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SOME PROPERTIES OF THE SET OF ALL STRONG UNIFORM CLUSTER POINTS

Serpil Pehlivan

Suleyman Demirel University, Faculty of Arts and Sciences
Department of Mathematics, 32260 Isparta, Turkey

Abstract. The aim of this paper is to establish the relationship between the set of strong uniform statistical cluster points and the set of strong statistical cluster points of a given sequence in the probabilistic normed space. To this aim, let the uniform density be on the positive integers $\mathbb{N}$ for a sequence in the probabilistic normed space, that is, cases as equal of the lower and upper uniform density of a subset of $\mathbb{N}$. We have introduced the concept of strong uniform statistical cluster points and given a new type convergence in the probabilistic normed space. Note that the set of strong uniform statistical cluster points is a non-empty compact set. We have also investigated some properties of the set all strong uniform cluster points of a sequence in the probabilistic normed space.
Keywords : lower and upper uniform density, strong uniform statistical convergence, strong uniform statistical cluster point, set of statistical cluster points, probabilistic normed space, $\mathcal{D}-$ bounded, statistically $\mathcal{D}-$ bounded sequence.

1. Introduction and Preliminaries

The theory of probabilistic normed spaces [15] originated from the concept of statistical metric spaces which was considered by Menger [11] and further studied by Schweizer and Sklar [14]. A probabilistic normed space is a generalization of an ordinary normed vector space and the norms of the vectors are represented by probability distribution functions, that is, it provides a method of generalizing

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Corresponding Author: Serpil Pehlivan, Suleyman Demirel University, Faculty of Arts and Sciences, Department of Mathematics, 32260 Isparta, Turkey | E-mail: serpilpehlivan@sdu.edu.tr
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the deterministic results of normed linear spaces. It has also very useful applications in optimal controls problems in discrete time [13, 20], convergence of random variables[7, 18], etc. In this paper we study the concept of strong uniform statistical cluster in a more general setting, that is, in the probabilistic normed space. We define the set of strong uniform statistical cluster points in the probabilistic normed space and prove some interesting results. We also define a new type convergence to a set for sequences in the probabilistic normed space and investigate some properties of the set of all strong uniform statistical cluster points of a sequence in a probabilistic normed space.

Šerstnev introduced the first definition of a probabilistic normed space in a series of papers [16]. A new definition of a probabilistic normed space was introduced in Alsina et al. [1]. It is also the definition that will be adopted in this paper. For more details, we refer to [2, 6, 8, 9, 19].

First, we recall some of the concepts related to the theory of probabilistic normed spaces.

A *distribution function* is a function F defined on the extended real line $\bar{\mathbb{R}} := [-\infty, +\infty]$ that is non decreasing, left - continuous on the set of real numbers $\mathbb{R}$ such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The set of all distribution functions is denoted by Δ. The elements of Δ are partially ordered via

$$F \leq G \text{ iff } \; F(t) \leq G(t) \text{ for all } t \in \bar{\mathbb{R}}.$$

For any a in $\bar{\mathbb{R}}$, ε_a, the *unit step at a,* is the function in Δ given by

$$\varepsilon_a(t) \quad := \quad \begin{cases} 0, & t \leq a \\ 1, & t > a \end{cases}$$

when $a \in \mathbb{R}$, while $\varepsilon_\infty(t) = 0$ for $t \in \mathbb{R}$ and $\varepsilon_\infty(+\infty) = 1$.

The Sibley [17] metric d_S was defined as follows if F and G in Δ and h is in $(0, 1)$, let $(F, G; h)$ denote the condition

$$F(x-h) - h \leq G(x) \leq F(x+h) + h \quad \text{ for all } \quad x \in (-\frac{1}{h}, \frac{1}{h}).$$

Then the Sibley metric was defined by

$$d_S(F, G) := \inf \{h \in (0,1) : \text{ both } \quad (F, G; h) \text{ and } \quad (G, F; h) \text{ hold}\}.$$

A *distance distribution function* is a non decreasing function F defined on $[0, +\infty]$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0, \infty)$. The set of all distance distribution functions is denoted by Δ^+. The set

$$\mathcal{D}^+ := \{F \in \Delta^+ : \ell^- F(+\infty) = 1$$

is a subset of Δ^+. Here $\ell^- F(+\infty)$ represents the left limit at $+\infty$.

A *triangle function* $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ is a binary operation that is associative, commutative, non decreasing in each place, and which has ε_0 as identity.

