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FORMATION OF TRANSFER FUNCTIONS FOR CONTROL SYSTEMS UNDER IMPLEMENTATION CONDITIONS*

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Anatoly R. Gaiduk[1], Nadežda M. Stojković[2]

[1] Southern Federal University, Department of Control Systems, Taganrog, Russia
[2] University of Niš, Faculty of Electronic Engineering, Department of General Education, Republic of Serbia

Abstract. *Implementation conditions of a transfer functions by the system with partially given structure are considered. It is supposed, that all coefficients (roots) of the denominator and partial coefficients of the numerators of these transfer functions can be appointed arbitrary, according to the desirable performance of the control system. The design problem of the control system has a mathematical solution and the controller can be implemented. The stated implementation conditions are necessary for the design of the control systems models which can be implemented precisely*.

Key words: *transfer function, control system, structure, plant, controller, performance, and implementation*.

1. INTRODUCTION

Control systems usually include some unit in which a controlled process proceeds. This unit is provided with gauges, control actuators and servo mechanisms. All these elements are usually called a plant and are described by a mathematical model. The design problem of a closed system will consist in the definition of structure and parameters of an automatic regulator (the control device or a controller) for this plant [1, 2 and 3]. As the plant is given, this design problem refers to the design problem of the closed control system with a partially given structure.

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Corresponding author: Anatoly R. Gaiduk
Department of Control Systems, Nekrasovskiy 44, 347928 Taganrog, Russia
E-mail: gaiduk_2003@mail.ru
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If the system is designed using feedback on the state variables, the implementation problem does not occur and the design problem is solved analytically [2 – 4]. It is difficult to provide the desirable performance of a control system such as astatic order, overshoot, settling time and the decrement at this approach was marked by Filimonov [5].

The desirable performance is determined by coefficients of the system transfer functions directly. Therefore the methods of control systems design on the basis of desirable transfer functions are preferable [2, 6 – 9]. The difficulties of this approach are caused by the fact that the desirable transfer functions should be formed under some restrictions. The design problem would have the mathematical solution; the control system would have a desirable performance and the control would be implemented.

Conditions on transfer functions of the system with the given plant under which the shown restrictions can be fulfilled refer to as "*implementation conditions of the transfer functions by a system with partially given structure*".

It will be shown below that the known implementation conditions of transfer functions by a system with partially given structure [2, p. 285] are not sufficient. In some cases under these conditions the design problem can not have the mathematical solution or the found control cannot be implemented.

Proposed below are the necessary and sufficient conditions of implementation of transfer functions by a system with a partially given structure that includes an additional condition on the order of the closed system. The problem of control system design always has the analytical solution under these conditions. Structure and parameters of the controller are determined by the solution of some linear equations system or polynomial equations.

The proposed conditions are focused on control systems with one reference input, one controlled output and, probably, with disturbances both measured and not measured within a plant These conditions are deduced from the analytical expressions including polynoms (operators) of the "input-output" equations of the closed system, plant and controller. Also it is supposed that the transfer functions are implemented by the system with control on "output and impact" suggested in [6].

The purpose of this article is to show restrictions, which are necessary to be taken into account in the process of the formation of transfer functions of the designed control system. These restrictions are caused by the plant, controller and a the possibility to appoint all coefficients of a denominator and partial coefficients of numerators of the transfer functions according to the desirable performance of system.

The paper is organized as follows. In section 2 the mathematical model of control system with partially given structure and also statement of the system design problem are presented. The implementation conditions of transfer functions by the system with partially given structure are discussed in Section 3. The concluding remarks are given in the last Section.

2. Statement of Implementation Problem

Let a plant described by the operational "input-output" equation

$$A(p)y = B_0(p)u + B_1(p)\widehat{f}_1 + B_2(p)\breve{f}_2, \tag{1}$$

where y is the controlled variable, u is control input, $\widehat{f}_1$, $\breve{f}_2$ are measured and unmeasured disturbances, respectively; $A(p)$, $B_j(p)$ are polynomials of degrees n, m_j, $j=0,1,2$ with known numerical coefficients. Here $A(p)$ is the characteristic polynomial of this plant, normalized on the highest degree p [1, 2, 6]. Note that the roots of polynomials $A(p)$ and $B_j(p)$ are referred as poles and zero of the plant (1).

