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ON THE HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRALS*

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Abstract. In this paper, we establish new Hermite-Hadamard inequalities involving Hadamard fractional integrals, which are described by series. To achieve our aim, we use established fractional integral identities and elementary inequalities based on convex functions and monotonicity introduced in our previous works. Finally, some applications to special means of real numbers are given.

1. Introduction

Fractional calculus has been recognized as one of the best tools to describe long-memory processes. The corresponding mathematical models of these processes are fractional systems. Due to the extensive applications of fractional systems in engineering and science, research in this area has grown significantly all around the world [1, 2, 3, 4, 5, 6, 7].

Recently, fractional integral inequalities have attracted many researchers. For example, Anastassiou [8] presented the first fractional differentiation inequalities of Opial type involving so-called balanced fractional derivatives and continued with right and mixed fractional differentiation Ostrowski inequalities in the univariate and multivariate cases; Set [9] studied fractional Ostrowski inequalities involving Riemann-Liouville fractional integrals; Sarikaya et al. [10] studied Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals. Strong motivated by the contributions of the above works, our researcher group established many fractional Hermite-Hadamard inequalities involving Riemann-Liouville and Hadamard fractional integrals for different convex functions [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

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However, there are few papers [12, 16] on Hermite-Hadamard inequalities for Hadamard fractional integrals, even though Hadamard fractional integrals were presented many years ago as Riemann-Liouville fractional integrals.

For $f \in L[a,b]$, Hadamard fractional integrals [3] $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}^+$ with $a \geq 0$ are defined by

$$({}_HJ^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ (0 < a < x \leq b),$$

and

$$({}_HJ^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ (0 < a \leq x < b),$$

where $\Gamma(\cdot)$ is Gamma function.

It seems that Wang et al. [12, 16] firstly established the following two powerful fractional integral identities involving Hadamard fractional integrals for once differential functions.

Lemma 1.1. *(see Lemma 3.1, [12]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^{\alpha}}[{}_HJ^{\alpha}_{a+}f(b) + {}_HJ^{\alpha}_{b-}f(a)]\\
=\ & \frac{\ln b - \ln a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]e^{t\ln a + (1-t)\ln b} f'(e^{t\ln a + (1-t)\ln b})dt\\
=\ & \frac{\ln b - \ln a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}]a^t b^{1-t} f'(a^t b^{1-t})dt.
\end{aligned}$$

Lemma 1.2. *(see Lemma 2.1, [16]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}
&\frac{(\ln x - \ln a)^{\alpha} + (\ln b - \ln x)^{\alpha}}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha+1)}{\ln b - \ln a}[{}_HJ^{\alpha}_{x-}f(a) + {}_HJ^{\alpha}_{x+}f(b)]\\
=\ & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a}\int_0^1 t^{\alpha} e^{t\ln x + (1-t)\ln a} f'(e^{t\ln x + (1-t)\ln a})dt\\
& - \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a}\int_0^1 t^{\alpha} e^{t\ln x + (1-t)\ln b} f'(e^{t\ln x + (1-t)\ln b})dt\\
=\ & \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a}\int_0^1 t^{\alpha} x^t a^{1-t} f'(x^t a^{1-t})dt\\
& - \frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a}\int_0^1 t^{\alpha} x^t b^{1-t} f'(x^t b^{1-t})dt,
\end{aligned}$$

for any $x \in (a,b)$.

In [16], the authors explored a new concept named by s-e-condition to overcome some essential difficulties from the singular kernels in Hadamard fractional integrals. Some new Ostrowski type inequalities for Hadamard fractional integrals are obtained. In the present paper, we will use Lemma 1.1 and Lemma 1.2 via elementary equalities and inequalities via convex functions to derive new type Hermite-Hadamard inequalities involving Hadamard fractional integrals. Here, we do not need the s-e-condition [16] by using Lemma and monotonicity. Finally, we give some applications to special means of real numbers.

2. Preliminaries

In this section, we recall the definition of convex functions and collect some elementary equalities and inequalities which will be used in the sequel.

