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BI-ROTATIONAL HYPERSURFACE WITH $\Delta\mathrm{x} = \mathcal{A}\mathrm{x}$ IN 4-SPACE

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Abstract. We introduce the bi-rotational hypersurface $\mathbf{x}(u, v, w)$ in the four dimensional Euclidean geometry $\mathbb{E}^4$. We obtain the i-th curvatures of the hypersurface. Moreover, we consider the Laplace–Beltrami operator of the bi-rotational hypersurface satisfying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$.
Key words: bi-rotational hypersurface, Eucledian geometry, curvative formulas.

1. Introduction

With the works of Chen [13, 14, 15, 16], the studies of the submanifolds of the finite type whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) by using a finite number of eigenfunctions of their Laplacian have been studied for almost a half century.

Takahashi [46] gave that a connected Euclidean submanifold is of 1-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$. Submanifolds of the finite type closest in simplicity to the minimal ones are the 2-type spherical submanifolds (where spherical means into a sphere). Some results of the 2-type spherical closed submanifolds were given by [9, 10, 14]. Garay studied [28] an extension of Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau introduced the hypersurfaces

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with constant scalar curvature; Chen and Piccinni [17] focused on the submanifolds with finite type Gauss map in $\mathbb{E}^m$. Dursun [23] considered the hypersurfaces with pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$.

In $\mathbb{E}^3$; Levi-Civita [40] worked the isoparametric surface family; Takahashi [46] proved the minimal surfaces and the spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez et al. [25] found the surfaces satisfying $\Delta H = AH$, $A \in Mat(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [20] classified the minimal helicoid in terms of the pointwise 1-type Gauss map of the first kind; Garay [27] studied the certain class of the finite type surfaces of revolution; Dillen et al. [21] obtained that the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [45] focused the surfaces of revolution satisfying $\Delta^{III} x = Ax$; Senoussi and Bekkar [44] gave the helicoidal surfaces M^2 which are of the finite type with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3,3)$; Kim et al. [37] introduced the Cheng–Yau's operator and the Gauss map of the surfaces of revolution.

In $\mathbb{E}^4$; Moore [41, 42] considered the general rotational surfaces; Hasanis and Vlachos [34] studied the hypersurfaces with the harmonic mean curvature vector field; Cheng and Wan [18] gave the complete hypersurfaces with CMC; Kim and Turgay [38] worked the surfaces with L_1-pointwise 1-type Gauss map; Arslan et al. [3] introduced the Vranceanu surface with the pointwise 1-type Gauss map; Arslan et al. [4] worked the generalized rotational surfaces; Arslan et al. [5] considered the tensor product surfaces with the pointwise 1-type Gauss map; Kahraman Aksoyak and Yaylı [35] studied the rotational surfaces with the pointwise 1-type Gauss map; Güler et al. [32] introduced the helicoidal hypersurfaces; Güler et al. [31] worked the Gauss map and the third Laplace–Beltrami operator of the rotational hypersurface; Güler and Turgay [33] studied the Cheng–Yau's operator and the Gauss map of the rotational hypersurfaces; Güler [30] obtained the rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4,4)$. He [29] also worked the fundamental form IV and the curvature formulas of the hypersphere.

In Minkowski 4-space $\mathbb{E}_1^4$; Ganchev and Milousheva [26] studied the analogue of surfaces of [41, 42]; Arvanitoyeorgos et al. [8] indicated if the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M_1^3 has CMC; Arslan and Milousheva introduced the meridian surfaces of elliptic or hyperbolic type with the pointwise 1-type Gauss map; Turgay considered some classifications of a Lorentzian surfaces with the finite type Gauss map; Dursun and Turgay worked the space-like surfaces with the pointwise 1-type Gauss map. Kahraman Aksoyak and Yaylı [36] gave the general rotational surfaces with the pointwise 1-type Gauss map in $\mathbb{E}_2^4$. Bektaş, Canfes, and Dursun [11] obtained surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}_2^5$. They [12] also considered pseudo-spherical submanifolds with the 1-type pseudo-spherical Gauss map. Arslan et al. [7] introduced the rotational λ-hypersurfaces in the Euclidean spaces.

