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# ON $f-$ STATISTICAL CONVERGENCE OF FRACTIONAL DIFFERENCE ON DOUBLE SEQUENCES

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**Abstract.** In this paper, using the fractional difference operator and a modulus function we introduce the concepts of $({}_2\Delta_{\beta}^{\tilde{\alpha}}, f)-$ statistical convergence, $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistical Cauchy and p-strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summability, $(0 < p < \infty)$ for double sequences. We also give some inclusion relations between $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistical convergence and p-strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summability $(0 < p < \infty)$.
**Key words:** Statistical convergence, difference sequence, Cesàro summability.

## 1. Introduction, Definitions and Preliminaries

The idea of statistical convergence was given by Zygmund [22] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [20], Fast [10] and later reintroduced by Schoenberg [19] independently. Over the years and under different names statistical convergence was discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Several writers have researched statistical convergence in the name of summability theory by Connor [7], Fridy [11], Akbaş and Işık ([2],[14], [15]) and many other writers.

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The statistical convergence depends on the density of subsets of the set $\mathbb{N}$. The natural density of a subset $K$ of $\mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}\right|$$

where $K(n) = |\{k \leqslant n : k \in K\}|$ denotes the number of elements $K \subset \mathbb{N}$ not exceeding $n$. It is clear that any finite subset of $\mathbb{N}$ have zero natural density and $\delta(K^c) = 1 - \delta(K)$

A sequence $x = (x_k)$ is said to be statistically convergent to $L$ if for every $\varepsilon > 0$

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}\right) = 0.$$

In this case we write

$$S - \lim_{k\to\infty} x_k = L.$$

The set of all statistically convergent sequences is denoted by $S$.

The definition of statistical convergence for double sequences was introduced by Mursaleen and Edely [17] as follows:

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(m,n) = \{j \leq m \text{ and } k \leq n : (j,k) \in K\}$. The double natural density of $K$ is defined by

$$\delta_2(K) = P - \lim_{m,n\to\infty} \frac{|K(m,n)|}{mn}$$

if the limit exists. A double sequence $x = (x_{jk})$ is said to be statistically convergent to a number $L$ if for every $\varepsilon > 0$ $\delta_2\left(\{(j,k); j \leqslant m \text{ and } k \leqslant n : |x_{jk} - L| \geqslant \varepsilon\}\right)$ has double natural density zero.

The idea of a modulus function was structured by Nakano [18]. Let us recall that $f : [0,\infty) \to [0,\infty)$ is said to be a modulus function if it satisfies:

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x+y) \leqslant f(x) + f(y)$ for $x, y \geqslant 0$ ,

iii) $f$ is increasing,

iv) $f$ is continuous from the right at 0.

With the help of an unbounded modulus function, Aizpuru et al. [1] defined a new concept of density, and as a result, they obtained a new concept of nonmatrix convergence that is intermediate between ordinary and statistical convergence, and agrees with statistical convergence when the modulus function is the identity mapping.

The $f-$ density of a set $K \subset \mathbb{N}$ is defined by

$$d^f(K) = \lim_{n\to\infty} \frac{f(|K(n)|)}{f(n)}$$

in case this limit exists.

A sequence $x = (x_k)$ is said to be $f-$ statistically convergent to $L$ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{f(n)} f\big( |\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}| \big) = 0$$

and it is denoted that

$$S^f - \lim_{k\to\infty} x_k = L$$

The idea of difference sequence was introduced by Kızmaz [16] and generalized by many authors such as Et and Çolak [9], Atabey and Çınar [3], Güngör and Et ([12], [13])

Let $r \notin \{0, -1, -2, -3, ...\}$ and $r$ be a reel number. We define the Gamma function as

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

We observe that

i) Let $r$ be any natural number, $\Gamma(r+1) = r!$. For example $\Gamma(1) = 1!$, $\Gamma(2) = 1!$, $\Gamma(3) = 2!$, ...

ii) Let $r \notin \{0, -1, -2, -3, \ldots\}$ and $r$ be a reel number, then $\Gamma(r+1) = r\Gamma(r)$.