The relative degrees of the transfer functions $W_{yu}(p)$, $W_{y f_1}(p)$, $W_{y f_2}(p)$ of the plant (1) are determined [2] by expressions:

$$\mu_{yu} = n - m_0\,,\quad \mu_{y f_1} = n - m_1,\quad \mu_{y f_2} = n - m_2\,, \tag{2}$$

and the relative order μ_{pl} of this plant is determined by equality

$$\mu_{pl} = \mu_{yu}\,. \tag{3}$$

Let's note the relative order of a plant equals the minimal order of the time derivative a plant output variable, which directly depends on control. For example, consider the transfer function $W_{yu}(p) = (\beta_0 + \beta_1 p)/(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + p^3)$, where $\beta_1 \neq 0$, $n=3$, $m_0 = 1$. In this case, according to (2) and (3) $\mu_{pl} = \mu_{yu} = 3-1 = 2$. On the other hand, it is well-known that the plant having this transfer function is described by the equations: $\dot{x}_1 = x_2$; $\dot{x}_2 = x_3$; $\dot{x}_3 = -\alpha_0 x_1 - \alpha_1 x_2 - \alpha_2 x_3 + u$; $y = \beta_0 x_1 + \beta_1 x_2$, where x_i is a state variable, $i = \overline{1, n}$ where $n=3$, $\dot{x} = dx/dt$. Here the first derivative of variable $y(t)$ is $\dot{y} = \beta_0 x_2 + \beta_1 x_3$ and the second derivative $\ddot{y} = -\alpha_0\beta_1 x_1 - \alpha_1\beta_1 x_2 + (\beta_0 - \alpha_0\beta_1)x_3 + \beta_1 u$. Evidently, $y(t)$ and $\dot{y}(t)$ do not depend on control $u(t)$ directly, but the second derivative $\ddot{y}(t)$ depends, hence $\mu_{pl} = \mu_{yu} = 2$.

Without the loss of generality, we will accept that the plant (1) is full, that means it is completely controlled and completely observable [3]. In view of the condition on the polynomial $A(p)$ accepted above, this assumption is equivalent to the condition

$$\mathrm{GCD}\{A(p), B_0(p)\} = Const\,, \tag{4}$$

or in the words the polynomials $A(p)$ and $B_0(p)$ have no equal roots [6]. Here GCD is the greatest common divider.

The controller equation is defined by the accepted principle of control. In this case according to the principle of "control on output, reference and disturbances (shortly, *control on output and impact*)" [6, 9], we will accept controller equation as

$$\bar{R}(p)u = Q_0(p)g - L(p)y + Q_1(p)\widehat{f}_1\,, \tag{5}$$

where g is the reference input of the closed system [2]; $\bar{R}(p) = R(p) + N(p)$ and $R(p)$, $N(p)$, $L(p)$, $Q_0(p)$, $Q_1(p)$ are some polynomials. Their degrees and coefficients should be determined during the design of the closed system (1), (5). Here $\bar{R}(p)$ is characteristic polynomial of controller (5), that is controller order $r = \deg \bar{R}(p)$.

Relative degrees of the controller transfer functions are determined by expressions: $\mu_{ug} = r - \deg Q_0(p)$, $\mu_{uy} = r - \deg L(p)$, $\mu_{u f_1} = r - \deg Q_1(p)$ and their relative order

$$\mu_{cu} = \min\{\mu_{ug}, \mu_{uy}, \mu_{u \hat{f}_1}\}\,. \tag{6}$$

Let's note that all measured signals are used in the equation (5). And this controller should be implemented as a uniform block with several inputs and with one output [6, 9]. Also if $L(p) \equiv Q_0(p)$, the equation (5) corresponds to the traditional principle of control on error $\varepsilon = g - y$ and disturbance $\widehat{f}_1$ [1, 8].

The controller (5) can be successfully implemented if

$$\mu_{cu} \geq \mu^*_{cu} \geq 0\,, \tag{7}$$

where μ^*_{cu} is the supposed value of the controller relative order [3, 6, 9].