We consider the bi-rotational hypersurface in the four dimensional Euclidean geometry $\mathbb{E}^4$. In Section 2, we give some basic notions of the four dimensional Euclidean geometry. We consider the curvature formulas of the hypersurfaces in $\mathbb{E}^4$, in Section 3. In Section 4, we define the bi-rotational hypersurface. We study the bi-rotational hypersurface satisfying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$ in $\mathbb{E}^4$ in Section 5. Finally, we give some results in the last section.

2. Preliminaries

In this section, giving some of basic facts and definitions, we describe the notations used the whole paper. Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by $\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \dots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi-Civita connections [40] of the manifold $\widetilde{M}$, and its submanifold M of $\mathbb{E}^m$ by $\widetilde{\nabla}$, ∇, respectively. We shall use letters X, Y, Z, W (resp., ξ, η) to denote vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \widetilde{\nabla}_X \xi &= -A_\xi(X) + D_X \xi, \end{aligned}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\begin{aligned} \langle R(X, Y,)Z, W \rangle &= \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \\ (\bar{\nabla}_X h)(Y, Z) &= (\bar{\nabla}_Y h)(X, Z), \end{aligned}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $\mathbf{S}$ its shape operator (i.e., the Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \dots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then, the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{2.1}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of M in $\mathbb{E}^{n+1}$ by ∇ and $\widetilde{\nabla}$, respectively. Then, from the Codazzi equation (2.1), we have

$$\begin{aligned} e_i(k_j) &= \omega_{ij}(e_j)(k_i - k_j), \\ \omega_{ij}(e_l)(k_i - k_j) &= \omega_{il}(e_j)(k_i - k_l) \end{aligned}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and the Gauss-Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$ (The curvature formulas sometimes are represented as the mean curvature H_i, and sometimes as the Gaussian curvature K_i by different writers, such as [1] and [39]. We will call it just the i-th curvature $\mathfrak{C}_i$ in this paper.), where $i = 0, \ldots, n$, firstly, we use the characteristic polynomial of $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k},$$

where $i = 0, \ldots, n$, $\mathcal{I}_n$ denotes the identity matrix of order n. Then, we get the curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$. Clearly, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \ldots, \binom{n}{n}\mathfrak{C}_n = s_n = K$.

For a Euclidean submanifold x: $M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called the *finite type,* if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e., $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ nonconstant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called *k-type.* See [14] for details.

Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an isometric immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. The triple vector product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}^4$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

For a hypersurface $\mathbf{x}$ in 4-space, (g_{ij}) and $(h_{ij})_{3\times 3}$, are the first, and the second fundamental form matrices, respectively, $g_{11} = \mathbf{x}_u \cdot \mathbf{x}_u$, $g_{12} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{22} = \mathbf{x}_v \cdot \mathbf{x}_v$, $g_{13} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{23} = \mathbf{x}_v \cdot \mathbf{x}_w$, $g_{33} = \mathbf{x}_w \cdot \mathbf{x}_w$, $h_{11} = \mathbf{x}_{uu} \cdot G$, $h_{12} = \mathbf{x}_{uv} \cdot G$, $h_{22} = \mathbf{x}_{vv} \cdot G$, $h_{13} = \mathbf{x}_{uw} \cdot G$, $h_{23} = \mathbf{x}_{vw} \cdot G$, $h_{33} = \mathbf{x}_{ww} \cdot G$. Here,

$$G = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|} \tag{2.2}$$

is the unit normal (i.e. the Gauss map) of the hypersurface $\mathbf{x}$. The product matrix $(g_{ij})^{-1} \cdot (h_{ij})$ gives the matrix of the shape operator $\mathbf{S}$ of the hypersurface $\mathbf{x}$ in 4-space. See [31, 32, 33] for details.

3. *i*-th Curvatures

In $\mathbb{E}^4$, to compute the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, 1, 2, 3$, we use the characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, and $P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda \mathcal{I}_3) = 0$.