Lastly, for a proper fraction $\alpha$, Baliarsingh [4] defined fractional difference operator $\Delta^\alpha : \omega \to \omega$ by

$$\begin{aligned}(\Delta^\alpha x)_k &= \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha - i + 1)} x_{k+i} \\ &= x_k - \alpha . x_{k+1} + \frac{\alpha(\alpha-1)}{2!} . x_{k+2} - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} . x_{k+3} + \ldots\end{aligned}$$

After, for a positive proper fraction $\tilde{\alpha}$, Baliarsingh [5] define double fractional operator $\Delta^{\tilde{\alpha}} : \omega^2 \to \omega^2$ defined by

$$\Delta^{\tilde{\alpha}}(x_{jk}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{\Gamma(\tilde{\alpha}+1)^2}{m!n!\Gamma(\tilde{\alpha} - m + 1)\Gamma(\tilde{\alpha} - n + 1)} x_{j+m,k+n}$$

In particular, for a double sequence $x = (x_{jk})$ and $\tilde{\alpha} = \frac{1}{2}$ we have

$$\Delta^{\frac{1}{2}}(x_{jk}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{\Gamma(\frac{1}{2}+1)^2}{m!n!\Gamma(\frac{1}{2} - m + 1)\Gamma(\frac{1}{2} - n + 1)} x_{j+m,k+n}$$

$$\begin{aligned}
&= x_{j,k} - \frac{1}{2}x_{j,k+1} - \frac{1}{8}x_{j,k+2} - \frac{1}{16}x_{j,k+3} - \frac{5}{128}x_{j,k+4} - \dots \\
&\quad -\frac{1}{2}x_{j+1,k} + \frac{1}{4}x_{j+1,k+1} + \frac{1}{16}x_{j+1,k+2} + \frac{1}{32}x_{j+1,k+3}\frac{1}{64}x_{j+1,k+4} \\
&\quad +\frac{5}{256}x_{j+1,k+5} + \dots - \frac{1}{2}x_{j+2,k} + \frac{1}{16}x_{j+2,k+1} + \frac{1}{64}x_{j+2,k+2} \\
&\quad +\frac{1}{128}x_{j+2,k+3} + \frac{1}{1024}x_{j+2,k+4} + \dots
\end{aligned}$$

The concept of ${}_2\Delta^{\tilde{\alpha}}-$ statistically convergence of double sequences was introduced by Atabey and Çınar [3] such as:

Let $\tilde{\alpha}$ be a proper fraction. A double sequence $x = (x_{jk})$ is said to be ${}_2\Delta^{\tilde{\alpha}}-$ statistically convergent to a number $L$ if for every $\varepsilon > 0$

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left|\{(j,k) : j \le m \,\text{and}\, k \le n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \ge \varepsilon\}\right| = 0.$$

The set of all ${}_2\Delta^{\tilde{\alpha}}-$ statistically convergent sequences will be denoted by $({}_2\Delta^{\tilde{\alpha}}(S_2))$. In this case we write $x_{jk} \to L({}_2\Delta^{\tilde{\alpha}}(S_2))$

Recently Altın et al. ([6],[21]) and Ercan [8] have studied the statistical convergence of double sequence.

## 2. Main Results

In this section, we will give the main results of this paper.

**Definition 2.1.** A double sequence $x = (x_{jk})$ is said to be $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent to a number $L$ if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{f(mn)} f\left(\left|\{(j,k) : j \le m \,\text{and}\, k \le n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \ge \varepsilon\}\right|\right) = 0.$$

The set of all $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent sequences will be denoted by $({}_2\Delta^{\tilde{\alpha}}(S_2^f))$. In this case we write $x_{jk} \to L({}_2\Delta^{\tilde{\alpha}}(S_2^f))$.

