment and curvature, which is one of the most important indicators of cross section behaviour. The parameters varied in this study are the amount and type of FRP reinforcement and the obtained results indicate a significant influence of additional FRP reinforcement both on the yielding and ultimate bending moment, and on the bending stiffness of the strengthened cross section.*

**Key words**: *reinforced concrete beams, cross-section analysis, fiber reinforced polymer materials, strengthening*

## 1. INTRODUCTION

Fiber reinforced polymer (FRP) materials are a subset of a class of materials called composites or composite materials. Composite materials are made of two or more materials that form a new material with improved properties that are superior to the properties of the individual components individually. FRP materials are relatively new, high-strength, low-

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weight composite materials made up of carbon (CFRP), glass (GFRP) or aramid (AFRP) fibers embedded in a polymer matrix [1].

Interest in the use of FRP materials in building structures is constantly increasing, so nowadays there is a large number of applications of these materials in structures around the world. Some of the most common applications in civil engineering include:

- strengthening and repair of structural elements made of reinforced concrete, steel, aluminium and wood.
- concrete reinforcement with bars and cables made of FRP material,
- production of structures from FRP material,
- production of hybrid constructions.

Strengthening of civil engineering infrastructure has gained significant attention due to deterioration problems of structures and need for meeting up-to-date design requirements [2]. One of the basic factors that causes the unsatisfactory condition of the existing infrastructure is corrosion of reinforced steel in concrete, which causes damage of concrete, loss of reinforcing steel and in some cases failure of construction [3]. Taking into consideration the existing concrete infrastructures both in Europe and worldwide, there is a large interest for the research in the field of strengthening of concrete structures. In addition, the most common reasons for strengthening the existing structures are damage to structures due to earthquakes, changes in the purpose of structures and the implementation of additional loads.

The two basic methods most commonly used in strengthening RC beams with FRP material are: strengthening by gluing laminates of FRP material on the surface of concrete beams—EB method; and strengthening by mounting bars or narrow strips of FRP material in grooves made in the cover of concrete—NSM method.

The largest number of researches of RC beams strengthened to bending by FRP reinforcement are experimental or numerical researches. Conclusions of the experimental researches presented in literature, indicate both increases in bearing capacity and reduction of deformations of strengthened beams [4, 5]. In addition to the papers that generally deal with experimental research of strengthened beam girders, there is a significant number of papers in which methods for modeling reinforced RC beams using FEM analysis are proposed [6, 7].

There are many analytical methods for analysing reinforced concrete (RC) beams strengthened with FRP materials [8, 9]. An analytical approach based on cross-sectional analysis can easily determine the ultimate load of a strengthened beam. The approach is based on the principles of strain compatibility, internal force balance and idealized constitutive relations for concrete, steel and FRP reinforcement. These idealized relations, together with the assumption that slip on the contact surface between concrete and FRP systems can be neglected, form the basis for the analysis of the ultimate state of strengthened RC beams [10].

The aim of this paper is the development of a mathematical model for the calculation of the load-bearing capacity of the cross section of a RC beam strengthened with fiber FRP materials, subjected to bending, at characteristic states of the beam girder:

- until cracks appear,
- from the appearance of cracks to the yielding of steel reinforcement
- after the occurrence of steel reinforcement yielding, up to failure.

**Notation**

The following symbols are used in this paper:

| | | | |
|---|---|---|---|
| $b$ | beam width | $h$ | beam height |
| $A_c$ | cross sectional area of concrete | $A_{cc}$ | compressed part of concrete cross-section |
| $A_{ct}$ | tensioned part of concrete cross-section | $A_{s1}$ | cross sectional area of tensioned steel reinforcement |
| $A_{s2}$ | cross sectional area of compressed steel reinforcement | $A_{frp}$ | cross sectional area of the FRP reinforcement |
| $E_c$ | modulus of elasticity of concrete | $E_s$ | modulus of elasticity of steel before yielding |
| $E_{sp}$ | modulus of elasticity of steel after yielding | $E_{frp}$ | modulus of elasticity of the FRP reinforcement |
| $\sigma_c$ | stress in the concrete | $\sigma_{cu}$ | stress in the concrete at the ultimate strain |
| $\sigma_{cc}$ | stress on the compressed edge of the concrete | $\sigma_{ct}$ | stress on tensioned edge of concrete |
| $\sigma_s$ | stress in the steel reinforcement | $\sigma_{s1}$ | stress in tensioned steel reinforcement |
| $\sigma_{s2}$ | stress in compressed steel reinforcement | $\sigma_{frp}$ | stress in the FRP reinforcement |
| $f_c'$ | compressive strength of concrete | $f_{ct}$ | tensile strength of concrete |
| $f_t$ | tensile strength of steel reinforcement | $f_y$ | yield strength of steel reinforcement |
| $f_{frp}$ | tensile strength of the FRP reinforcement | | |
| $\varepsilon_c$ | strain in concrete | $\varepsilon_{c1}$ | compressive strain in the concrete at the peak stress $f_c'$ |
| $\varepsilon_{cu}$ | ultimate compressive strain in concrete | $\varepsilon_{cc}$ | compressive strain in the concrete |
| $\varepsilon_o$ | appropriate strain in the concrete | $\varepsilon_s$ | strain in steel reinforcement |
| $\varepsilon_{su}$ | ultimate tensile steel strain | $\varepsilon_y$ | steel strain at the steel yield strength |
| $\varepsilon_{s1}$ | strain in tensioned steel reinforcement | $\varepsilon_{s2}$ | strain in compressed steel reinforcement |
| $\varepsilon_{frp}$ | strain in FRP reinforcement | $\varepsilon_{frp,u}$ | ultimate tensile strain in FRP reinforcement |
| $M$ | bending moment | $M_{cr}$ | cracking moment of concrete |
| $M_y$ | yielding moment of tensile steel reinforcement | $M_u$ | ultimate bending moment |
| $M_{ext}$ | external bending moment | | |
| $C_c$ | compressive force in the concrete | $C_{s2}$ | force in the compressed steel reinforcement |
| $T_c$ | tensile force in the concrete | $T_{s1}$ | force in the tensioned steel reinforcement |
| $T_{frp}$ | force in the FRP reinforcement | | |
| $y$ | neutral axis distance from the corresponding force | $y_{Cs2}$ | neutral axis distance from the force $C_{s2}$ |
| $y_{Tfrp}$ | neutral axis distance from the force $T_{frp}$ | $y_{Ts1}$ | neutral axis distance from the force $T_{s1}$ |
| $\kappa$ | curvature | $\kappa_y$ | yielding curvature |
| $\kappa_u$ | ultimate curvature | | |
| $\alpha_1$ | compressive stress reduction coefficient in the concrete | $\beta_1$ | coefficient of reduction of the height of the stress diagram |
| $\lambda$ | factor for calculating the bulk density of the concrete | | |