Practically, value μ^*_{cu} depends on properties of those technical elements on the basis of which the controller device will be implemented. For example, if the used operational amplifiers are broadband, it is possible to take $\mu^*_{cu} = 0$. Otherwise, it is necessary to take $\mu^*_{cu} \geq 1$. It is caused by that if $\mu^*_{cu} = 0$, then direct channel is formed between the inputs and output of the controller. When $\mu^*_{cu} \geq 1$, such channel is not formed.

The closed system "input-output" equation following from expressions (1) and (5) is

$$D(p)y = H_0(p)g + H_1(p)\widehat{f}_1 + H_2(p)\breve{f}_2\,. \tag{8}$$

Operators of this equation are determined by the expressions:

$$D(p) = A(p)\bar{R}(p) + B_0(p)L(p)\,, \tag{9}$$

$$H_0(p) = B_0(p)Q_0(p)\,, \tag{10}$$

$$H_1(p) = B_0(p)Q_1(p) + B_1(p)\bar{R}(p)\,, \tag{11}$$

$$H_2(p) = B_2(p)\bar{R}(p)\,. \tag{12}$$

From (8) and (9) the order of the closed system (1), (5) it follows that $n_{\text{sys}} = \deg D(p) = n + r$. The relative degree μ_{yg} of the transfer function $W_{yg}(p) = H_0(p)/D(p)$ of the system (8) and its relative order μ_{sys} are defined (when $\mu_{cu} = \mu_{uy}$) by expression $\mu_{\text{sys}} = \mu_{yg} = \deg D(p) - \deg H_0(p) = n + r - [\deg B_0(p) + \deg Q_0(p)]$ or

$$\mu_{sys} = \mu_{\text{pl}} + \mu_{cu}\,. \tag{13}$$

In particular, if $\mu_{cu} = 0$ from (13) we have $\mu_{sys} = \mu_{pl}$ that is typical for systems with $u = k_c\varepsilon$ or $u = k_1x_1 + k_2x_2 + \ldots + k_nx_n$.

The expressions (9)–(12) are equations which connect polynomials from the "input-output" equations of the closed system, plant and the controller. After replacement $D(p)$ on $D^*(p)$ and $H_j(p)$ on $H^*_j(p)$, $j = 0,1,2$ the expressions (9)–(12) become *resolving equations* of the analytical design problem of the control system with a partially given structure. In these equations the polynomials $A(p)$, $B_j(p)$ are known as the plant is given. Also, polynomials $D^*(p)$, $H^*_j(p)$, $j = 0,1,2$, can be formed according to the requirements of stability and the desirable performance of the designed system. Polynomials from the equation (5) are unknown in these equations. It is very important that all the polynomial equations (9)–(12) are equivalent to the systems of linear algebraic equations with relatively unknown coefficients of polynomials $\bar{R}(p)$, $L(p)$, $Q_0(p)$ and $Q_1(p)$ [6, 9].

Thus, for the design of a control system, first of all it is necessary to form polynomials $D^*(p)$ and $H^*_j(p)$, $j = 0,1,2$ which should satisfy some implementation conditions of transfer functions by the system with a partially given structure. These conditions should provide:

i) all coefficients (roots) of the polynomial $D^*(p)$ and partially of the polynomials $H^*_j(p)$, $j = 0,1,2$ can be appointed according to the desirable performance of the control system (1), (5);

ii) the control system (1), (5) or (8) has a desirable performance, if $D(p) = D^*(p)$ and $H_j(p) = H^*_j(p)$, $j = 0,1,2$;

iii) the equations system (9) – (12) has the mathematical solution relative to polynomials $\bar{R}(p)$, $L(p)$, $Q_0(p)$, $Q_1(p)$;

iv) the polynomials $\bar{R}(p)$, $L(p)$, $Q_0(p)$, $Q_1(p)$ satisfy the conditions (6), (7) which are the implementation conditions of the controller (5).