Definition 2.1. Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ and I is a convex set. A function f is said to be convex on I if for every $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y). \tag{2.1}$$

Lemma 2.1. *(see Lemma 2.1, [13]) For $\alpha > 0$ and $k > 0$,we have*

$$J(\alpha) := \int_0^1 t^{\alpha-1}k^t dt = k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty$$

where

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1).$$

Lemma 2.2. *(see Lemma 2.2, [13]) For $\alpha > 0$ and $k > 0, z > 0$,we have*

$$\int_0^z t^{\alpha-1}k^t dt = z^\alpha k^z \sum_{i=0}^{\infty}\frac{(-z\ln k)^{i-1}}{(\alpha)_i} < +\infty.$$

where

$$(\alpha)_i = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1).$$

Lemma 2.3. *(see [15]) For $A > B > 0$, it holds*

$$(A-B)^\theta \leq A^\theta - B^\theta \quad \text{when } \theta \geq 1,$$
$$(A-B)^\theta \geq A^\theta - B^\theta \quad \text{when } 0 < \theta \leq 1.$$

Lemma 2.4. *(see Lemma 2.4, [13]) For $A \geq 0, B \geq 0$, it holds*

$$(A+B)^\theta \leq 2^{\theta-1}(A^\theta + B^\theta) \quad \text{when } \theta \geq 1,$$
$$(A+B)^\theta \leq A^\theta + B^\theta \quad \text{when } 0 < \theta \leq 1.$$

Lemma 2.5. *(see Lemma 2.5, [18]) For $t \in [0,1]$, we have*

$$(1-t)^n \leq 2^{1-n} - t^n \quad \textit{when } n \in [0,1],$$
$$(1-t)^n \geq 2^{1-n} - t^n \quad \textit{when } n \in [1,\infty).$$

Lemma 2.6. *(see Lemma 2.5, [19]) For $t \in [0,1]$, $x, y > 0$, we have*

$$y^{1-t}x^t \leq tx + (1-t)y.$$

3. Main results

In this section, we will use convex functions via the results in Section 2 to derive our main results in this paper.

3.1. Main results via Lemma 1.1

Theorem 3.1. *Let $f : (0,b) \to \mathbb{R}$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is convex and increasing function on $(0,b)$ for some fixed $\alpha \in (0,\infty)$, $0 < a < b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha}\left[{}_HJ^\alpha_{a+}f(b) + {}_HJ^\alpha_{b-}f(a)\right]\right|$$
$$\leq \max\left\{\frac{\ln b - \ln a}{2}b\left[(2^{1-\alpha}+1)\left((|f'(a)|-|f'(b)|)\frac{1}{4}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(2)_i}\right.\right.\right.$$
$$\left. +|f'(b)|\frac{(ab^{-1})^{\frac{1}{2}}-1}{\ln(ab^{-1})}\right) - 4\left((|f'(a)|-|f'(b)|)\left(\frac{1}{2}\right)^{\alpha+2}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}\right.$$
$$\left. +|f'(b)|\left(\frac{1}{2}\right)^{\alpha+1}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\right)$$
$$+2\left((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}\right.$$
$$\left. +|f'(b)|(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\right)$$
$$\left.-\left((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)|\frac{(ab^{-1})-1}{\ln(ab^{-1})}\right)\right],$$
$$\frac{\ln b - \ln a}{2}b\left[(2^{1-\alpha}+1)\left((|f'(a)|-|f'(b)|)\frac{1}{4}(ab^{-1})^{\frac{1}{2}}\right.\right.$$
$$\left.\cdot\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(2)_i} + |f'(b)|\frac{(ab^{-1})^{\frac{1}{2}}-1}{\ln(ab^{-1})}\right)$$

$$
\begin{aligned}
&-4\Bigg((|f'(a)|-|f'(b)|)\left(\frac{1}{2}\right)^{\alpha+2}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}\\
&+|f'(b)|\left(\frac{1}{2}\right)^{\alpha+1}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\Bigg)\\
&+2\Bigg((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}\\
&+|f'(b)|(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\Bigg)\\
&-2^{1-\alpha}\left((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(2)_i}+|f'(b)|\frac{(ab^{-1})-1}{\ln(ab^{-1})}\right)\Bigg]\Bigg\},
\end{aligned}
$$

where $(\alpha+1)_i=(\alpha+1)(\alpha+2)\cdots(\alpha+i)$.