## 2. Basic Postulates

The assumptions introduced in the cross-sectional analysis of RC beams strengthened with FRP reinforcement are as follows [9]:

1. The distribution of strains by section height is linear - Bernoulli's hypothesis of straight sections.
2. No slipping between longitudinal reinforcing steel and concrete;
3. No slipping between FRP system and concrete;
4. Beam failure occurs either due to reaching the ultimate strain of concrete under compression or due to failure of the FRP strengthening system.

When designing the strengthening system, the influence of the previous load should be taken into account. The distribution of dilatations in the cross section of the RC beam during the action of the moment before the installation of the strengthening system ($M_o$) can be determined on the basis of the theory of elasticity. As the moment ($M_o$) is usually greater than the moment of crack appearance ($M_{cr}$), the calculation should be based on the cracked cross section. If the moment $M_o$ is less than the moment $M_{cr}$, its influence in the design of the strengthening system can be neglected [1].

To illustrate the state of strains in the cross section of RC beams strengthened with FRP reinforcement, they can be shown as a superposition of strains before and after the installation of FRP strengthening system (Fig. 1)

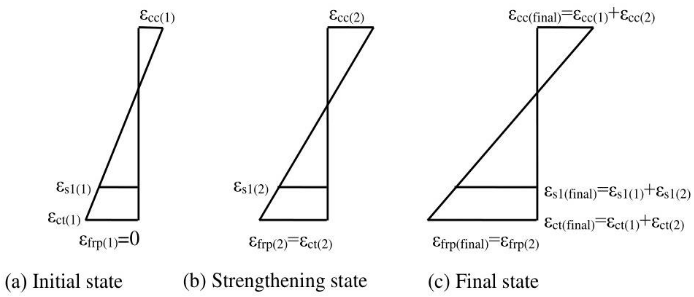



**Fig. 1** Superposition of strain in EB strengthening method [11]

The strain states shown are:

1. Initial state, before the installation of the strengthening system, when strains $\varepsilon_{(1)}$ due to load are considered at the time of installation of the strengthening system, Fig. 2a;

2. Strengthened state, after installation of strengthening system, when strains $\varepsilon_{(2)}$ due to load applied after installation of strengthening system (hypothetical situation) are considered, Fig. 2b;

3. Final state, after the installation of the strengthening system when strains $\varepsilon_{final}$ are considered due to the load that exists at the time of installation of the strengthening system as well as from the additional load, Fig. 2c.

The final state of strains is obtained by superposition of the initial strains and strains that occur in FRP reinforcement, whereby the neutral axis changes position [11].

## 3. Adopted Models of Constituent Materials

### 3.1. Concrete

In the analysis of the cross section of the RC beam strengthened with the FRP reinforcement, the parabolic relation between stress and strain for concrete was adopted

(Fig. 2). The mathematical formulation of this relation can be described by the expressions (1-3) [12]:

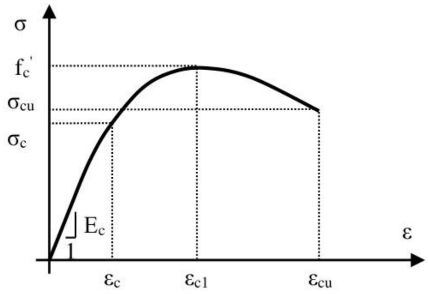



**Fig. 2** Idealised stress- strain curve for concrete at axial pressure

$$\sigma_c = f_c^{'} \left[ \frac{2\varepsilon_c}{\varepsilon_{c1}} - \left( \frac{\varepsilon_c}{\varepsilon_{c1}} \right)^2 \right] \quad (1)$$