3. Solution of the Task

The mentioned above implementation conditions of transfer functions essentially depend on how the roots of a characteristic polynomial (system's poles) are appointed. If the system poles are appointed without taking into account the plant poles or zeros, the system is referred to as "a *system with independent poles*". In this case the implementation conditions of transfer functions are rather rigid. If poles are appointed so that a part of them is equal to zero or poles of the plant, the system is referred to as "*system with coordinated poles*". In this case implementation conditions of a transfer functions on referent input are the least rigid. Therefore, following [1, 2], we will consider only the closed systems with coordinated poles.

For a system stability it is necessary that all roots of the polynomial (9) have a negative real part. Therefore, the factorization of the polynomials $A(p)$ $A(p)$ and $B_0(p)$ is carried out as follows:

$$A(p) = A^-(p)A^+(p), \quad B_0(p) = \beta_{m_0} B^-(p)B^+(p), \tag{14}$$

where $A^-(p), A^+(p)$ and $B^-(p), B^+(p)$ are the polynomials normalized on the highest degree p; β_{m_0} is coefficient of the polynomial $B_0(p)$ at the highest degree p. Here $A^-(p), B^-(p)$ are the polynomials whose roots are equal to roots of the polynomials $A(p)$ and $B_0(p)$ with strictly negative real parts (located in strictly left half-plane). Thus it is supposed that all roots of the polynomials $A^-(p)$ and $B^-(p)$ are included in the characteristic polynomial of the closed system. Generally, each of the polynomials $A^-(p)$, $A^+(p)$ and $B^-(p)$, $B^+(p)$ can be equal to 1. The choice of the roots of the polynomials $A(p)$ or (and) $B_0(p)$ which are included in the characteristic polynomial of a system, should be done in the view of the designed system properties.

For systems with coordinated poles the polynomials from (5) should have the following form:

$$\bar{R}(p) = B^-(p)\tilde{R}(p), \quad L(p) = A^-(p)\tilde{L}(p), \quad Q_0(p) = A^-(p)M^-(p)\tilde{Q}_0(p), \tag{15}$$

where $\tilde{R}(p)$, $\tilde{L}(p)$, $\tilde{Q}_j(p)$, $M^-(p)$ are polynomials which are defined during the design of a control system [6, 9]. From expressions (9), (10) and (15) it follows that polynomials

$H_0(p)$ and $D(p)$ of the system (1), (5) or (8), by which the transfer function on reference $W^*_{yg}(p) = H^*_0(p) / D^*(p)$ is implemented, look like

$$H_0(p) = B^-(p)A^-(p)H_0^*(p)M^-(p)\,, \tag{16}$$

$$D(p) = A^-(p)B^-(p)D^*(p)M^-(p)\,. \tag{17}$$

The polynomials $\tilde{R}(p)$, $\tilde{L}(p)$, $\tilde{Q}_0(p)$ in equalities (15) are determined by the solution of linear algebraic equations, following from the equations (9), (10) in the view of equality (14)–(17) [6, 9]. The examples of these systems are given below. The polynomial $M^-(p)$ in (16), (17) is necessary to increase the order of system (1), (5) up to necessary value in some cases. The designed system (1), (5) includes not controlled and not observable subsystem with polynomials $A^-(p)$, $B^-(p)$ and $M^-(p)$. These polynomials influence the properties of the system only in transients if initial conditions are non zero.

3.1. Implementation conditions of transfer function on reference

According to [2] the transfer function $W^*_{yg}(p) = H^*_0(p)/D^*(p)$ is implemented by the system with a partially given structure if the following conditions are satisfied

$$\mu^*_{yg} = \deg D^*(p) - \deg H_0^*(p) \ge \mu_{pl} + \mu^*_{cu}\,, \quad H_0^*(p) = B^+(p)\bar{H}_0^*(p)\,. \tag{18}$$

As it will be shown below, these conditions are necessary but insufficient for conditions i) – iv) to be fulfilled. Usually, the coefficients (roots) of the polynomials $D^*(p)$ and $H_0^*(p)$ are appointed arbitrarily, according to the desirable performance of the designed system. Therefore in some cases the equations (9) and (10) have no solution relative to polynomials $\bar{R}(p)$, $L(p)$, $Q_0(p)$ or these polynomials do not satisfy the conditions (6), (7), though conditions (18) are fulfilled.