We usually take $s, t, u, v \in (0, 1]$ and write $\beta$ instead of $(s, t)$, $\theta$ instead of $(u, v)$. We also define

$$\begin{gathered}
\beta \preceq \theta \Leftrightarrow s \le u \,\text{and}\, t \le v \\
\beta \prec \theta \Leftrightarrow s < u \,\text{and}\, t < v \\
\beta \cong \theta \Leftrightarrow s = u \,\text{and}\, t = v \\
\beta \in (0,1] \Leftrightarrow s, t \in (0,1] \\
\theta \in (0,1] \Leftrightarrow u, v \in (0,1] \\
\beta \cong 1 \,\text{in case of}\,\ s = t = 1 \\
\theta \cong 1 \,\text{in case of}\,\ u = v = 1
\end{gathered}$$

.

**Definition 2.2.** Let $f$ be a modulus function. A double sequence $x = (x_{jk})$ is said to be $({}_2\Delta^{\tilde{\alpha}}_{\beta}, f)-$ statistically convergent to a number $L$ if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{f(m^s n^t)} f\left(\left|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geq \varepsilon\}\right|\right) = 0.$$

The set of all $({}_2\Delta^{\tilde{\alpha}}_{\beta}, f)-$ statistically convergent sequences will be denoted by $({}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f))$. In this case we write $x_{jk} \to L({}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f))$. We write ${}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f)$ for ${}_2\Delta^{\tilde{\alpha}}_{(s,t)}(S_2^f)$ and ${}_2\Delta^{\tilde{\alpha}}_{\theta}(S_2^f)$ for ${}_2\Delta^{\tilde{\alpha}}_{(u,v)}(S_2^f)$.

**Definition 2.3.** A double sequence $x = (x_{jk})$ is said to be $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically Cauchy, if for $\forall\varepsilon > 0$ there exist $M_0 \in \mathbb{N}$ and $N_0 \in \mathbb{N}$ such that for all $j \geqslant M_0$ and $k \geqslant N_0$,

$$\lim_{m,n\to\infty} \frac{1}{f(mn)} f(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk} - x_{M_0 N_0})| \geqslant \varepsilon\}|) = 0.$$

As it can be seen in the example below, the set of $({}_2\Delta^{\tilde{\alpha}}_{\beta}, f)-$ statistically convergent sequences is not empty.

**Example 2.1.** Let us consider $(x_{jk}) = 1$ for all $j, k \in \mathbb{N}$ and $f$ be an any modulus function and $\beta \cong 1$. Then we obtain $x_{jk} \to 0({}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f))$.

As it can be seen in the example below, a double sequence is $({}_2\Delta^{\tilde{\alpha}}_{\beta}, f)-$ statistically convergent but need not be $({}_2\Delta^{\tilde{\alpha}})-$ bounded, that is $x \notin \ell_\infty({}_2\Delta^{\tilde{\alpha}})$, where $\ell_\infty({}_2\Delta^{\tilde{\alpha}}) = \{x = (x_{jk}) : \sup_{j,k} |\Delta^{\tilde{\alpha}}(x_{jk})| < \infty\}$.

**Example 2.2.** Let $f(x) = \sqrt{x}$ and defined $x = (x_{jk})$ such that

$$\Delta^{\tilde{\alpha}}(x_{jk}) = \begin{cases} j+k & , j = m^2 \,\text{and}\, k = n^2 \\ 0 & , \text{otherwise} \end{cases} \qquad m, n = 0, 1, 2, \ldots$$

then $x = (x_{jk})$ is $({}_2\Delta^{\tilde{\alpha}}_{\beta}, f)-$ statistical convergent, but is not $({}_2\Delta^{\tilde{\alpha}})-$ bounded, for $\beta \cong 1$.