Proof. To achieve our aim, we divide our proof into two cases.

Case 1: $\alpha\in(0,1)$. By using Definition 2.1, Lemma 1.1, Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.5, Lemma 2.6 and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[{}_HJ^{\alpha}_{a+}f(b)+{}_HJ^{\alpha}_{b-}f(a)]\right|\\
=&\left|\frac{\ln b-\ln a}{2}\int_0^1[(1-t)^{\alpha}-t^{\alpha}]a^tb^{1-t}f'(a^tb^{1-t})dt\right|\\
\le&\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|a^tb^{1-t}|f'(a^tb^{1-t})|dt\\
\le&\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|a^tb^{1-t}|f'(at+(1-t)b)|dt\\
\le&\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
=&\frac{\ln b-\ln a}{2}\Big[\int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&+\int_{\frac{1}{2}}^1(-(1-t)^{\alpha}+t^{\alpha})a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\Big]\\
=&\frac{\ln b-\ln a}{2}\Big[\int_0^{\frac{1}{2}}(1-t)^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&-\int_0^{\frac{1}{2}}t^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&-\int_{\frac{1}{2}}^1(1-t)^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt
\end{aligned}
$$

$$
\begin{aligned}
& + \int_{\frac{1}{2}}^{1} t^{\alpha} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \Big] \\
\leq \; & \frac{\ln b - \ln a}{2} \Big[\int_{0}^{\frac{1}{2}} (2^{1-\alpha} - t^{\alpha}) a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& - \int_{0}^{\frac{1}{2}} t^{\alpha} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& - \int_{\frac{1}{2}}^{1} (1 - t^{\alpha}) a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& + \int_{\frac{1}{2}}^{1} t^{\alpha} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \Big] \\
= \; & \frac{\ln b - \ln a}{2} \Big[\int_{0}^{\frac{1}{2}} 2^{1-\alpha} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& -2 \int_{0}^{\frac{1}{2}} t^{\alpha} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \\
& - \int_{\frac{1}{2}}^{1} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt + 2 \int_{\frac{1}{2}}^{1} t^{\alpha} a^{t} b^{1-t} (t|f'(a)| + (1-t)|f'(b)|) dt \Big] \\
= \; & \frac{\ln b - \ln a}{2} \left[(2^{1-\alpha} + 1) \int_{0}^{\frac{1}{2}} a^{t} b^{1-t} \left(t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \right. \\
& -4 \int_{0}^{\frac{1}{2}} t^{\alpha} a^{t} b^{1-t} \left(t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \\
& +2 \int_{0}^{1} t^{\alpha} a^{t} b^{1-t} \left(t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \\
& \left. - \int_{0}^{1} a^{t} b^{1-t} \left(t(|f'(a)| - |f'(b)|) + |f'(b)| \right) dt \right] \\
= \; & \frac{\ln b - \ln a}{2} b \left[(2^{1-\alpha} + 1) \left(\int_{0}^{\frac{1}{2}} \left(\frac{a}{b} \right)^{t} t(|f'(a)| - |f'(b)|) dt + \int_{0}^{\frac{1}{2}} \left(\frac{a}{b} \right)^{t} |f'(b)| dt \right) \right. \\
& -4 \left(\int_{0}^{\frac{1}{2}} t^{\alpha+1} \left(\frac{a}{b} \right)^{t} (|f'(a)| - |f'(b)|) dt + \int_{0}^{\frac{1}{2}} t^{\alpha} \left(\frac{a}{b} \right)^{t} |f'(b)| dt \right) \\
& +2 \left(\int_{0}^{1} t^{\alpha+1} \left(\frac{a}{b} \right)^{t} (|f'(a)| - |f'(b)|) dt + \int_{0}^{1} t^{\alpha} \left(\frac{a}{b} \right)^{t} |f'(b)| dt \right) \\
& \left. - \left(\int_{0}^{1} \left(\frac{a}{b} \right)^{t} t(|f'(a)| - |f'(b)|) dt + \int_{0}^{1} \left(\frac{a}{b} \right)^{t} |f'(b)| dt \right) \right] \\
= \; & \frac{\ln b - \ln a}{2} b \left[(2^{1-\alpha} + 1) \left((|f'(a)| - |f'(b)|) \frac{1}{4} (ab^{-1})^{\frac{1}{2}} \sum_{i=0}^{\infty} \frac{(-\frac{1}{2} \ln(ab^{-1}))^{i-1}}{(2)_i} \right. \right.
\end{aligned}
$$