$$\varepsilon_{c1} = \frac{2 f_c^{'}}{E_c} \quad (2)$$

$$E_c = 4500 \sqrt{f_c^{'}} \quad (3)$$

A simplified distribution of compressive stresses in the concrete cross section shown in Fig. 3 was given by Collins and Mitchell [12]. Coefficients of rectangular compressive stress distribution in concrete ($\alpha_1$ and $\beta_1$) are determined according to the following expressions:

$$\alpha_1 \beta_1 = \frac{\varepsilon_{cc}}{\varepsilon_0} - \frac{1}{3} \left( \frac{\varepsilon_{cc}}{\varepsilon_0} \right)^2 \quad (4)$$

$$\beta_1 = \frac{4 - \frac{\varepsilon_{cc}}{\varepsilon_0}}{6 - \frac{2\varepsilon_{cc}}{\varepsilon_0}} \quad (5)$$

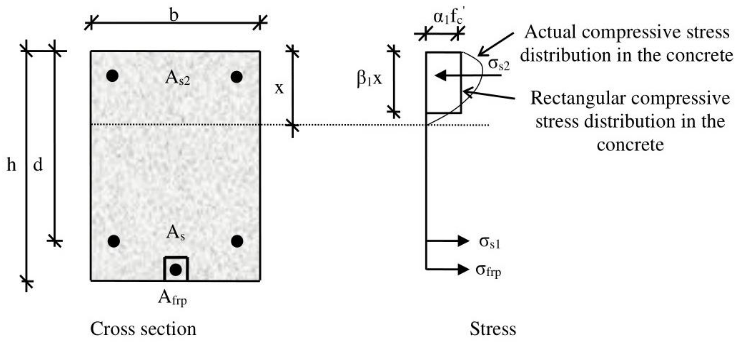



**Fig. 3** Stress distribution in the cross section of the RC beam strengthened with FRP reinforcement

The compressive force in concrete ($C_c$) and its position relative to the neutral axis ($y_{Cc}$) are given by the following expressions [12]:

$$C_c = \alpha_1 \beta_1 f_c^{'} x b \tag{6}$$

$$y_{Cc} = x - \frac{1}{2}\beta_1 x \tag{7}$$

When the stress on the tensioned edge of concrete is less than the tensile strength of concrete, the value of the tensile force in concrete and its position relative to the neutral axis are given in terms of:

$$T_c = \frac{1}{2} b \sigma_{ct} (h - x) \; , \tag{8}$$

$$y_{Tc} = \frac{2}{3}(h - x) \; , \tag{9}$$

$$f_{ct} = 0.6\lambda \sqrt{f_c^{'}} \; . \tag{10}$$

### 3.2. Steel reinforcement

In the analysis of the cross section of the RC beam strengthening with FRP reinforcement, a bilinear dependence between stress and strain for reinforcing steel (elastoplastic behaviour) with a 1% slope inclination was adopted (Fig. 4). The mathematical formulation of this dependence can be described by expressions (11-12).

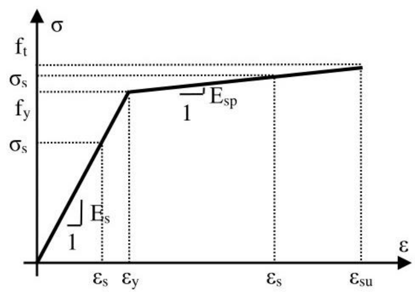



**Fig. 4** Idealised stress-strain curve for reinforcing steel

$$\sigma_s = \begin{cases} \varepsilon_s E_s & for \quad \varepsilon_s \le \varepsilon_y \\ f_y + E_{sp}(\varepsilon_s - \varepsilon_y) & for \quad \varepsilon_s \ge \varepsilon_y \end{cases} \tag{11}$$

$$E_{xp} = 0.01 E_s \tag{12}$$

### 3.3. FRP reinforcement

In the analysis of the cross section of the RC beam strengthening with FRP reinforcement, the linear-elastic relation between stress and strain for FRP reinforcement was adopted, as in Fig. 5. The mathematical formulation of this dependence can be described by the expression (13).

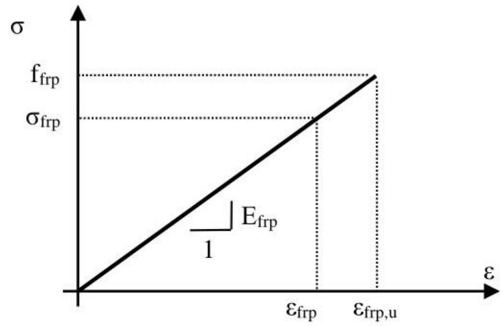



**Fig. 5** Idealised stress-strain curve for FRP reinforcement

$$\sigma_{frp} = \varepsilon_{frp} E_{frp} \tag{13}$$

## 4. STRESS-STRAIN STAGES OF THE CROSS SECTION OF THE RC BEAM STRENGTHENED WITH FRP REINFORCEMENT

Figure 6 shows a diagram of the dependence between the bending moment (M) and the curvature (κ) in the cross section of the RC beam strengthened with the FRP reinforcement, which is subjected to bending. The dependence is idealized by a nonlinear curve consisting of three parts:

- zone before cracks appear,
- zone after the appearance of cracks and before the appearance of steel yielding and
- zone after the occurrence of steel yielding to the cross-section failure.