Clearly,

$$\Delta^{\tilde{\alpha}}(x_{jk}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 4 & 0 & \ldots \\ 1 & 2 & 0 & 0 & 5 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 4 & 5 & 0 & 0 & 8 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and since

$$\left|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - 0| \geq \varepsilon\}\right| \leq \sqrt{m}\sqrt{n}$$

we have

$$\lim_{m,n\to\infty}\frac{1}{f(mn)}\left|\{(j,k): j\le m \,\text{and}\, k\le n; |\Delta^{\tilde\alpha}(x_{jk})-0|\ge\varepsilon\}\right| \le \lim_{m,n\to\infty}\frac{1}{f(mn)}f(\sqrt{m}\sqrt{n})$$

$$=\lim_{m,n\to\infty}\frac{\sqrt[4]{mn}}{\sqrt{mn}}=0$$

As a result, double sequence $(x_{jk})$ is $({}_2\Delta^{\tilde\alpha},f)-$ statistically convergent to 0,but is not $({}_2\Delta^{\tilde\alpha})-$ bounded.

As it can be seen in the following example, a double sequence is $({}_2\Delta^{\tilde\alpha})-$ bounded, but need neither $({}_2\Delta^{\tilde\alpha},f)-$ statistical convergent nor $({}_2\Delta^{\tilde\alpha}_{\tilde\beta},f)-$ statistical convergent.

**Example 2.3.** Let us consider the sequence $x=(x_{jk})=\dfrac{1+(-1)^{j+k}}{2}$ and choose $f(x)=x^p$, for $1\leqslant p<\infty$. Then we have:

$$\begin{aligned}
\Delta^{\tilde\alpha}(x_{jk}) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\frac{\Gamma(\tilde\alpha+1)^2}{m!n!\Gamma(\tilde\alpha-m+1)\Gamma(\tilde\alpha-n+1)}x_{j+m,k+n}\\
&= \sum_{m=0}^{\infty}\Bigg((-1)^{m+0}\frac{\Gamma(\tilde\alpha+1)^2}{m!0!\Gamma(\tilde\alpha-m+1)\Gamma(\tilde\alpha-0+1)}x_{j+m,k+0}\\
&\quad+(-1)^{m+1}\frac{\Gamma(\tilde\alpha+1)^2}{m!1!\Gamma(\tilde\alpha-m+1)\Gamma(\tilde\alpha-1+1)}x_{j+m,k+1}\\
&\quad+(-1)^{m+2}\frac{\Gamma(\tilde\alpha+1)^2}{m!2!\Gamma(\tilde\alpha-m+1)\Gamma(\tilde\alpha-2+1)}x_{j+m,k+2}\\
&\quad+(-1)^{m+3}\frac{\Gamma(\tilde\alpha+1)^2}{m!3!\Gamma(\tilde\alpha-m+1)\Gamma(\tilde\alpha-3+1)}x_{j+m,k+3}+\dots\Bigg)\\
&=\begin{cases} 2^{2\tilde\alpha-1} & , \ j+k \text{ even}\\ -2^{2\tilde\alpha-1} & , \ j+k \text{ odd}\end{cases}
\end{aligned}$$

That is,

$$\Delta^{\tilde\alpha}(x_{jk})=\begin{pmatrix} 2^{2\tilde\alpha-1} & -2^{2\tilde\alpha-1} & 2^{2\tilde\alpha-1} & \dots\\ -2^{2\tilde\alpha-1} & 2^{2\tilde\alpha-1} & -2^{2\tilde\alpha-1} & \dots\\ 2^{2\tilde\alpha-1} & -2^{2\tilde\alpha-1} & 2^{2\tilde\alpha-1} & \dots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