All conditions i) – iv) can be fulfilled, if in addition to the conditions (18) the polynomial $D^*(p)$ satisfies [6, 9] the inequality

$$\deg D^*(p) \ge n - \deg B^-(p) + \mu^*_{cu} \tag{19}$$

and the degree of the polynomial $M^-(p)$ from (15) – (16) is determined by the expression

$$\deg M^-(p) = \max\{0;\ 2n - 1 + \mu^*_{cu} - \deg[A^-(p)B^-(p)D^*(p)]\}\,. \tag{20}$$

The expressions (18)–(20) represent the necessary and sufficient implementation conditions of a transfer function $W^*_{yg}(p)$ by a system with a partially given structure. Practically, the expression (20) is equivalent to the inequality $r \ge n - 1 + \mu^*_{cu}$ that is the restriction on the order of the control device (5). This inequality, evidently, corresponds to the known W.R. Ashby principle of "a necessary diversity".

Let's show the importance of the conditions (18)–(20) on some examples. Preliminary notice is that if $\deg D^*(p)$ is equal to the minimally possible value $n - \deg B^- + \mu^*_{cu}$ (19), the numerator of $W^*_{yg}(p)$ is equal to $\eta^* B^+(p)$, where η^* is the unique appointed coefficient. Hence, to increase the number of appointed coefficients of the polynomial $H_0^*(p)$, it is necessary to increase the degree of the polynomial $D^*(p)$ over the value $n - \deg B^- + \mu^*_{cu}$.

3.2. Necessity of the first condition (18)

Consider the transfer function $W^*_{yg}(p) = (\eta^*_0 + \eta^*_1 p + \eta^*_2 p^2)/(\delta^*_0 + \delta^*_1 p + \delta^*_2 p^2)$, where the coefficients η^*_i, δ^*_i, $i = 0,\ 1,\ 2$ have the arbitrary values caused by the desirable performance of the designed control system. It is necessary to implement this function using (1), (5), where polynomials $B_0(p) = \beta_0$, $A(p) = \alpha_0 + \alpha_1 p$ and $\beta_0 \neq 0$, $\alpha_1 \neq 0$. The controller (5) is implemented at $\mu^*_{cu} = 0$. In this case the plant has $\mu_{pl} = 1$ and the given function $W^*_{yg}(p)$ has $\mu^*_{yg} = 0$, that means the first condition (18) is not fulfilled. If we do not take this fact into account, and we substitute the polynomials $A(p)$, $B_0(p)$ and $D(p) = D^*(p) = \delta^*_0 + \delta^*_1 p + \delta^*_2 p^2$ in (9), it is easy to establish [6, 9] that with $\bar{R}(p) = \rho_0 + \rho_1 p$ and $L(p) = \lambda_0$ this equation is equivalent to the system

$$\begin{bmatrix} \beta_0 & \alpha_0 & 0 \\ 0 & \alpha_1 & \alpha_0 \\ 0 & 0 & \alpha_1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \rho_0 \\ \rho_1 \end{bmatrix} = \begin{bmatrix} \delta^*_0 \\ \delta^*_1 \\ \delta^*_2 \end{bmatrix}.$$

The solution of this system determines values of the coefficients λ_0, ρ_0 and ρ_1. Similarly, the polynomials $H_0(p) = H^*_0(p) = \eta^*_0 + \eta^*_1 p + \eta^*_2 p^2$ and $B_0(p) = \beta_0$ are substituted in the equation (10). Its solution gives $Q_0(p) = \beta_0^{-1}(\eta^*_0 + \eta^*_1 p + \eta^*_2 p^2)$. Hence, in this case $\mu_{uy} = 1 - 0 = 1$ and $\mu_{ug} = 1 - 2 = -1$, that is according to (6) $\mu_{cu} = \min\{1;\ -1\} = -1$. Evidently, the condition (7) is not fulfilled and the corresponding controller (5) cannot be implemented precisely.