$$+|f'(b)|\frac{(ab^{-1})^{\frac{1}{2}}-1}{\ln(ab^{-1})}\Bigg)-4\Bigg((|f'(a)|-|f'(b)|)\left(\frac{1}{2}\right)^{\alpha+2}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}$$

$$+|f'(b)|\left(\frac{1}{2}\right)^{\alpha+1}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\Bigg)$$

$$+2\Bigg((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}$$

$$+|f'(b)|(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\Bigg)$$

$$-\Bigg((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(2)_i}+|f'(b)|\frac{(ab^{-1})-1}{\ln(ab^{-1})}\Bigg)\Bigg].$$

Case 2: $\alpha\in[1,\infty)$. By using Definition 2.1, Lemma 1.1, Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.5, Lemma2.6 and Hölder inequality again, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}[{}_HJ^{\alpha}_{a^+}f(b)+{}_HJ^{\alpha}_{b^-}f(a)]\right|\\
=\ &\left|\frac{\ln b-\ln a}{2}\int_0^1[(1-t)^{\alpha}-t^{\alpha}]a^tb^{1-t}f'(a^tb^{1-t})dt\right|\\
\leq\ &\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|a^tb^{1-t}|f'(a^tb^{1-t})|dt\\
\leq\ &\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|a^tb^{1-t}|f'(at+(1-t)b)|dt\\
\leq\ &\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
=\ &\frac{\ln b-\ln a}{2}\Big[\int_0^{\frac{1}{2}}((1-t)^{\alpha}-t^{\alpha})a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&+\int_{\frac{1}{2}}^1(-(1-t)^{\alpha}+t^{\alpha})a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\Big]\\
=\ &\frac{\ln b-\ln a}{2}\Big[\int_0^{\frac{1}{2}}(1-t)^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&-\int_0^{\frac{1}{2}}t^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&-\int_{\frac{1}{2}}^1(1-t)^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\\
&+\int_{\frac{1}{2}}^1t^{\alpha}a^tb^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\Big]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{\ln b-\ln a}{2}\Big[\int_0^{\frac{1}{2}}(1-t^\alpha)a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
& -\int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
& -\int_{\frac{1}{2}}^1 (2^{1-\alpha}-t^\alpha)a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
& +\int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\Big] \\
=\ & \frac{\ln b-\ln a}{2}\Big[\int_0^{\frac{1}{2}} a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
& -2\int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
& -\int_{\frac{1}{2}}^1 2^{1-\alpha} a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt \\
& +2\int_{\frac{1}{2}}^1 t^\alpha a^t b^{1-t}(t|f'(a)|+(1-t)|f'(b)|)dt\Big] \\
=\ & \frac{\ln b-\ln a}{2}\Bigg[(2^{1-\alpha}+1)\int_0^{\frac{1}{2}} a^t b^{1-t}(t(|f'(a)|-|f'(b)|)+|f'(b)|)dt \\
& -4\int_0^{\frac{1}{2}} t^\alpha a^t b^{1-t}(t(|f'(a)|-|f'(b)|)+|f'(b)|)dt \\
& +2\int_0^1 t^\alpha a^t b^{1-t}(t(|f'(a)|-|f'(b)|)+|f'(b)|)dt \\
& -\int_0^1 2^{1-\alpha} a^t b^{1-t}(t(|f'(a)|-|f'(b)|)+|f'(b)|)dt\Bigg] \\
=\ & \frac{\ln b-\ln a}{2}b\Bigg[(2^{1-\alpha}+1)\left(\int_0^{\frac{1}{2}}\left(\frac{a}{b}\right)^t t(|f'(a)|-|f'(b)|)dt+\int_0^{\frac{1}{2}}\left(\frac{a}{b}\right)^t|f'(b)|dt\right) \\
& -4\left(\int_0^{\frac{1}{2}} t^{\alpha+1}\left(\frac{a}{b}\right)^t(|f'(a)|-|f'(b)|)dt+\int_0^{\frac{1}{2}} t^\alpha\left(\frac{a}{b}\right)^t|f'(b)|dt\right) \\
& +2\left(\int_0^1 t^{\alpha+1}\left(\frac{a}{b}\right)^t(|f'(a)|-|f'(b)|)dt+\int_0^1 t^\alpha\left(\frac{a}{b}\right)^t|f'(b)|dt\right) \\
& -\left(\int_0^1 2^{1-\alpha}\left(\frac{a}{b}\right)^t t(|f'(a)|-|f'(b)|)dt+\int_0^1 2^{1-\alpha}\left(\frac{a}{b}\right)^t|f'(b)|dt\right)\Bigg] \\
=\ & \frac{\ln b-\ln a}{2}b\Big[(2^{1-\alpha}+1)\Big((|f'(a)|-|f'(b)|)\frac{1}{4}(ab^{-1})^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(2)_i}+|f'(b)|\frac{(ab^{-1})^{\frac{1}{2}}-1}{\ln(ab^{-1})}\Bigg)\\
&-4\Bigg((|f'(a)|-|f'(b)|)\left(\frac{1}{2}\right)^{\alpha+2}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}\\
&+|f'(b)|\left(\frac{1}{2}\right)^{\alpha+1}(ab^{-1})^{\frac{1}{2}}\sum_{i=0}^{\infty}\frac{(-\frac{1}{2}\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\Bigg)\\
&+2\Bigg((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+2)_i}\\
&+|f'(b)|(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(\alpha+1)_i}\Bigg)\\
&-2^{1-\alpha}\left((|f'(a)|-|f'(b)|)(ab^{-1})\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(ab^{-1}))^{i-1}}{(2)_i}+|f'(b)|\frac{(ab^{-1})-1}{\ln(ab^{-1})}\right)\Bigg].
\end{aligned}
$$