Obviously $x=(x_{jk})$ is $({}_2\Delta^{\tilde\alpha})-$ bounded,but not $({}_2\Delta^{\tilde\alpha},f)-$ statistically convergent, for $f(x)=x^p$, $1\le p<\infty$, since

$$\lim_{m,n\to\infty}\frac{1}{f(mn)}f\Big(|\{(j,k): j\le m \,\text{and}\, k\le n; |\Delta^{\tilde\alpha}(x_{jk})-2^{2\tilde\alpha-1}|\geqslant\varepsilon\}|\Big)$$

$$=\lim_{m,n\to\infty}\frac{f(mn/2)}{f(mn)}\quad=\frac{1}{2^p}\neq 0.$$

**Definition 2.4.** Let $p$ be a positive real number. A double sequence $x = (x_{jk})$ is said to be $p-$strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summable to a number $L$ if

$$\lim_{m,n\to\infty} \frac{1}{f(mn)} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(|{}_2\Delta^{\tilde{\alpha}} x_{jk} - L|^p\right) = 0.$$

We denote the set of all double $p-$strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summable sequences by $w_p^2({}_2\Delta^{\tilde{\alpha}}, f)$.

In case of $p = 1$ we shall write $[C^f, 1, 1]({}_2\Delta^{\tilde{\alpha}})$ instead of $w_p^2({}_2\Delta^{\tilde{\alpha}}, f)$.

## 3. The Inclusion Theorems

In this section we give some inclusion relations.

**Lemma 3.1.** *Let $f$ be an unbounded modulus function and $K \subset \mathbb{N} \times \mathbb{N}$. If $0 \preceq \beta \preceq \theta \preceq 1$ then $\delta_\beta^f(K) \leq \delta_\theta^f(K)$.*

*Proof.* Let $0 < s \leq t \leq u \leq v \leq 1$. Since $m^s n^t \leq m^u n^v$ for all $m, n \in \mathbb{N} \times \mathbb{N}$ and $f$ is increasing, we can write $\dfrac{1}{m^u n^v} \leq \dfrac{1}{m^s n^t}$ and $\dfrac{1}{f(m^u n^v)} \leq \dfrac{1}{f(m^s n^t)}$. Then

$$\frac{1}{m^u n^v} |\{j \leq m \,\text{and}\, k \leq n : (j,k) \in K\}| \leq \frac{1}{m^s n^t} |\{j \leq m \,\text{and}\, k \leq n : (j,k) \in K\}|$$

$$\begin{aligned} &\frac{1}{f(m^u n^v)} f(|\{j \leq m \,\text{and}\, k \leq n : (j,k) \in K\}|) \\ &\leq \frac{1}{f(m^s n^t)} f(|\{j \leq m \,\text{and}\, k \leq n : (j,k) \in K\}|) \end{aligned}$$

so $\delta_\beta^f(K) \leq \delta_\theta^f(K)$. □

**Theorem 3.1.** *Let $f$ be an unbounded modulus function. $x = (x_{jk})$ and $y = (y_{jk})$ be any sequences of real (or complex) numbers. Then*

*(i) if ${}_2\Delta^{\tilde{\alpha}}(S_2^f) - \lim x_{jk} = x_0$ and $c$ be real (or complex) number, then ${}_2\Delta^{\tilde{\alpha}}(S_2^f) - \lim c.x_{jk} = cx_0$*

*(ii) if ${}_2\Delta^{\tilde{\alpha}}(S_2^f) - \lim x_{jk} = x_0$ and ${}_2\Delta^{\tilde{\alpha}}(S_2^f) - \lim y_{jk} = y_0$, then ${}_2\Delta^{\tilde{\alpha}}(S_2^f) - \lim(x_{jk} + y_{jk}) = x_0 + y_0$*

*Proof.* Proof is omitted. □

**Theorem 3.2.** *A double sequence $x = (x_{jk})$ is $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent, then $x = (x_{jk})$ is $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically Cauchy sequence.*