3.3. Necessity of the second condition (18)

This condition follows from resolvability of the equation (10) concerning polynomial $Q_0(p)$ and stability of the closed system (1), (5). Really, the polynomial $B^+(p)$ cannot be a multiplier of the polynomial $D(p)$ on the stability conditions of this system. Therefore polynomial $D(p)$ can be as (17) and we have from expressions (8), (10), (14)

$$W_{yg}(p) = \frac{H_0(p)}{D(p)} = \frac{\beta_{m_0} B^+(p) B^-(p) Q_0(p)}{A^-(p) B^-(p) D^*(p) M^-(p)}.$$

Setting here $Q_0(p) = \tilde{Q}_0(p) A^-(p) M^-(p)$, we will obtain

$$W_{yg}(p) = \frac{H_0(p)}{D(p)} = \frac{\beta_{m_0} B^+(p) \tilde{Q}_0(p)}{D^*(p)}. \tag{21}$$

General view of the operator (16) and the necessity of the second condition (18) follow from here with $\tilde{Q}_0(p) = \beta_{m_0}^{-1} \bar{H}^*_0(p)$; otherwise the polynomial equation (10) with $H_0(p) = H^*_0(p)$ has no solution.

Note, that the right part of the expression (21) with $\tilde{Q}_0(p) = \beta_{m_0}^{-1} \bar{H}^*_0(p)$ is a transfer function which can be implemented by the system with a partially given structure and with coordinated poles.

3.4. Necessity of the condition (20)

Let the full plant be described by the equation (1) where polynomials $A(p) = (p+4)(p^2-2p)$ and $B_0(p) = 4(2+p)(1.5-p)$, that is $n=3$, $m_0=2$. Also, let $A^-(p) = p+4$, $A^+(p) = p^2-2p$, $B^-(p) = p+2$, $B^+(p) = p-1.5$ and $\mu^*_{cu}=0$. Then, $\deg A^-(p)=1$, $\deg A^+(p)=2$, $\beta_2=-4$, $\deg B^-(p)=1$, $\deg B^+(p)=1$, $\mu_{pl}=1$. The given transfer function

$$W^*_{yg}(p) = 2(-p+1.5)/(p^2+\delta^*_1 p+\delta^*_2).$$

Hence in this case $H^*_0(p) = 2(-p+1.5)$, $D^*(p) = p^2+\delta^*_1 p+\delta^*_2$, $\deg D^*(p)=2$, $\mu^*_{yg}=1$, $\bar{H}^*_0(p)=-2$. Both conditions (18) are evidently fulfilled. The coefficients δ^*_1, δ^*_2 have arbitrary values caused by desirable performance of the designed control system.

If ignoring the condition (20) to put a polynomial $M^-(p)=1$ in (15)–(17) we will obtain $D(p) = (p+2)(p+4)(p^2+\delta^*_1 p+\delta^*_0)$, that is $n_{\text{sys}} = \deg D(p) = 4$. As $r = n_{\text{sys}} - n$ we have $r = 4-3 = 1$, and from (15) we find: $\deg\tilde{R}(p) = \deg\tilde{L}(p) = 0$, $\bar{R}(p) = \rho_0(p+2)$, $L(p) = \lambda_0(p+4)$. The equation (9) with the resulted polynomials after simplification looks like this: $(p^2-2p)\rho_0 + (-4p+6)\lambda_0 = p^2+\delta^*_1 p+\delta^*_0$. To this polynomial equation the following algebraic system corresponds:

$$\begin{bmatrix} 6 & 0 \\ -4 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \rho_0 \end{bmatrix} = \begin{bmatrix} \delta^*_0 \\ \delta^*_1 \\ 1 \end{bmatrix}. \tag{22}$$

The system (22) evidently does not have solution for all values of the coefficients δ^*_0, δ^*_1. On the other hand, if $\delta^*_1 = -2(\delta^*_0+3)/3$, the system (22) formally has a solution, but it is impossible to make the closed system stable.

Generally, the design task of a low order controller certainly can be obtained [10, 11]. This task consists in finding such solution of a system similar to (22) at which properties of the closed system are comprehensible. However, the solution of such problem exists in very rare cases.