Definition 1.1. [1, 9] A *probabilistic normed space (for short, PNS)*, is a quadruple (E, ν, τ, τ^*) where E is a linear space, τ and τ^* are continuous triangle functions, and the mapping $\nu : E \to \Delta^+$ satisfies, for all p and q in E, the conditions

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$, (θ is the null vector in E);

(N2) $\forall p \in E,\ \nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau\left(\nu_p, \nu_q\right)$;

(N4) $\forall \alpha \in [0, 1]$ $\nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$.

The function ν is called the *probabilistic normed.* A probabilistic metric $\mathcal{F} : E \times E \to \Delta^+$ is defined via $\mathcal{F}(p, q) = F_{pq} = \nu_{p-q}$. Let (E, ν, τ, τ^*) be a PNS. For any $p \in E$ and $t > 0$, the set

$$N_p(t) := \{q \in E : \nu_{p-q}(t) > 1 - t\} = \{q \in E : d_S\left(F_{pq}, \varepsilon_0\right) < t\}$$

is called the *strong $t-$neighborhood* of p. A sequence $(q_k)_{k \in \mathbb{N}}$ in E is said to be strongly convergent to a point $q \in E$, and we write $q_k \to q$, if for each $t > 0$, there exists a positive integer m such that $q_k \in N_q(t)$ for $k \geq m$. Moreover, since the triangle function τ is continuous,the system of neighborhoods

$$\mathcal{N} := \bigcup_{q \in E} \bigcup_{t > 0} \{N_q(t)\ :\ q \in E\},$$

is called the *strong neighborhood system.* This determines a Hausdorff and first countable topology on E, called the *strong topology.*

Now for any $q \in E$ and any non-empty subset B of E, let $F_{q,B}$ be the function defined on the extended real line $\bar{\mathbb{R}}$ by

$$F_{q,B}(x) := \sup\left\{\nu_{q-p}(x) : p \in B\right\}.$$

Then $F_{q,B}$ is a distance distribution function, and it is called the *probabilistic distance from the point q to the set B*. Moreover, for any $t > 0$, the open set

$$N_B(t) := \{q \in E : F_{q,B}(t) > 1 - t\}$$

is the *strong $t-$neighborhood* of B . For the following definitions we refer to Lafuerza-Guillén and Harikrishnan [9].

Definition 1.2. Let (E, ν, τ, τ^*) be a PNS. The *probabilistic radius R_A* of a nonempty set A in E is defined by

$$R_A = \begin{cases} \ell^- \varphi_A(t), & t \in [0, +\infty) \\ 1, & t = +\infty \end{cases}$$

where $\varphi_A(t) = \inf\ \left\{\nu_p(t) : p \in A\right\}$.

Clearly, the probabilistic radius R_A of every non-empty set A in a PNS is equal to the probabilistic radius of its closure $\overline{A}$. Different kinds of bounded sets may be introduced in a PNS. Next definition presents different concepts of boundedness in a PNS.

Definition 1.3. A non-empty set A in (E, ν, τ, τ^*) is said to be:

(i) *certainly bounded*, if $R_A(t_0) = 1$ for some $t_0 \in (0, +\infty)$;

(ii) *perhaps bounded*, if one has $R_A(t) < 1$ for every $t \in (0, +\infty)$, but

$$\lim_{t \to +\infty} R_A(t) = 1;$$

(iii) *perhaps unbounded*, if $R_A(t_0) > 0$ for some $t_0 \in (0, +\infty)$ and

$$\lim_{t \to +\infty} R_A(t) \in (0, 1);$$

(iv) *certainly unbounded*, if

$$\lim_{t \to +\infty} R_A(t) = 0, \text{ i.e.} R_A = \varepsilon_\infty.$$

Moreover, the set A is said to be *distributionally bounded* (for short, $\mathcal{D}-$*bounded*) if either (i) or (ii) holds, i.e., if $R_A \in \mathcal{D}^+$; otherwise, i.e., if R_A belongs to $\Delta^+ \diagdown \mathcal{D}^+$ then A is said to be $\mathcal{D}-$*unbounded*. Let (E, ν, τ, τ^*) be a PNS. A non-empty set A in E is $\mathcal{D}-$ bounded if, and only if, there exists a distribution function $H \in \mathcal{D}^+$ such that $\nu_p \geq H$ for every $p \in A$.

Closely related with the concept of $\mathcal{D}-$boundedness in a PNS is the concept of compactness. A compact set in a PNS is always closed since the strong topology is Hausdorff, but a compact set needs not be $\mathcal{D}-$bounded.