*Proof.* Assume for $\forall\varepsilon > 0$ $x = (x_{jk})$ be $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent. Then, we can write $|\Delta^{\tilde{\alpha}}(x_{jk}) - L| < \frac{\varepsilon}{2}$ for almost all $j, k \in \mathbb{N}$ and choosen for $M_0, N_0 \in \mathbb{N}$ we have $|\Delta^{\tilde{\alpha}}(x_{M_0N_0}) - L| < \frac{\varepsilon}{2}$. Now for almost all $j, k \in \mathbb{N}$, we have

$$\begin{aligned} |\Delta^{\tilde{\alpha}}(x_{jk}) - \Delta^{\tilde{\alpha}}(x_{M_0N_0})| &= |\Delta^{\tilde{\alpha}}(x_{jk}) - L - (\Delta^{\tilde{\alpha}}(x_{M_0N_0}) - L)| \\ &\leqslant |\Delta^{\tilde{\alpha}}(x_{jk}) - L| + |\Delta^{\tilde{\alpha}}(x_{M_0N_0}) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This mean that $x = (x_{jk})$ is $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically Cauchy sequence. □

**Theorem 3.3.** *Let $0 < p < q < \infty$. Then $w_q^2({}_2\Delta^{\tilde{\alpha}}, f) \subset w_p^2({}_2\Delta^{\tilde{\alpha}}, f)$*

*Proof.* Let $x = (x_{jk}) \in w_q^2({}_2\Delta^{\tilde{\alpha}}, f)$ and $0 < p < q < \infty$. Then this means that

$$\lim_{m,n\to\infty} \frac{1}{f(mn)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^q) = 0$$

and since

$$|\Delta^{\tilde{\alpha}} x_{jk} - L|^p \quad \leq \quad |\Delta^{\tilde{\alpha}} x_{jk} - L|^q$$

we have

$$\begin{aligned} &\lim_{m,n\to\infty} \frac{1}{f(mn)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^p) \\ \leq\ &\lim_{m,n\to\infty} \frac{1}{f(mn)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^q) = 0. \end{aligned}$$

Therefore $x = (x_{jk}) \in w_p^2({}_2\Delta^{\tilde{\alpha}}, f)$. □

**Theorem 3.4.** $w_p^2({}_2\Delta^{\tilde{\alpha}}, f) \subset {}_2\Delta^{\tilde{\alpha}}(S_2^f)$

*Proof.* Let $x = (x_{jk}) \in w^2({}_2\Delta^{\tilde{\alpha}}, f)$, $\varepsilon > 0$ and $f$ be an any modulus function. Then we have

$$\begin{aligned} &\frac{1}{f(mn)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^p) \\ &= \frac{1}{f(mn)} \sum \sum_{|\Delta^{\tilde{\alpha}} x_{jk} - L| \geqslant \varepsilon} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^p) \\ &\quad + \frac{1}{f(mn)} \sum \sum_{|\Delta^{\tilde{\alpha}} x_{jk} - L| < \varepsilon} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^p) \\ &\geqslant \frac{1}{f(mn)} \sum \sum_{|\Delta^{\tilde{\alpha}} x_{jk} - L| \geqslant \varepsilon} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|^p) \\ &\geqslant \frac{1}{f(mn)} f(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geqslant \varepsilon\}|).f(\varepsilon)^p \end{aligned}$$

and so $x=(x_{jk})\in {}_2\Delta^{\tilde{\alpha}}(S_2^f)$. $\square$

**Theorem 3.5.** *If $f$ is bounded, then ${}_2\Delta^{\tilde{\alpha}}(S_2^f)\subset w_p^2({}_2\Delta^{\tilde{\alpha}},f)$.*

*Proof.* Because $f$ is bounded, we have an integer $M$ such that $|f(x)|<M$, for every $x>0$.