Then, we obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1} H = -\frac{b}{a}$, $\binom{3}{2} \mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3} \mathfrak{C}_3 = K = -\frac{d}{a}$. Therefore, we find the following i-th curvature folmulas in 4-space:

Theorem 3.1. *Any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{ \begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})g_{33} + (g_{11}g_{22} - g_{12}^2)h_{33} \\ -g_{23}^2 h_{11} - g_{13}^2 h_{22} - 2(g_{13}h_{13}g_{22} - g_{23}h_{13}g_{12} \\ -g_{13}h_{23}g_{12} + g_{11}g_{23}h_{23} - g_{13}g_{23}h_{12}) \end{array} \right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{\left\{ \begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})\, h_{33} + \left(h_{11}h_{22} - g_{12}^2\right) g_{33} \\ -g_{11}h_{23}^2 - g_{22}h_{13}^2 - 2\left(g_{13}h_{13}h_{22} - g_{23}h_{13}h_{12}\right. \\ \left. -g_{13}h_{23}h_{12} + g_{23}h_{23}h_{11} - h_{13}h_{23}g_{12}\right) \end{array} \right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{\left(h_{11}h_{22} - h_{12}^2\right) h_{33} - h_{11}h_{23}^2 + 2h_{12}h_{13}h_{23} - h_{22}h_{13}^2}{(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2}. \tag{3.3}$$

See [29] for details.

4. Bi-Rotational Hypersurface

In this section, we define the rotational hypersurface, then find its differential geometric properties in the Euclidean 4-space $\mathbb{E}^4$. We would like to note that the definition of the rotational hypersurfaces in the Riemannian space forms were defined in [22]. A rotational hypersurface $M \subset \mathbb{E}^{n+1}$ generated by a curve γ around an axis that does not meet γ is obtained by taking the orbit of γ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed (See [22, Remark 2.3]).

By using the curve $\gamma(u) = (\mathbf{f}(u), 0, \mathbf{g}(u), 0)$ with the rotation matrix

$$\begin{pmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & \cos w & -\sin w \\ 0 & 0 & \sin w & \cos w \end{pmatrix},$$

we give the following definition.

Definition 4.1. A bi-rotational hypersurface in $\mathbb{E}^4$ is defined by

$$\mathbf{x}(u, v, w) = (\mathbf{f}(u)\cos v, \mathbf{f}(u)\sin v, \mathbf{g}(u)\cos w, \mathbf{g}(u)\sin w), \tag{4.1}$$

where $\mathbf{f}, \mathbf{g}$ are the differentiable functions, and $0 \leq v, w \leq 2\pi$.

Remark 4.1. While $\mathbf{f}(u) = \mathbf{g}(u) = 1$ in (4.1), we obtain the Clifford torus in $\mathbb{E}^4$. See [2, 48] for details. Moreover, when $v = w$ in (4.1), we get the tensor product surface in $\mathbb{E}^4$. See [5, 43] for details.

Considering the first derivatives of (4.1) with respect to u, v, w, respectively,

$$\mathbf{x}_u = \begin{pmatrix} \mathbf{f}'\cos v \\ \mathbf{f}'\sin v \\ \mathbf{g}'\cos w \\ \mathbf{g}'\sin w \end{pmatrix}, \ \mathbf{x}_v = \begin{pmatrix} -\mathbf{f}\sin v \\ \mathbf{f}\cos v \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{x}_w = \begin{pmatrix} 0 \\ 0 \\ -\mathbf{g}\sin w \\ \mathbf{g}\cos w \end{pmatrix},$$

we find the following first quantities of (4.1):

$$(g_{ij}) = diag\left(\mathbf{f}'^2 + \mathbf{g}'^2, \mathbf{f}^2, \mathbf{g}^2\right).$$

Here,

$$g = \det(g_{ij}) = \mathbf{f}^2\mathbf{g}^2\left(\mathbf{f}'^2 + \mathbf{g}'^2\right).$$

Using the (2.2), we get the following Gauss map of the bi-rotational hypersurface (4.1):

$$G = \frac{1}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\left(-\mathbf{g}'\cos v, -\mathbf{g}'\sin v, \mathbf{f}'\cos w, \mathbf{f}'\sin w\right), \tag{4.2}$$