In our case the previous polynomial equation has a solution with arbitrary values of the coefficients δ^*_0, δ^*_1, if $\bar{R}(p) = \rho_0(p+2)$ and $L(p) = (\lambda_1 p+\lambda_0)(p+4)$. The algebraic system equivalent to the equation $(p^2-2p)\rho_0 + (-4p+6)(\lambda_1 p+\lambda_0) = p^2+\delta^*_1 p+\delta^*_0$ will be written down as follows:

$$\begin{bmatrix} 6 & 0 & 0 \\ -4 & 6 & -2 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \rho_0 \end{bmatrix} = \begin{bmatrix} \delta^*_0 \\ \delta^*_1 \\ 1 \end{bmatrix},$$

and it has a solution with arbitrary values of the coefficients δ^*_0, δ^*_1. But the corresponding controller (5) has $\mu_{cu} = \mu_{uy} = \deg\bar{R}(p) - \deg L(p) = -1$ and cannot be implemented precisely.

Practically, the condition (20) allows for the fulfillment of the conditions iiii) and iv) with arbitrary roots (or coefficients) of the polynomial $D^*(p)$. Let us assume , that in the example considered above, the desirable transfer function is described with $W_{yg}^*(p)=2(-p+1.5)/(p^2+5p+3)$. Now with the account condition (20) we have - $\deg M^-(p)=1$. Let - $M^-(p)=p+1$, then from (17) $n_{\text{sys}}=\deg D(p)=5$ and $r=5-3=2$. From (15), since $\mu_{cu}^*=0$, it follows that $\deg\tilde{R}(p)=1$, $\deg\tilde{L}(p)=1$ or $\tilde{L}(p)=\lambda_1 p+\lambda_0$, $\tilde{R}(p)=\rho_1 p+\rho_0$ and the product $D^*(p)M^-(p)=p^3+6p^2+8p+3$. Now the system, equivalent to the polynomial equation (9), looks like the following:

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ -4 & 6 & -2 & 0 \\ 0 & -4 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \rho_0 \\ \rho_1 \end{bmatrix}=\begin{bmatrix} 3 \\ 8 \\ 6 \\ 1 \end{bmatrix},$$

The solution of this system is $\lambda_0=0.5$, $\lambda_1=-13$, $\rho_0=-44$, $\rho_1=1$ and polynomials $\bar{R}(p)=(p+2)(p-44)$, $L(p)=(p+4)(-13p+0.5)$. From (10), (16), (14) and the equality (18) we find $Q_0(p)=\beta_{m_0}^{-1}A^-(p)M^-(p)\bar{H}_0^*(p)$, therefore $Q_0(p)=0.5(p+4)(p+1)$. Hence, the controller according to (5) is described by the following equation:

$$(p^2-42p-88)u=(0,5p^2+2.5p+2)g-(-13p^2-51.5p+2)y.$$

The equations of this controller in state space are:

$$\dot{z}=\begin{bmatrix} 0 & 88 \\ 1 & 42 \end{bmatrix}z+\begin{bmatrix} 46 \\ 23.5 \end{bmatrix}g+\begin{bmatrix} 1142 \\ 597.5 \end{bmatrix}y,\ \ u=\begin{bmatrix} 0 & 1 \end{bmatrix}z+0.5g+13y. \tag{23}$$

where $z=[z_1 \quad z_2]^T$ is the state vector.

The closed system with the given plant and the controller (23) has transfer function

$$W_{yg}(p)=\frac{2(p+2)(p+4)(1.5-p)(p+1)}{(p+2)(p+4)(p^2+5p+3)(p+1)}=\frac{2(1.5-p)}{p^2+5p+3}=W_{yg}^*(p).$$

This example shows that the expressions (18) – (20) are the implementation conditions of transfer functions on the reference by the system with partially given structure. The transfer function $W_{unf}(p)=(p+2)(p+4)(p+1)/(p+2)(p+4)(p+1)$ represents here an incomplete subsystem. Evidently, this subsystem does not influence the properties of the system in the steady state mode.