Now we recall some basic concepts related to uniform density and uniform statistical convergence.

Let $C \subseteq \mathbb{N}$; $n, j \in \mathbb{Z}$ and $n \geq 0$, $j \geq 1$. Let $C(n+1, n+j)$ denote the cardinality of the set $C \cap [n+1, n+j]$. Then the numbers

$$\underline{u}(C) := \lim_{j \to \infty} j^{-1} \min_{n \geq 0} C(n+1, n+j)$$

and

$$\overline{u}(C) := \lim_{j \to \infty} j^{-1} {}_{\max_{n \geq 0}} C(n+1, n+j)$$

are called *lower uniform density* and *upper uniform density* of C, respectively. If $\underline{u}(C) = \overline{u}(C) = u(C)$, then $u(C)$ is called *uniform density* or *Banach density* of a set $C \subseteq \mathbb{N}$ [3]. In the special case n = 0 in the definitions of lower and upper uniform densities, one has the lower and upper asymptotic density of a set C.

Given a set $C \subset \mathbb{N}$, its *lower* and *upper asymptotic (or natural) densities*, denoted by $\underline{d}(C)$ and $\overline{d}(C)$ respectively satisfy the relation

$$0 \leq \underline{u}(C) \leq \underline{d}(C) \leq \overline{d}(C) \leq \overline{u}(C) \leq 1.$$

The principal characteristic of the uniform density is that it is more sensitive to local density in any interval, not necessarily initial, than the asymptotic density. For instance the set $C = \bigcup_{j\geq 1}\{j!+1, j!+2, \ldots, j!+j\}$ having rare but sufficiently long blocks of consecutive integers has asymptotic density zero while its upper uniform density $\overline{u}(C) = 1$ and so its uniform density does not exist [5].

ξ is called a uniform statistical cluster point of $x = (\xi_j)$ if for every $\varepsilon > 0$ the set $\{j \in \mathbb{N} \ : |\xi_j - \xi| < \varepsilon\}$ does not have uniform density 0 i.e. $\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0 \leq i \leq j \,:\, |\xi_{i+n} - \xi| < \varepsilon\}| > 0$. Let Γ_x^u denotes the set of all uniform statistical cluster points of x and Γ_x denotes the set of all statistical cluster points. If we put $i = 0$ in the definition of the set of uniform statistical cluster points then we get the set of all statistical cluster points. It means that $\Gamma_x \subseteq \Gamma_x^u$. It is clear that $\Gamma_x \subsetneq \Gamma_x^u$, for instance if $x = \{0, 2, 0, 0, 2, 2, 0, 0, 0, 0, 2, 2, 2, , 0, 0, 0, 0, 0, 0, 0, 0, 2, \ldots, 0, 0, 2, 2, 2, \ldots\}$ where segments of 0 of length $2^j, j = 0, 1, 2\ldots$ and 2 of length $j+1$, $j = 0, 1, \ldots$, alternated. Then we have, $\Gamma_x = \{0\}$ and $\Gamma_x^u = \{0, 2\}$.

The aim of this note is present and comment two equivalent definitions of the so-called strong uniform statistical convergence and strong statistical convergence in a PNS. We define SUSC points of the sequence $q = (q_k)$ in a PNS. Finally, we list some of the basic concepts related to strong uniform statistical convergence and strong uniform statistical cluster (for short, *SUSC*) point. We define *SUSC* points of the sequence $q = (q_k)$ in a PNS. We also study some properties of the set *SUSC* points of the sequence $q = (q_k)$ in a PNS.

We give some notations and basic definitions used in this paper.

Definition 1.4. Let (E, ν, τ, τ^*) be a PNS. A sequence $q = (q_k)$ in E is said to be strongly uniformly statistically convergent to a point $\gamma \in E$ provided that

$$\lim_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0 \leq i \leq j \,:\, q_{i+n} \notin N_\gamma(t)\}| = 0$$

or

$$u(\{i \in \mathbb{N} \,:\, q_{i+n} \notin N_\gamma(t)\ \}) = 0$$

for each $t > 0$. Then γ is called the strong uniform statistical limit of a sequence $q = (q_k)$ in a PNS.