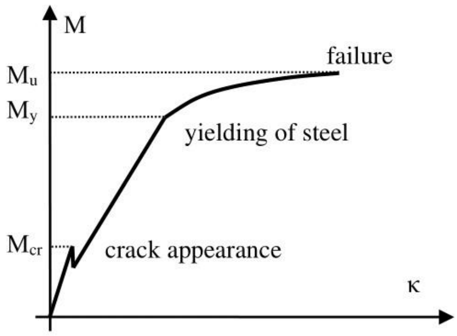



**Fig. 6** Diagram of dependence between bending moment (M) and curvature (κ) in the cross-section of RC beam strengthened with FRP reinforcement

In the cross-sectional analysis of reinforced concrete beams strengthened with the FRP reinforcement, the principles of calculation of complex (composite) cross-sections used in structural theory are applied in this paper [13]. The distribution of strains and stresses by section height is shown in Fig. 7 [14]:

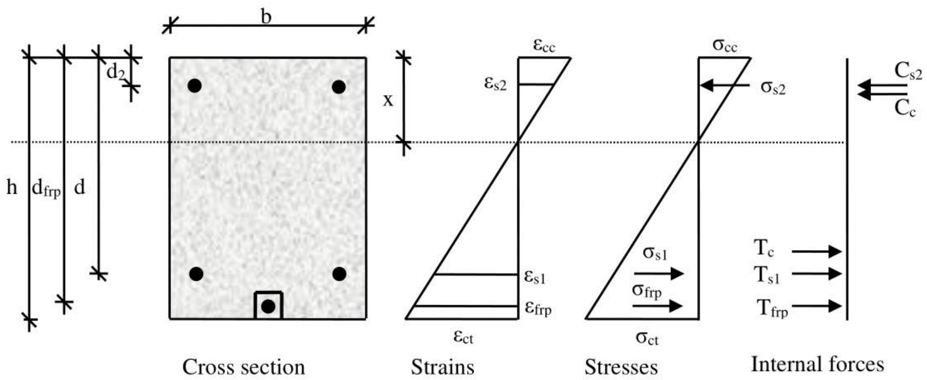



**Fig. 7** Assumed distribution of strains, stresses and internal forces in the cross section of RC beams strengthened with FRP reinforcement (with neglection of initial strains)

The corresponding equations of equilibrium of internal forces are based on the assumption that the resultant of internal forces in the cross section is equal to zero (ΣX=0):

$$\int_{A_{cc}} \sigma_c dA_{cc} + \int_{A_{s2}} \sigma_{s2} dA_{s2} - \int_{A_{ct}} \sigma_c dA_{ct} - \int_{A_{s1}} \sigma_{s1} dA_{s1} - \int_{A_{frp}} \sigma_{frp} dA_{frp} = 0\,, \tag{14}$$

that is:

$$C_c + C_{s2} - T_c - T_{s1} - T_{frp} = 0\,, \tag{15}$$

as well as that the moment of internal forces is equal to the external bending moment ($\Sigma M=M_{ext}$):

$$\int_{A_{cc}} \sigma_c y dA_{cc} + \int_{A_{s2}} \sigma_{s2} y dA_{s2} - \int_{A_{ct}} \sigma_c y dA_{ct} - \int_{A_{s1}} \sigma_{s1} y dA_{s1} - \int_{A_{frp}} \sigma_{frp} y dA_{frp} = M_{ext}\,, \tag{16}$$

that is:

$$C_c y_c + C_{s2} y_{s2} - T_c y_{Tc} - T_{s1} y_{Ts1} - T_{frp} y_{Tfrp} = M_{ext}\,, \tag{17}$$

### 4.1. Pre-cracking stage

Before the appearance of cracks, the tensile stress on the tensioned edge of the concrete is less than the tensile strength of the concrete, so the entire cross-section participates in accepting the external load. Therefore, the moment of inertia of the whole cross section ($I_g$) is used for the calculation.

For the stage before the appearance of cracks, the equations of equilibrium of internal forces ($\Sigma X = 0$ and $\Sigma M = M_{ext}$) can be written in the following form:

$$bx\frac{\sigma_{cc}}{2} + A_{s2}\sigma_{s2} - b(h-x)\frac{\sigma_{ct}}{2} - A_{s1}\sigma_{s1} - A_{frp}\sigma_{frp} = 0 \tag{18}$$

$$\frac{1}{3}bx^2\sigma_{cc} + A_{s2}\sigma_{s2}(x-d_2) - \frac{1}{3}b(h-x)^2\sigma_{ct} - A_{s1}\sigma_{s1}(d-x) - A_{frp}\sigma_{frp}(d_{frp}-x) = M_{ext} \tag{19}$$

### 4.2. Pre-yielding stage

When the tensile stress on the tensioned edge of the concrete exceeds the tensile strength of the concrete, cracks appear due to bending, and the beam is viewed as if it were composed of cracked cross-sections.