3.5. Transfer functions on the disturbances are applied to plant

Note, generally it is possible to take the polynomials $\tilde{L}(p)$, $\tilde{Q}_0(p)$, $\tilde{R}(p)$ in (15) and $Q_1(p)$ in (5) as $\tilde{L}(p)-\tilde{Q}_0(p)=G(p)\ddot{L}(p)$, $\tilde{R}(p)=\Phi(p)\ddot{R}(p)$ and $Q_1(p)=F_1(p)\ddot{Q}_1(p)$, where $\Phi(p)=\text{LCM}\{G(p), F_1(p), F_2(p)\}$. Here the polynomials $G(p)$, $F_1(p)$ and $F_2(p)$ are $K(p)$ images of the reference signal $g(t)$, disturbances $\hat{f}_1(t)$ and $\breve{f}_2(t)$ accordingly; LCM is the least common multiplier. Note, if $\hat{f}_1(t)\equiv 0$ and (or) $\breve{f}_2(t)\equiv 0$ then $F_1(p)=1$

and (or) $F_2(p)=1$. Practically, $K(p)$ - an image of some function $f(t)$ is equal to the denominator of Laplace transformation of this function [9].

From the expressions (8)–(11) and (14)–(17), according to the entered designations it follows that the transfer function on the error from reference and the transfer function on the *measured* disturbance $\widehat{f}_1(t)$, which are implemented by the system (1), (5), are:

$$W_{\varepsilon g}(p)=\frac{G(p)\ddot{H}_0(p)}{D^*(p)}, \quad W_{y\hat{f}_1}(p)=\frac{\tilde{H}_1^*(p)F_1(p)}{A(p)D^*(p)M^-(p)},$$

where $\ddot{H}_0(p)=A^+(p)\Phi_0(p)\ddot{R}(p)+\beta_{m_0}B^+(p)\ddot{L}(p)$ and $\tilde{H}_1(p)=\beta_{m_0}B^+(p)\ddot{Q}_1(p)+B_1(p)\Phi_1(p)\ddot{R}(p)$ are polynomials whose coefficients can partly be appointed so that the closed system has desirable performances of channels $g\to y$ and $\widehat{f}_1\to y$. Here - $\Phi_0(p)=G^{-1}(p)\Phi(p)$, $\Phi_1(p)=F_1^{-1}(p)\Phi(p)$.

The transfer function on the *unmeasured* disturbance $\breve{f}_2(t)$, which is implemented by the system (1), (5) with coordinated poles, is

$$W_{y\hat{f}_2}(p)=\frac{\ddot{R}_2(p)B_2(p)F_2(p)}{A(p)D^*(p)M^-(p)},$$

where $\ddot{R}_2(p)=F_2^{-1}(p)\tilde{R}(p)$.

Note that the polynomials $G(p)$ and $F_1(p)$, $F_2(p)$ can be used for obtaining some astatic order to a reference $g(t)$ and disturbances $\widehat{f}_1(t)$, $\breve{f}_2(t)$ of the system [12]. Application of the *control principle on output and impact* excludes the known difficulties connected with the stability of a system having the high astatic order [6, 9].

To design systems with the desirable degree of stability, the factorization of polynomials $A(p)$ and $B_0(p)$ is necessary to be carried out with respect to some area Ω [6]. The area Ω is entirely in the left-half of the complex plane and poles of the designed system (1), (5) should be placed only in this area according to the desirable performance of system. The equalities (14) are replaced by equalities $A(p)=A_\Omega(p)A_{\bar{\Omega}}(p)$, $B_0(p)=\beta_{m_0}B_\Omega(p)B_{\bar{\Omega}}(p)$ and all subsequent expressions and conditions do not change, except for evident replacements of polynomials in this case. Here $A_\Omega(p)$ and $B_\Omega(p)$ are the normalized polynomials, whose roots are equal to those roots of the polynomials $A(p)$ and $B_0(p)$ located in area Ω and are included in poles of the designed system.

4. Conclusion

The implementation conditions of the transfer functions are proposed here. All coefficients (roots) of the denominator and partly of the numerator of the transfer functions can be appointed arbitrary under these conditions. Practically, these coefficients are appointed according to the desirable performance of a control system in the transitive and steady state mode. The submitted conditions include restrictions on the relative degree and zeros of the implemented transfer functions and on the order of the implementing system. They provide implementation of the transfer function by the closed system with partially given structure and the coordinated poles.

The conditions of implementation can be applied to form transfer functions of the designed control system with the given plant and desirable performance for which design problem has analytical solution.

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