$$\begin{aligned}
&\frac{1}{f(mn)}\sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}}x_{jk}-L|^p)\\
&\leqslant \frac{1}{f(mn)}\sum\sum_{|\Delta^{\tilde{\alpha}}x_{jk}-L|\geqslant\varepsilon} f(|\Delta^{\tilde{\alpha}}x_{jk}-L|^p)\\
&\quad+\frac{1}{f(mn)}\sum\sum_{|\Delta^{\tilde{\alpha}}x_{jk}-L|<\varepsilon} f(|\Delta^{\tilde{\alpha}}x_{jk}-L|^p)\\
&\leqslant \frac{1}{f(mn)}M^p|\{(j,k): j\le m \,\text{and}\, k\le n; |\Delta^{\tilde{\alpha}}(x_{jk})|\geqslant\varepsilon\}|+f(\varepsilon)^p
\end{aligned}$$

therefore $x=(x_{jk})\in w_p^2({}_2\Delta^{\tilde{\alpha}},f)$. $\square$

**Theorem 3.6.** *If $f$ is bounded, then ${}_2\Delta^{\tilde{\alpha}}(S_2^f)=w_p^2({}_2\Delta^{\tilde{\alpha}},f)$.*

*Proof.* From Theorem 3.4 and Theorem 3.5 we have ${}_2\Delta^{\tilde{\alpha}}(S_2^f)=w_p^2({}_2\Delta^{\tilde{\alpha}},f)$. $\square$

**Theorem 3.7.** *Let $f$ be an unbounded modulus function, $\beta,\theta\in(0,1]$ and $\beta\preceq\theta$ be given. Therefore ${}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f)\subseteq {}_2\Delta^{\tilde{\alpha}}_{\theta}(S_2^f)$.*

*Proof.* Since $f$ is increasing, $\beta,\theta\in(0,1]$ and $\beta\preceq\theta$, we write $s\le u$ and $t\le v$. Then

$$\begin{aligned}
&\frac{1}{f(m^u n^v)}f(|\{(j,k): j\le m \text{ and } k\le n; |\Delta^{\tilde{\alpha}}(x_{jk})-L|\ge\varepsilon\}|)\\
\le\ &\frac{1}{f(m^s n^t)}f(|\{(j,k): j\le m \,\text{and}\, k\le n; |\Delta^{\tilde{\alpha}}(x_{jk})-L|\ge\varepsilon\}|)
\end{aligned}$$

for $\forall\varepsilon>0$. This gives that ${}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f)\subseteq {}_2\Delta^{\tilde{\alpha}}_{\theta}(S_2^f)$. $\square$

**Corollary 3.1.** *Let $f$ be an unbounded modulus function and $\beta,\theta\in(0,1]$. Then*

*(i)* ${}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f)={}_2\Delta^{\tilde{\alpha}}_{\theta}(S_2^f)$ *if and only if* $\beta\cong\theta$.

*(ii)* ${}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f)={}_2\Delta^{\tilde{\alpha}}(S_2^f)$ *if and only if* $\beta\cong 1$.

**Corollary 3.2.** *Let $f$ be an unbounded modulus function and $\beta\in(0,1]$. If $x=(x_{jk})\to L({}_2\Delta^{\tilde{\alpha}}_{\beta}(S_2^f))$ then $x=(x_{jk})\to L({}_2\Delta^{\tilde{\alpha}}(S_2^f))$ and the inclusion is strict.*

**Theorem 3.8.** *Let $f$ be an unbounded modulus function, $\beta, \theta \in (0,1]$, $\beta \preceq \theta$ and $p$ positive real number. Then we have $w_p^2({}_2\Delta_\beta^{\tilde{\alpha}}, f) \subseteq w_p^2({}_2\Delta_\theta^{\tilde{\alpha}}, f)$ and the inclusion is strict.*

*Proof.* Let take $x = (x_{jk}) \in w_p^2({}_2\Delta_\beta^{\tilde{\alpha}}, f)$. Hence

$$\begin{aligned} &\lim_{m,n\to\infty} \frac{1}{f(m^u n^v)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|{}_2\Delta^{\tilde{\alpha}} x_{jk} - L|^p) \\ &\leq \lim_{m,n\to\infty} \frac{1}{f(m^s n^t)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|{}_2\Delta^{\tilde{\alpha}} x_{jk} - L|^p) = 0. \end{aligned}$$