The equations of equilibrium of internal forces ($\Sigma X = 0$ and $\Sigma M = M_{ext}$) before the occurrence of yielding of the reinforcement take the following form:

$$\alpha_1 \beta_1 b x f_c^{'} + A_{s2}\sigma_{s2} - A_{s1}\sigma_{s1} - A_{frp}\sigma_{frp} = 0 \qquad (20)$$

$$\alpha_1 \beta_1 b x^2 f_c^{'}(x - \frac{1}{2}\beta_1) + A_{s2}\sigma_{s2}(x - d_2) - A_{s1}\sigma_{s1}(d - x) - A_{frp}\sigma_{frp}(d_{frp} - x) = M_{ext} \qquad (21)$$

### 4.3. Post-yielding stage

After the beginning of the yielding of the steel reinforcement, a state arises in which the equations of equilibrium of internal forces have the following form:

$$\alpha_1 \beta_1 b x f_c^{'} + A_{s2}\sigma_{s2} - A_{s1}[f_y + 0.01E_{sp}(\varepsilon_{z1} - \varepsilon_y)] - A_{frp}\sigma_{frp} = 0 \qquad (22)$$

$$\begin{aligned} &\alpha_1 \beta_1 b x^2 f_c^{'}(x - \frac{1}{2}\beta_1) + A_{s2} f_{s2}(x - d_2) - A_{s1}[f_y + 0.01E_{sp}(\varepsilon_{z1} - \varepsilon_y)](d - x) - \\ &- A_{frp}\sigma_{frp}(d_{frp} - x) = M_{ext} \end{aligned} \qquad (23)$$

## 5. Analysis of the Moment-Curvature Relationship in the Cross Section of the RC Beam Strengthened with FRP Reinforcement

### 5.1. M_k.m program for determining bending moment-curvature relationship

The shape of the moment-curvature diagram is an important characteristic of the cross-section on which the behaviour of the RC beam as a whole significantly depends [15].

The relationship between the moment and curvature can be most easily determined by varying the values of edge strains in concrete. Based on the strain on the compressed edge of concrete ($\varepsilon_{cc}$), other characteristic values of strain can be determined according to the following expressions (Fig. 7):

$$\varepsilon_{s2} = \varepsilon_{cc}\frac{x - d_2}{x} \quad \varepsilon_{s1} = \varepsilon_{cc}\frac{d - x}{x} \quad \varepsilon_{frp} = \varepsilon_{cc}\frac{d_{frp} - x}{x} \quad \varepsilon_{ct} = \varepsilon_{cc}\frac{h - x}{x} \qquad (24)$$

By substituting the expressions (24) in the equilibrium condition (14), the distance of the neutral axis from the compressed edge of the concrete (x) can be determined, after which the equilibrium condition (16) can be determined. and the bending moment (M) corresponding to the strain at the compressed edge of the concrete ($\varepsilon_{cc}$).

Using the straight section hypothesis, the magnitude of the curvature in the cross section ($\kappa$) can be determined by dividing the strain of the compressed edge of the concrete section ($\varepsilon_{cc}$) by the distance of that edge from the neutral axis (x):

$$\kappa = \frac{\varepsilon_{cc}}{x} \tag{25}$$

In this way, the relationship between the bending moment and the curvature (M-κ) in the cross section of the RC beam strengthened with FRP reinforcement is obtained. The diagram of dependence between the moment and curvature can be determined by incremental increase of strain of the compressed edge of concrete until one of the following conditions is fulfilled:

1. The value of the strain on the compressed edge of concrete is equal to the strain of concrete crushing,
2. The strain value in the FRP reinforcement is equal to its ultimate strain.

In the paper [16] for the analysis of the cross section of the RC beam strengthened with FRP reinforcement, the program M_k.m in Matlab (MATLAB R2014a) was written.

The program is based on the previously described procedure for determining the dependence between moment and curvature in the cross section of an RC beam strengthened with FRP reinforcement, subjected to bending. Execution of this program determined the curves of dependence between bending moment and curvature. The strain of the compressed edge of the concrete cross section was increased in each step by 10 microstrains (0.00001 mm / mm).

### 5.2. Numerical example

The focus of the analytical investigation in this paper is on the RC beam with a cross section of 120/200 mm.

Mechanical characteristics of concrete and steel reinforcement are:

- Compressive strength of concrete is $f_{c'}$ = 40 MPa;
- Beam is reinforced in both compressed and tensioned zones with steel reinforcement 2BØ8 ($A_s$ = 100 mm2, $\mu_s \approx 0.5\%$, $f_y$ = 400 MPa, $E_s$ = 210 GPa).

In order to analyse the influence of the FRP reinforcement on the magnitude of the cracking moment ($M_{cr}$), yielding moment ($M_y$) and ultimate bending moment ($M_u$), using the program M_k.m, diagrams of the dependence between the moment and the curvature were obtained for:

1. Cross section without FRP reinforcement;
2. Cross sections strengthened with different amount of CFRP reinforcement ($A_{cfrp}$=10-100mm$^2$) with tensile strength $f_{frp,u}$ = 2000 MPa, modulus of elasticity $E_{frp}$ = 150 GPa and ultimate strain $\varepsilon_{frp,u}$ = 0.0133;
3. Cross sections strengthened with different amount of GFRP reinforcement ($A_{gfrp}$=10-100mm$^2$) with tensile strength $f_{frp,u}$ = 760 MPa, elastic modulus $E_{frp}$ = 40.8 GPa and ultimate strain $\varepsilon_{frp,u}$ = 0.0186.

### 5.3. Results

The values of the yielding moment, as well as the ultimate bending moment obtained by the program M_k.m are shown in Table 1.