The proof is done. □

Theorem 3.2. *Let $f:(0,b)\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f'|^q$ is measurable and $|f'|^q$ is convex and $|f'|$ is the increasing function on $(0,b)$ for some fixed $\alpha\in(0,\infty)$, $0<a<b$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}\left[{}_HJ^\alpha_{b-}f(a)+{}_HJ^\alpha_{a+}f(b)\right]\right|\\
\leq\ &\frac{\ln b-\ln a}{2}b\left(\frac{2-2(\frac{1}{2})^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|\left(\left(\frac{a}{b}\right)^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(q\ln\frac{a}{b})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right.\\
&\left.+|f'(b)|\left(\frac{(\frac{a}{b})^q-1}{q\ln\frac{a}{b}}-\left(\frac{a}{b}\right)^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(q\ln\frac{a}{b})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 2.1, Lemma 1.1, Lemma 2.3, Lemma 2.5, Lemma2.6 and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^\alpha}[{}_HJ^\alpha_{a+}f(b)+{}_HJ^\alpha_{b-}f(a)]\right|\\
=\ &\left|\frac{\ln b-\ln a}{2}\int_0^1[(1-t)^\alpha-t^\alpha]a^tb^{1-t}f'(a^tb^{1-t})dt\right|\\
\leq\ &\frac{\ln b-\ln a}{2}\int_0^1|(1-t)^\alpha-t^\alpha|a^tb^{1-t}|f'(a^tb^{1-t})|dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\ln b - \ln a}{2} b \int_0^1 |(1-t)^\alpha - t^\alpha| \left(\frac{a}{b}\right)^t |f'(at+(1-t)b)| dt \\
&\leq \frac{\ln b - \ln a}{2} b \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} |f'(at+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&= \frac{\ln b - \ln a}{2} b \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)^p dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)^p dt\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} |f'(at+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{\ln b - \ln a}{2} b \left(\int_0^{\frac{1}{2}} ((1-t)^{\alpha p} - t^{\alpha p}) dt + \int_{\frac{1}{2}}^1 (t^{\alpha p} - (1-t)^{\alpha p}) dt\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} |f'(at+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{\ln b - \ln a}{2} b \left(\int_0^{\frac{1}{2}} ((1-t)^{\alpha p} - t^{\alpha p}) dt + \int_{\frac{1}{2}}^1 (t^{\alpha p} - (1-t)^{\alpha p}) dt\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} (t|f'(a)|^q + (1-t)|f'(b)|^q) dt\right)^{\frac{1}{q}} \\
&= \frac{\ln b - \ln a}{2} b \left(\int_0^{\frac{1}{2}} (1-t)^{\alpha p} dt - \int_0^{\frac{1}{2}} t^{\alpha p} dt + \int_{\frac{1}{2}}^1 t^{\alpha p} dt - \int_{\frac{1}{2}}^1 (1-t)^{\alpha p} dt\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} t|f'(a)|^q dt + \int_0^1 \left(\frac{a}{b}\right)^{qt} (1-t)|f'(b)|^q dt\right)^{\frac{1}{q}} \\