Definition 1.5. Let $q = (q_k)$ be a sequence in (E, ν, τ, τ^*). Then an element $\beta \in E$ is a *SUSC* point of $q = (q_k)$ provided that

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0 \leq i \leq j \,:\, q_{i+n} \in N_\beta(t)\}| > 0$$

for each $t > 0$. By $\Gamma_s^u(q)$ we denote the set of all *SUSC* points of the sequence $q = (q_k)$. In the special case n = 1 of the above definition, one has the strong statistical cluster *(for short, SSC)* point of $q = (q_k)$ in a PNS [12]. We denote the set of all *SSC* points of $q = (q_k)$ by $\Gamma_s(q)$.

Note that every *SSC* point is also *SUSC* point. So we have $\Gamma_s(q) \subset \Gamma_s^u(q) \subset L_s(q)$ in a PNS.

Finally, we list some of the basic concepts related to statistical $\mathcal{D}-$boundedness.

Definition 1.6. Let (E, ν, τ, τ^*) be a PNS. A sequence $q = (q_k)$ in E is said to be *statistically* $\mathcal{D}$-bounded provided that there exists a set $K = \{k_j : k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and (q_{k_j}) is $\mathcal{D}$-bounded, that is, $R_A \in \mathcal{D}^+$ where $A = \{q_{k_j} : j \in \mathbb{N}\}$.

Note that a sequence (q_k) in E is statistically $\mathcal{D}$-bounded iff almost all terms of (q_k) yields a $\mathcal{D}$-bounded subset of E. Moreover, a $\mathcal{D}$-bounded sequence is always statistically $\mathcal{D}$-bounded, but the converse may not hold in general [18].

Example 1.1. Let $(\mathbb{R}, \nu, \tau, \mathbf{M})$ be a PNS, where $\mathbb{R}$ is the real number field, τ is a triangle function such that $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$ $(a, b > 0)$, $\mathbf{M}$ is the maximal triangle function and the probabilistic norm $\nu : \mathbb{R} \longrightarrow \Delta^+$ is defined by $\nu_x = \varepsilon_{\frac{|x|}{t+|x|}}$ for every $x \in \mathbb{R}$ and for a fixed $t > 0$ (see [8]). Now let $t = 1$ and (x_k) be a sequence in $(\mathbb{R}, \nu, \tau, \mathbf{M})$, defined as $x_k = 2k$ for every $k \in \mathbb{N}$.

Thus, the set $\{x_k : k \in \mathbb{N}\}$ is $\mathcal{D}$-bounded, it is also statistically $\mathcal{D}$-bounded. However, it is not statistically bounded in the sense of [4] in the topology generated by the Euclidean norm.

In [13], it was established that if $x = (x_k)$ is a statistically bounded sequence in the finite dimensional space $\mathbb{R}^m$, then the set of statistical cluster points $\Gamma(x)$ is non-empty and compact.

2. Main Results

Throughout this section, (E, ν, τ, τ^*) denotes an arbitrary PNS and $q = (q_k)$ denotes a sequence in this space. We will sometimes abbreviate the quadruple (E, ν, τ, τ^*) as E.

In this section we present a modification of the classical results of the set $\Gamma_s^u(q)$ which is non-empty and closed set. We define the set of *SUSC* points of the sequence $q = (q_k)$ in a PNS. We also study the concept of the set of *SUSC* points of the sequence $q = (q_k)$ and investigate necessary and sufficient conditions for gamma strong uniform convergence to compact set of cluster points of the sequence $q = (q_k)$ in a PNS.

First we give the following two lemmas to the theorem.

Lemma 2.1. *Let (E, ν, τ, τ^*) be a PNS and $q = (q_k)$ be a $\mathcal{D}-$bounded sequence in the PNS such that $\Gamma_s^u(q) \neq \emptyset$. Then $\Gamma_s^u(q)$ is a closed set with respect to the strong topology in E.*

Proof. Let $w = (w_k) \in \Gamma_s^u(q)$ and $w_k \rightarrow w_0$ in strong topology. We will show that $w_0 \in \Gamma_s^u(q)$. Let $w_0 \in \overline{\Gamma_s^u(q)}$. Then there exists a number k' such that $w_{k'} \in N_{w_0}(t)$.