In addition to the value of the yielding moment of steel reinforcement and the ultimate bending moment, as it was said, the diagram of the dependence between the moment and the curvature speaks a lot about the behaviour of the RC section. Among other things, the change in the bending stiffness of the cross-section can be clearly seen on it with different amounts of added FRP reinforcement.

**Table 1** Yielding and ultimate bending moment for the examined cross-sections

| | CFRP | | GFRP | |
|---|---|---|---|---|
| $A_{frp}$ [$mm^2$] | $M_y$ | $M_u$ | $M_y$ | $M_u$ |
| control | 6.00 | 6.78 | 6.00 | 6.78 |
| 10 | 6.58 | 10.19 | 6.18 | 8.44 |
| 20 | 7.17 | 13.47 | 6.35 | 9.69 |
| 30 | 7.75 | 16.72 | 6.53 | 10.93 |
| 40 | 8.34 | 19.29 | 6.70 | 11.95 |
| 50 | 8.93 | 21.05 | 6.79 | 12.86 |
| 60 | 9.52 | 22.58 | 6.96 | 13.71 |
| 70 | 10.00 | 23.95 | 7.14 | 14.50 |
| 80 | 10.60 | 25.17 | 7.31 | 15.24 |
| 90 | 11.19 | 26.29 | 7.49 | 15.94 |
| 100 | 11.78 | 27.31 | 7.68 | 18.50 |

A diagram of dependence between bending moment and curvature for control, unstrengthen cross-section is shown in Fig. 8, while diagrams for cross-sections strengthened with the CFRP and GFRP reinforcement are shown in Fig. 9 and Fig. 10 respectively.

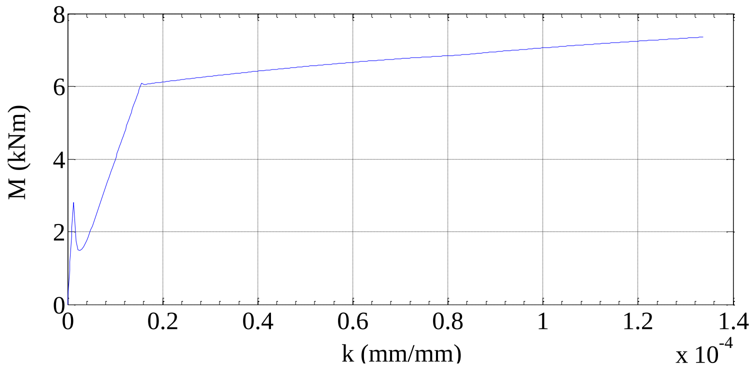
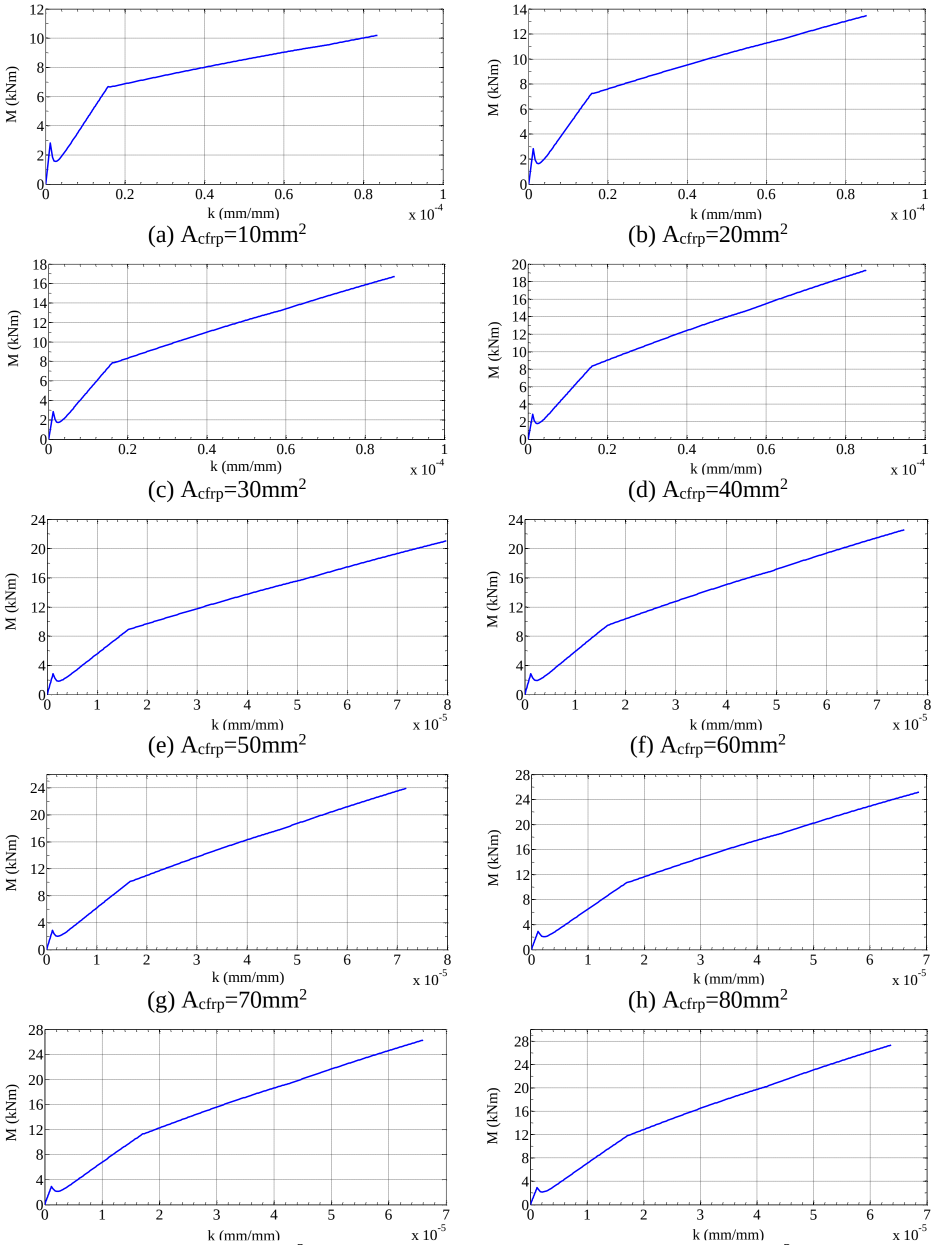