Therefore we can write $w_p^2({}_2\Delta_\beta^{\tilde{\alpha}}, f) \subseteq w_p^2({}_2\Delta_\theta^{\tilde{\alpha}}, f)$. □

**Corollary 3.3.** *Let $f$ be an unbounded modulus function, $\beta, \theta \in (0,1]$, $\beta \preceq \theta$ and $p$ positive real number. Then*

*(i)* $w_p^2({}_2\Delta_\beta^{\tilde{\alpha}}, f) = w_p^2({}_2\Delta_\theta^{\tilde{\alpha}}, f)$ *if and only if* $\beta \cong \theta$,

*(ii)* $w_p^2({}_2\Delta_\beta^{\tilde{\alpha}}, f) \subseteq w_p^2({}_2\Delta^{\tilde{\alpha}}, f)$.

**Theorem 3.9.** *Let $\beta$ and $\theta$ be two real numbers satisfying the condition $0 < \beta \leqslant \theta \leqslant 1$, $f$ be unbounded modulus such that there is a positive constant $c$ such that $f(x.y) \geqslant c.f(x)f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, if a double sequence $x = (x_{jk}$ is strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summable of order $\beta$ to $L$, then it is $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent of order $\theta$ to $L$( or if $x_{jk} \to L[C_\beta^f, 1, 1]({}_2\Delta^{\tilde{\alpha}})$ then $x_{jk} \to L({}_2\Delta_\beta^{\tilde{\alpha}}(S_2^f))$.*

*Proof.* For any sequence $x = (x_{jk})$ and $\varepsilon > 0$, using by definition of modulus functions, we can write

$$\begin{aligned} & \frac{1}{f(mn)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|) \\ \geqslant\ & \frac{1}{f(mn)} f(\sum_{j=1}^{m}\sum_{k=1}^{n} |\Delta^{\tilde{\alpha}} x_{jk} - L|) \\ \geqslant\ & f(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geq \varepsilon\}|.\varepsilon) \\ \geqslant\ & cf(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geq \varepsilon\}|)f(\varepsilon) \end{aligned}$$

and $\beta \leqslant \theta$

$$\frac{1}{f(m^s n^t)} \sum_{j=1}^{m}\sum_{k=1}^{n} f(|\Delta^{\tilde{\alpha}} x_{jk} - L|)$$

$$\begin{aligned} &\geqslant \frac{cf(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geq \varepsilon\}|)f(\varepsilon)}{f(m^s n^t)} \\ &\geqslant \frac{cf(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geq \varepsilon\}|)f(\varepsilon)}{f(m^u n^v)} \\ &= \frac{cf(|\{(j,k) : j \leq m \,\text{and}\, k \leq n; |\Delta^{\tilde{\alpha}}(x_{jk}) - L| \geq \varepsilon\}|)f(\varepsilon)f(m^u n^v)}{m^u n^v f(m^u n^v)}. \end{aligned}$$

So the theorem is proved. □

According to Theorem 3.9 we can obtain the following results

**Corollary 3.4.** *Let $\beta$ be real number and $0 < \beta \leqslant 1$, $f$ be unbounded modulus such that there is a positive constant c such that $f(x.y) \geqslant c.f(x)f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, if a double sequence $x = (x_{jk})$ is strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summable of order $\beta$ to $L$, then it is $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent of order $\beta$ to $L$.*

If $\beta = 1$ is taken we can give the following Corollary 3.5.

**Corollary 3.5.** *Let $f$ be unbounded modulus such that there is a positive constant c such that $f(x.y) \geqslant c.f(x)f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, if a double sequence $x = (x_{jk})$ is strongly $({}_2\Delta^{\tilde{\alpha}}, f)-$ Cesàro summable to $L$, then it is $({}_2\Delta^{\tilde{\alpha}}, f)-$ statistically convergent to $L$.*

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