&= \frac{\ln b - \ln a}{2} b \left(-\frac{(\frac{1}{2})^{\alpha p+1} - 1}{\alpha p+1} - \frac{(\frac{1}{2})^{\alpha p+1}}{\alpha p+1} + \frac{1-(\frac{1}{2})^{\alpha p+1}}{\alpha p+1} + \frac{(\frac{1}{2})^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} t|f'(a)|^q dt + \int_0^1 \left(\frac{a}{b}\right)^{qt} (1-t)|f'(b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{\ln b - \ln a}{2} b \left(\frac{2-2(\frac{1}{2})^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}} \left[\left(\int_0^1 \left(\frac{a}{b}\right)^{qt} t|f'(a)|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} (1-t)|f'(b)|^q dt\right)^{\frac{1}{q}}\right] \\
&= \frac{\ln b - \ln a}{2} b \left(\frac{2-2(\frac{1}{2})^{\alpha p+1}}{\alpha p+1}\right)^{\frac{1}{p}} \left[|f'(a)| \left(\left(\frac{a}{b}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q\ln\frac{a}{b})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right. \\
&\quad \left. + |f'(b)| \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} dt - \int_0^1 t\left(\frac{a}{b}\right)^{qt} dt\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$= \frac{\ln b - \ln a}{2} b \left(\frac{2 - 2(\frac{1}{2})^{\alpha p+1}}{\alpha p + 1}\right)^{\frac{1}{p}} \Bigg[|f'(a)| \left(\left(\frac{a}{b}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}$$
$$+|f'(b)| \left(\frac{(\frac{a}{b})^q - 1}{q \ln \frac{a}{b}} - \left(\frac{a}{b}\right)^q \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(q \ln \frac{a}{b})^{i-1}}{(2)_i}\right)^{\frac{1}{q}} \Bigg].$$

The proof is done. □

3.2. Main results via Lemma 1.2

Theorem 3.3. *Let $f : (0, b) \to \mathbb{R}$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is convex and increasing function on $(0, b)$ for some fixed $\alpha \in (0, \infty)$, $0 < a < b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{(\ln x - \ln a)^{\alpha} + (\ln b - \ln x)^{\alpha}}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} \left[{}_H J^{\alpha}_{x-} f(a) +_H J^{\alpha}_{x+} f(b)\right]\right|$$
$$\leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \Bigg(x(|f'(x)| - |f'(a)|) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha + 2)_i}$$
$$+x|f'(a)| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xa^{-1}))^{i-1}}{(\alpha + 1)_i} \Bigg)$$
$$+\frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \Bigg(x(|f'(x)| - |f'(b)|) \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha + 2)_i}$$
$$+x|f'(b)| \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln(xb^{-1}))^{i-1}}{(\alpha + 1)_i} \Bigg)$$

for any $x \in (a, b)$.