**Fig. 8** Diagram of dependence between bending moment and curvature for control, unstrengthen cross-section [16]




(a) $A_{cfrp}$=10mm$^2$ (b) $A_{cfrp}$=20mm$^2$

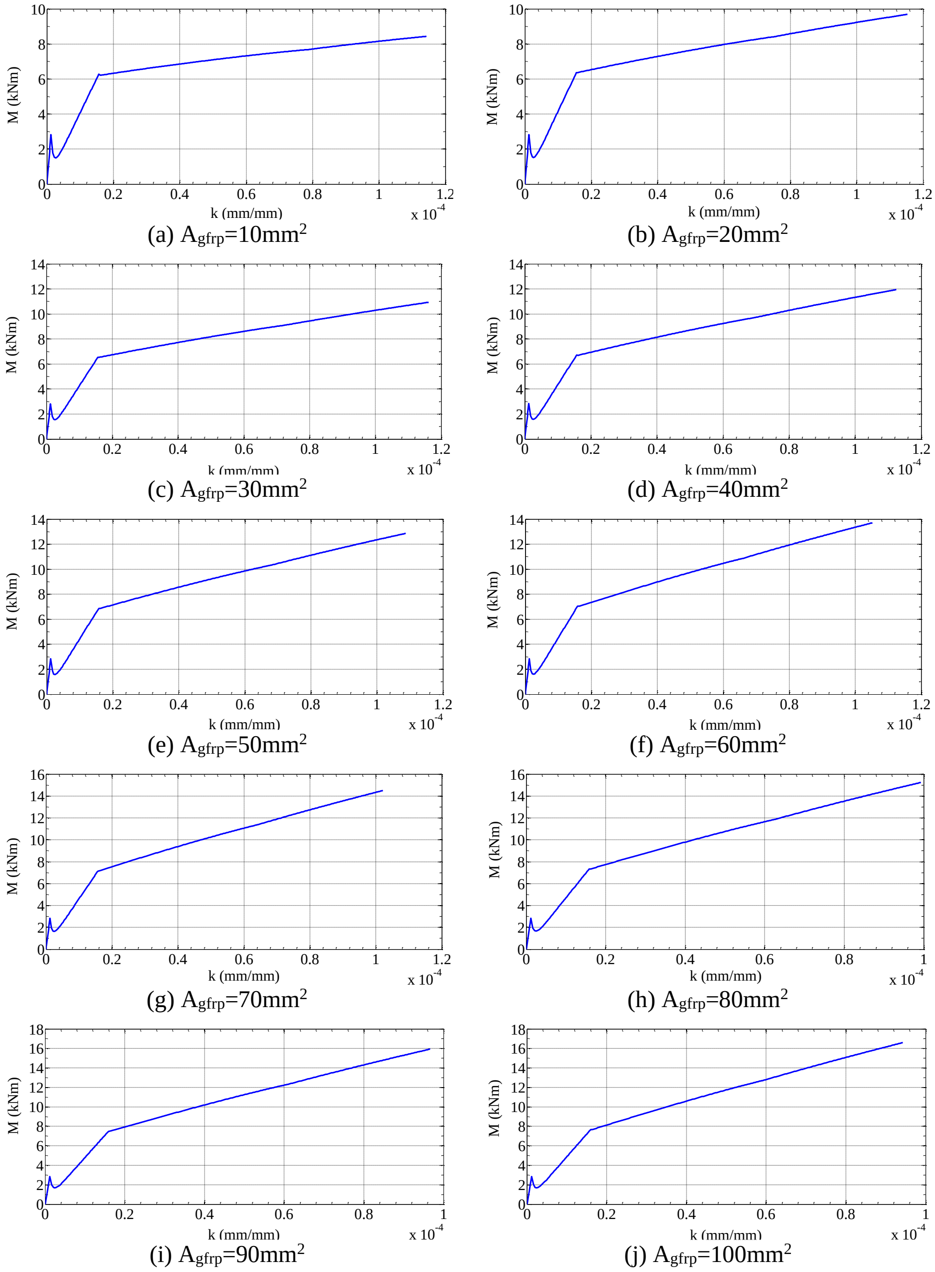
(c) $A_{cfrp}$=30mm$^2$ (d) $A_{cfrp}$=40mm$^2$