Since $w_{k'} \in \Gamma_s^u(q)$ by definition we have

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j \,:\, q_{i+n}\in N_{w_{k'}}(t)\}|>0.$$

But for every n we choose $\alpha > 0$ such that $N_{w_{k'}}(\alpha) \subset N_{w_0}(t)$. Then we have for every n

$$\{0\leq i\leq j \,:\, q_{i+n}\in N_{w_0}(t)\}\supset\{0\leq i\leq j \,:\, q_{i+n}\in N_{w_{k'}}(\alpha)\}$$

and therefore for each $t > 0$

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j \,:\, q_{i+n}\in N_{w_0}(t)\}|>0.$$

which implies that $w_0 \in \Gamma_s^u(q)$. Then the proof is complete. □

Lemma 2.2. *Let $\Gamma_s^u(q)$ be a set of SUSC points of the sequences $q = (q_k)$ and M be a non-empty compact subset of E. If*

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j \,:\, q_{i+n}\in M\}|>0 \tag{2.1}$$

then

$$\Gamma_s^u(q)\cap M\neq\emptyset.$$

Proof. Assume that $\Gamma_s^u(q) \cap M = \emptyset$ and $\eta \in M$. Since $\eta \notin \Gamma_s^u(q)$ there exists a positive number $t = t(\eta) > 0$ such that

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j \,:\, q_{i+n}\in N_{\eta}(t)\}|=0. \tag{2.2}$$

Since M is a non-empty compact subset of E such that $M \subset \bigcup_{r=1}^{p} N_{\eta_r}(t_r)$. Then for every n we have

$$|\{0\leq i\leq j \,:\, q_{i+n}\in M\}|\leq\sum_{r=1}^{p}|\{0\leq i\leq j \,:\, q_{i+n}\in N_{\eta_r}(t_r)\}|.$$

According to the assumption (2.2), it follows that

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j \,:\, q_{i+n}\in M\}|=0$$

which contradict to (2.1) and this completes the proof. □

Theorem 2.1. *Let $q = (q_k)$ be a $\mathcal{D}$-bounded sequence in a PNS.*

(i) $\Gamma_s^u(q)$ is a non-empty compact set,

(ii) $\lim_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j\,:\,q_{i+n}\notin N_{\Gamma_s^u(q)}(t)\}|=0.$

Proof. Let $q=(q_k)$ be a $\mathcal{D}$-bounded sequence and M be a compact set such that $\{q_k\,:\,k\in\mathbb{N}\}\subset M$. We have

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j\,:\,q_{i+n}\in M\}|>0.$$

Then from Lemma 2.2, $\Gamma_s^u(q)\cap M\neq\emptyset$ i.e. the set $\Gamma_s^u(q)$ is non-empty. In this case from Lemma 2.1, we have $\Gamma_s^u(q)$ is also a compact set.

Now we show (ii). Let us consider the set

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j\,:\,q_{i+n}\notin N_{\Gamma_s^u(q)}(t)\}|>0$$

then for the set $\tilde{M}=M\setminus N_{\Gamma_s^u}(t)$ we have

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j\,:\,q_{i+n}\in\tilde{M}\}|>0.$$

By Lemma 2.2, we get $\Gamma_s^u(q)\cap\tilde{M}\neq\emptyset$, which is a contradiction with the definition of $\tilde{M}$. This completes the proof. □

Next example shows that if $q=(q_k)$ is a $\mathcal{D}$-unbounded Theorem 2.1 may be not true.

Example 2.1. Let $(\mathbb{R},\nu,\tau_\pi,\tau_{\pi^*})$ be a PNS, where $\nu:\mathbb{R}\longrightarrow\Delta^+$ is defined by

$$\nu_q(x)=\begin{cases}0, & \text{if } x\leq 0\\ \exp\left(-|q|^{1/2}\right), & \text{if } 0<x<+\infty\\ 1, & \text{if } x=\infty\end{cases}$$

and $\nu_0=\varepsilon_0$ [1]. Note that $\nu_q\in\Delta^+\diagdown\mathcal{D}^+$ for every $q\neq 0$ and, in this space, the only $\mathcal{D}-$bounded set is the singleton $\{0\}$. For instance, given $a\in\mathbb{R}\backslash\{0\}$, although the singleton $\{a\}$ is compact in $(\mathbb{R},\nu,\tau_\pi,\tau_{\pi^*})$, it is not $\mathcal{D}-$bounded. On the other hand, the sequence (q_k) for every $k\in\mathbb{N}$ defined by

$$q_k=\begin{cases}0, & \text{if } k \text{ is even}\\ k, & \text{if } k \text{ is odd}\end{cases}$$

is $\mathcal{D}-$unbounded. Clearly we have $\Gamma_s^u(q)=\{0\}\neq\emptyset$. But for every $t>0$,

$$\limsup_{j\to\infty}(j+1)^{-1}\max_{n\geq 1}|\{0\leq i\leq j\,:\,q_{i+n}\notin N_{\Gamma_s^u(q)}(t)\}|=1/2>0.$$

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