Proof. By using Definition 2.1, Lemma 1.2, Lemma 2.1 and Lemma 2.6, we have

$$\left|\frac{(\ln x - \ln a)^{\alpha} + (\ln b - \ln x)^{\alpha}}{\ln b - \ln a} f(x) - \frac{\Gamma(\alpha + 1)}{\ln b - \ln a} \left[{}_H J^{\alpha}_{x-} f(a) +_H J^{\alpha}_{x+} f(b)\right]\right|$$
$$= \Bigg|\frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t a^{1-t} f'\left(x^t a^{1-t}\right) dt$$
$$-\frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t b^{1-t} f'\left(x^t b^{1-t}\right) dt\Bigg|$$
$$\leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t a^{1-t} |f'\left(x^t a^{1-t}\right)| dt$$
$$+\frac{(\ln b - \ln x)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t b^{1-t} |f'\left(x^t b^{1-t}\right)| dt$$
$$\leq \frac{(\ln x - \ln a)^{\alpha+1}}{\ln b - \ln a} \int_0^1 t^{\alpha} x^t a^{1-t} |f'(tx + (1 - t)a)| dt$$

$$
\begin{aligned}
& +\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t b^{1-t}|f'(tx+(1-t)b)|dt \\
\leq\; & \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t a^{1-t}(t|f'(x)|+(1-t)|f'(a)|)dt \\
& +\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t b^{1-t}(t|f'(x)|+(1-t)|f'(b)|)dt \\
=\; & \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 \Big(t^{\alpha+1}x^t a^{1-t}|f'(x)|+t^\alpha x^t a^{1-t}|f'(a)|-t^{\alpha+1}x^t a^{1-t}|f'(a)|\Big)dt \\
& +\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 \Big(t^{\alpha+1}x^t b^{1-t}|f'(x)|+t^\alpha x^t b^{1-t}|f'(b)|-t^{\alpha+1}x^t b^{1-t}|f'(b)|\Big)dt \\
=\; & \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\Big(a|f'(x)|\int_0^1 t^{\alpha+1}(xa^{-1})^t dt+a|f'(a)|\int_0^1 t^\alpha (xa^{-1})^t dt \\
& -a|f'(a)|\int_0^1 t^{\alpha+1}(xa^{-1})^t dt\Big)+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\Big(b|f'(x)|\int_0^1 t^{\alpha+1}(xb^{-1})^t dt \\
& +b|f'(b)|\int_0^1 t^\alpha (xb^{-1})^t dt-b|f'(b)|\int_0^1 t^{\alpha+1}(xb^{-1})^t dt\Big) \\
=\; & \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\Big(a|f'(x)|xa^{-1}\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+2)_i} \\
& +a|f'(a)|xa^{-1}\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+1)_i}-a|f'(a)|xa^{-1}\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+2)_i}\Big) \\
& +\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\Big(b|f'(x)|xb^{-1}\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+2)_i} \\
& +b|f'(b)|xb^{-1}\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+1)_i}-b|f'(b)|xb^{-1}\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+2)_i}\Big) \\
=\; & \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\Big(x(|f'(x)|-|f'(a)|)\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+2)_i} \\
& +x|f'(a)|\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xa^{-1}))^{i-1}}{(\alpha+1)_i}\Big) \\
& +\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\Big(x(|f'(x)|-|f'(b)|)\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+2)_i} \\
& +x|f'(b)|\sum_{i=1}^\infty (-1)^{i-1}\frac{(\ln(xb^{-1}))^{i-1}}{(\alpha+1)_i}\Big).
\end{aligned}
$$

The proof is completed. □

Theorem 3.4. *Let $f:(0,b)\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If*