With the help of the second derivatives with respect to u, v, w of the (4.1), and the Gauss map (4.2) of the bi-rotational hypersurface (4.1), we have the following second quantities

$$(h_{ij}) = diag\left(\frac{\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, \frac{\mathbf{f}\mathbf{g}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, -\frac{\mathbf{g}\mathbf{f}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\right). \tag{4.3}$$

So, we get

$$h = \det\left(h_{ij}\right) = -\frac{\mathbf{f}\mathbf{g}\mathbf{f}'\mathbf{g}'\left(\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}'\right)}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{3/2}}.$$

By using (4.2) and (4.3), we calculate the following shape operator matrix of the bi-rotational hypersurface (4.1):

$$\mathbf{S} = diag\left(\frac{\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{3/2}}, \frac{\mathbf{g}'}{\mathbf{f}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, -\frac{\mathbf{f}'}{\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\right).$$

Finally, by using (3.1), (3.2) and (3.3), with (4.2), (4.3), respectively, we find the following curvatures of the bi-rotational hypersurface (4.1):

Corollary 4.1. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1). *The* $\mathbf{x}$ *has the following curvatures*

$$\begin{aligned}
\mathfrak{C}_1 &= \frac{\left(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''\right)\mathbf{f}\mathbf{g} - \left(\mathbf{f}'^2 + \mathbf{g}'^2\right)\left(\mathbf{f}\mathbf{f}' - \mathbf{g}\mathbf{g}'\right)}{3\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{3/2}}, \\
\mathfrak{C}_2 &= \frac{\left(\mathbf{f}\mathbf{f}' - \mathbf{g}\mathbf{g}'\right)\left(\mathbf{g}'\mathbf{f}'' - \mathbf{f}'\mathbf{g}''\right) - \left(\mathbf{f}'^2 + \mathbf{g}'^2\right)\mathbf{f}'\mathbf{g}'}{3\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^2}, \\
\mathfrak{C}_3 &= -\frac{\mathbf{f}'\mathbf{g}'\left(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''\right)}{\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{5/2}}.
\end{aligned}$$

Example 4.1. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of $\mathbf{x}$ is parametrized by the arc length, and $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, the bi-rotational hypersurface has the following curvatures

$$\mathfrak{C}_i = 1,$$

where $i = 1, 2, 3$.

Example 4.2. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of $\mathbf{x}$ is parametrized with $\mathbf{f}(u) = \mathbf{g}(u) = \frac{u}{\sqrt{2}}$, then bi-rotational hypersurface has the following

$$\begin{aligned}
\mathfrak{C}_1 &= 0, \\
\mathfrak{C}_2 &= -\frac{1}{3u^2}, \\
\mathfrak{C}_3 &= 0.
\end{aligned}$$

5. Bi-Rotational Hypersurface Satisfying $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$

In this section, we give the Laplace–Beltrami operator of a smooth function. Then, calculate the Laplace–Beltrami operator of the bi-rotational hypersurface.

The inverse of the matrix

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

is given by

$$\frac{1}{g}\begin{pmatrix} g_{22}g_{33} - g_{23}g_{32} & -(g_{12}g_{33} - g_{13}g_{32}) & g_{12}g_{23} - g_{13}g_{22} \\ -(g_{21}g_{33} - g_{31}g_{23}) & g_{11}g_{33} - g_{13}g_{31} & -(g_{11}g_{23} - g_{21}g_{13}) \\ g_{21}g_{32} - g_{22}g_{31} & -(g_{11}g_{32} - g_{12}g_{31}) & g_{11}g_{22} - g_{12}g_{21} \end{pmatrix},$$

where

$$\begin{aligned} g &= \det(g_{ij}) \\ &= g_{11}g_{22}g_{33} - g_{11}g_{23}g_{32} + g_{12}g_{31}g_{23} - g_{12}g_{21}g_{33} + g_{21}g_{13}g_{32} - g_{13}g_{22}g_{31}. \end{aligned}$$

Definition 5.1. On $\mathbf{D} \subset \mathbb{R}^3$ the Laplace–Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3)\ |_{\mathbf{D}}$ of class C^3 with respect to the first fundamental form is the operator Δ defined by

$$\Delta\phi = \frac{1}{g^{1/2}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i}\left(g^{1/2} g^{ij} \frac{\partial \phi}{\partial x^j}\right), \tag{5.1}$$

where $\left(g^{ij}\right) = (g_{kl})^{-1}$, and $g = \det(g_{ij})$.