(e) $A_{cfrp}$=50mm$^2$ (f) $A_{cfrp}$=60mm$^2$

(g) $A_{cfrp}$=70mm$^2$ (h) $A_{cfrp}$=80mm$^2$

(i) $A_{cfrp}$=90mm$^2$ (j) $A_{cfrp}$=100mm$^2$

**Fig. 9** Diagram of dependence between bending moment and curvature for cross-section strengthened with CFRP reinforcement [16]




(a) $A_{gfrp}$=10mm$^2$ (b) $A_{gfrp}$=20mm$^2$

(c) $A_{gfrp}$=30mm$^2$ (d) $A_{gfrp}$=40mm$^2$

(e) $A_{gfrp}$=50mm$^2$ (f) $A_{gfrp}$=60mm$^2$

(g) $A_{gfrp}$=70mm$^2$ (h) $A_{gfrp}$=80mm$^2$

(i) $A_{gfrp}$=90mm$^2$ (j) $A_{gfrp}$=100mm$^2$

**Fig. 10** Diagram of dependence between bending moment and curvature for cross-section strengthened with GFRP reinforcement [16]

## 6. Conclusion

Based on the obtained diagrams of the dependence between the moment and the curvature (Figure 5-12, 5-13 and 5-14), and the obtained values of the cracking moment ($M_{cr}$), the yielding moment ($M_y$) and the curvature of the section at the beginning of the yielding of steel reinforcement ($\kappa_y$), the ultimate bending moment (Mu) and the curvature of the section at the moment of failure ($\kappa_u$), as well as their relations, the following conclusions were drawn:

1. The external FRP reinforcement does not have a significant effect on the value of the cracking moment, as well as on the bending stiffness of the section before cracking:
   - in the case of beams strengthened with the CFRP reinforcement, the increase in the cracking moment ($M_{cr}$) is a maximum of 4.41%, in the case when the percentage of reinforcement with the external FRP reinforcement is equal to the percentage of reinforcement with internal steel reinforcement,
   - for beams reinforced with the GFRP reinforcement, the increase in the cracking moment ($M_{cr}$) is a maximum of 0.24%, in the case when the percentage of reinforcement with the external FRP reinforcement is equal to the percentage of reinforcement with internal steel reinforcement.
2. The external FRP reinforcement affects the value of the yielding moment, as well as the bending stiffness of the section:
   - for beams strengthened with the CFRP reinforcement, the increase in yielding moment (My) is a maximum of 96.33%, while the maximum increase in bending stiffness at the beginning of the yielding of steel reinforcement is 77.56%,
   - for beams strengthened with the GFRP reinforcement, the increase in yielding moment (My) is a maximum of 26.17%, while the maximum increase in bending stiffness at the moment of the start of yielding of steel reinforcement is 22.30%.
3. The external FRP reinforcement significantly affects the ultimate bending moment, as well as the bending stiffness of the cross section:
   - in the case of a beam strengthened with the CFRP reinforcement, the increase in the ultimate bending moment ($M_u$) is a maximum of 302.8%, while the maximum increase in bending stiffness at the moment of failure is 298.07%,
   - in the case of a beam strengthened with the GFRP reinforcement, the increase in the ultimate bending moment ($M_u$) is a maximum of 144.84%, while the maximum increase in bending stiffness at the moment of failure is 64.70%.

The obtained results indicate a significant influence of FRP reinforcement on the cross - sectional behaviour of the strengthened RC beam, considering that even with small amounts of additional reinforcement, the bearing capacity, as well as the bending stiffness, significantly increases.

However, the required percentage of the additional reinforcement should be determined in each individual case, given that in the case when the compressive strength of the concrete is small, as well as in the case of high percentage of steel reinforcement, the use of additional FRP reinforcement loses importance because due to concrete crushing, with little utilization of the load-bearing capacity of FRP reinforcement. In general, strengthening of an RC beam is possible only if there is an additional capacity of the compressed part of the concrete cross-section that would allow an increase in the internal bending moment in the cross-section [16].

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# ANALITIČKI PRORAČUN PRESEKA AB GREDE OJAČANE NA SAVIJANJE FRP MATERIJALIMA

*Ojačavanje betonskih konstrukcija se primenjuje kao rešenje pri rešavanju različitih problema u građevinskoj praksi. U radu je prikazano analitičko istraživanje ponašanja poprečnog preseka armirano-betonske (AB) grede, ojačane na savijanje vlaknima armiranim polimernim (eng. FRP) materijalima. Korišćenjem uslova ravnoteže unutrašnjih sila u poprečnom preseku grede, kroz sve faze naprezanja kroz koje ona prolazi, napisan je program u softveru MATLAB, čijim se izvršenjem dobija kriva zavisnosti između momenta savijanja i krivine, koja predstavlja jedan od osnovnih pokazatelja ponašanja poprečnog preseka grede. Parametri koji su varirani u ovom istraživanju su količina i vrsta FRP armature, a dobijeni rezultati ukazuju na značajan uticaj dodate FRP armature kako na moment tečenja i granični moment savijanja, tako i na krutost na savijanje ojačanog poprečnog preseka.*

Ključne reči: *armirano-betonske grede, analiza poprečnog preseka, vlaknima armirani polimerni materijali, ojačavanje*