Clearly, we can write (5.1) as follows

$$\Delta\phi = \frac{1}{|g|^{1/2}} \left\{ \begin{array}{l} \frac{\partial}{\partial x^1}\left(|g|^{1/2} g^{11} \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^1}\left(|g|^{1/2} g^{12} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(|g|^{1/2} g^{13} \frac{\partial \phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^2}\left(|g|^{1/2} g^{21} \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(|g|^{1/2} g^{22} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^2}\left(|g|^{1/2} g^{23} \frac{\partial \phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(|g|^{1/2} g^{31} \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^3}\left(|g|^{1/2} g^{32} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(|g|^{1/2} g^{33} \frac{\partial \phi}{\partial x^3}\right) \end{array} \right\}.$$

When $i \neq j$, $g_{ij} = 0$ for any rotational hypersurface. Hence, we can re-write $\Delta\phi$ as follows

$$\Delta\phi = \frac{1}{|g|^{1/2}} \left\{ \frac{\partial}{\partial x^1}\left(|g|^{1/2} g^{11} \frac{\partial \phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(|g|^{1/2} g^{22} \frac{\partial \phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(|g|^{1/2} g^{33} \frac{\partial \phi}{\partial x^3}\right) \right\}.$$

Therefore, more clear form of the Laplace–Beltrami operator of any rotational hypersurface $\mathbf{x}(u, v, w)$ is given by

$$\Delta\mathbf{x} = \frac{1}{|g|^{1/2}} \left\{ \frac{\partial}{\partial u}\left(\frac{g_{22}g_{33}}{|g|^{1/2}} \mathbf{x}_u\right) + \frac{\partial}{\partial v}\left(\frac{g_{11}g_{33}}{|g|^{1/2}} \mathbf{x}_v\right) + \frac{\partial}{\partial w}\left(\frac{g_{11}g_{22}}{|g|^{1/2}} \mathbf{x}_w\right) \right\}. \tag{5.2}$$

Finally, by taking the derivatives of the $\frac{g_{22}g_{33}}{|g|^{1/2}}\mathbf{x}_u$, $\frac{g_{11}g_{33}}{|g|^{1/2}}\mathbf{x}_v$, $\frac{g_{11}g_{22}}{|g|^{1/2}}\mathbf{x}_w$, with respect to u, v, w, respectively, and substituting them into (5.2), we have the following.

Theorem 5.1. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. The Laplace–Beltrami operator of the bi-rotational hypersurface* $\mathbf{x}(u, v, w)$ *is given by*

$$\Delta\mathbf{x} = \begin{pmatrix} \Delta\mathbf{x}_1 \\ \Delta\mathbf{x}_2 \\ \Delta\mathbf{x}_3 \\ \Delta\mathbf{x}_4 \end{pmatrix} = \begin{pmatrix} \mathfrak{f}(u)\cos v \\ \mathfrak{f}(u)\sin v \\ \mathfrak{g}(u)\cos w \\ \mathfrak{g}(u)\sin w \end{pmatrix},$$

where

$$\begin{aligned} \mathfrak{f}(u) &= \frac{\left(\mathbf{f}'\mathbf{g} + \mathbf{f}\mathbf{g}'\right)\mathbf{f}'}{\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)} - \frac{\left(\mathbf{f}'\mathbf{f}'' + \mathbf{g}'\mathbf{g}''\right)\mathbf{f}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^2} + \frac{\mathbf{f}''}{\mathbf{f}'^2 + \mathbf{g}'^2} - \frac{1}{\mathbf{f}}, \\ \mathfrak{g}(u) &= \frac{\left(\mathbf{f}'\mathbf{g} + \mathbf{f}\mathbf{g}'\right)\mathbf{g}'}{\mathbf{f}\mathbf{g}\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)} - \frac{\left(\mathbf{f}'\mathbf{f}'' + \mathbf{g}'\mathbf{g}''\right)\mathbf{g}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^2} + \frac{\mathbf{g}''}{\mathbf{f}'^2 + \mathbf{g}'^2} - \frac{1}{\mathbf{g}}. \end{aligned}$$

6. Conclusion

Taking into account all findings, we serve the following.

Corollary 6.1. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. The bi-rotational hypersurface* $\mathbf{x}$ *satisfies* $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$, *where*

$$\mathcal{A} = diag\left(\frac{\mathfrak{f}}{\mathbf{f}}\mathcal{I}_2, \frac{\mathfrak{g}}{\mathbf{g}}\mathcal{I}_2\right),$$

and $\mathcal{A} \in Mat\,(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Corollary 6.2. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.1)*. When the curve* γ *of the bi-rotational hypersurface* $\mathbf{x}$ *is parametrized by the arc length, the* $\mathbf{x}$ *holds* $\Delta\mathbf{x} = \mathcal{B}\mathbf{x}$, *where* $\mathcal{B} = diag\,(\mathfrak{p}\mathcal{I}_2, \mathfrak{q}\mathcal{I}_2)$, *and*

$$\begin{aligned} \mathfrak{p}\,(u) &= \frac{\mathbf{f}''}{\mathbf{f}} + \frac{\mathbf{f}'\left(\log\left(\mathbf{f}\mathbf{g}\right)\right)'}{\mathbf{f}} - \frac{1}{\mathbf{f}^2}, \\ \mathfrak{q}\,(u) &= \frac{\mathbf{g}''}{\mathbf{g}} + \frac{\mathbf{g}'\left(\log\left(\mathbf{f}\mathbf{g}\right)\right)'}{\mathbf{g}} - \frac{1}{\mathbf{g}^2}, \end{aligned}$$

with $\mathcal{B} \in Mat\,(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Example 6.1. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of the $\mathbf{x}$ is parametrized by $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, the bi-rotational hypersurface $\mathbf{x}$ supplies $\Delta\mathbf{x} = \mathcal{C}\mathbf{x}$, where $\mathcal{C} = diag\,(\mathfrak{a}\mathcal{I}_2, \mathfrak{b}\mathcal{I}_2)$, $\mathfrak{a}(u) = -3$, $\mathfrak{b}(u) = -3$, and $\mathcal{I}_2$ is the identity matrix. Briefly, $\mathcal{C} = -3\mathcal{I}_4$, where $\mathcal{I}_4$ is the identity matrix.

Example 6.2. Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.1). When the curve γ of the $\mathbf{x}$ is parametrized by $\mathbf{f}(u) = \mathbf{g}(u) = \frac{u}{\sqrt{2}}$, the bi-rotational hypersurface $\mathbf{x}$ has $\Delta\mathbf{x} = \mathcal{D}\mathbf{x}$, where $\mathcal{D} = diag\,(\mathfrak{c}\mathcal{I}_2, \mathfrak{d}\mathcal{I}_2) = O_4$, and $\mathfrak{c}(u) = 0$, $\mathfrak{d}(u) = 0$. Hence, the bi-rotational hypersurface is the 1-minimal harmonic hypersurface.

Example 6.3. Taking the hypersphere $S^3(r) := \left\{\xi \in \mathbb{E}^4 \mid \langle \xi, \xi \rangle = r^2\right\}$ (for radius $r > 0$) as

$$(6.1) \qquad \xi(u, v, w) = (r \cos u \cos v, r \cos u \sin v, r \sin u \cos w, r \sin u \sin w),$$

we get $\mathbf{S} = \frac{1}{r}\mathcal{I}_3$, and we obtain the following curvatures

$$\mathfrak{C}_0 = 1,\ \mathfrak{C}_1 = H = \frac{1}{r},\ \mathfrak{C}_2 = \frac{1}{r^2},\ \mathfrak{C}_3 = K = \frac{1}{r^3}.$$

Here, $(\mathfrak{C}_1)^2 = \mathfrak{C}_2$, $\mathfrak{C}_1\mathfrak{C}_2 = \mathfrak{C}_3$, $(\mathfrak{C}_1)^3 = \mathfrak{C}_3$, i.e., $H^3 = K$. Therefore, the hypersurface (6.1) is the bi-rotational umbilical hypersphere.

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