$|f'|^q$ is measurable and $|f'|^q$ is convex and $|f'|$ is the increasing function on $(0,b)$ for some fixed $\alpha \in (0,\infty)$, $0<a<b$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]\right|\\
\leq\ &\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[a|f'(x)|\left(\left(\frac{x}{a}\right)^{q}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{a})^{q})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right.\\
&\left.+x|f'(a)|\left(\left(\frac{a}{x}\right)^{q}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{a}{x})^{q})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right]\\
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[b|f'(x)|\left(\left(\frac{x}{b}\right)^{q}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{b})^{q})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right.\\
&\left.+x|f'(b)|\left(\left(\frac{b}{x}\right)^{q}\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{b}{x})^{q})^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Definition 2.1, Lemma 1.2, Lemma 2.1, Lemma 2.4, Lemma 2.6 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]\right|\\
=\ &\left|\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^{t}a^{1-t}f'\left(x^{t}a^{1-t}\right)dt\right.\\
&\left.-\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^{t}b^{1-t}f'\left(x^{t}b^{1-t}\right)dt\right|\\
\leq\ &\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^{t}a^{1-t}|f'(x^{t}a^{1-t})|dt\\
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}x^{t}b^{1-t}|f'\left(x^{t}b^{1-t}\right)|dt\\
\leq\ &\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left(x^{t}a^{1-t}|f'\left(x^{t}a^{1-t}\right)|\right)^{q}dt\right)^{\frac{1}{q}}\\
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left(x^{t}b^{1-t}|f'\left(x^{t}b^{1-t}\right)|\right)^{q}dt\right)^{\frac{1}{q}}\\
\leq\ &\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 (x^{t}a^{1-t})^{q}|f'(tx+(1-t)a)|^{q}dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 (x^t b^{1-t})^q|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\\
\leq\;& \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 x^{qt}a^{q(1-t)}(t|f'(x)|^q+(1-t)|f'(a)|^q)dt\right)^{\frac{1}{q}}\\
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1 x^{qt}b^{q(1-t)}(t|f'(x)|^q+(1-t)|f'(b)|^q)dt\right)^{\frac{1}{q}}\\
\leq\;& \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left[\left(\int_0^1 x^{tq}a^{(1-t)q}t|f'(x)|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 x^{tq}a^{(1-t)q}(1-t)|f'(a)|^q dt\right)^{\frac{1}{q}}\right]\\
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left[\left(\int_0^1 x^{tq}b^{(1-t)q}t|f'(x)|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 x^{tq}b^{(1-t)q}(1-t)|f'(b)|^q dt\right)^{\frac{1}{q}}\right]\\
=\;& \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[a|f'(x)|\left(\int_0^1\left(\frac{x}{a}\right)^{tq}tdt\right)^{\frac{1}{q}}\right.\\
&\left.+x|f'(a)|\left(\int_0^1\left(\frac{a}{x}\right)^{t'q}t'dt'\right)^{\frac{1}{q}}\right]+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\\
&\cdot\left[b|f'(x)|\left(\int_0^1\left(\frac{x}{b}\right)^{tq}tdt\right)^{\frac{1}{q}}+x|f'(b)|\left(\int_0^1\left(\frac{b}{x}\right)^{t'q}t'dt'\right)^{\frac{1}{q}}\right]\\
=\;& \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[a|f'(x)|\left(\left(\frac{x}{a}\right)^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{a})^q)^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right.\\
&\left.+x|f'(a)|\left(\left(\frac{a}{x}\right)^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{a}{x})^q)^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right]\\
&+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[b|f'(x)|\left(\left(\frac{x}{b}\right)^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{x}{b})^q)^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right.\\
&\left.+x|f'(b)|\left(\left(\frac{b}{x}\right)^q\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln(\frac{b}{x})^q)^{i-1}}{(2)_i}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

The proof is done. □

4. Applications to special means

Consider the following special means (see [26]) for arbitrary real numbers $x, y, x \neq y$ as follows:

(i) $A(x,y) = \frac{x+y}{2}$, $x, y \in \mathbb{R}$.

(ii) $L(x,y) = \frac{y-x}{\ln|y| - \ln|x|}$, $|x| \neq |y|, xy \neq 0$.

(iii) $L_n(x,y) = \left[\frac{y^{n+1}-x^{n+1}}{(n+1)(y-x)}\right]^{\frac{1}{n}}$, $n \in \mathbb{Z} \setminus \{-1, 0\}, x, y \in \mathbb{R}, x \neq y$.

Using the results in Section 3., we give some applications to special means of real numbers.

Proposition 4.1. *Let $a, b \in \mathbb{R}^+ \setminus \{0\}, 0 < a < b, x \in [0, b]$. Then*

$$
\begin{aligned}
& \left| A(a,b) - L(a,b) \right| \\
\leq \quad & \frac{a(\ln a - \ln b) - (\sqrt{a} - \sqrt{b})^2}{\ln b - \ln a} + \frac{a+b}{2}; \\
& \left| A(a,b) - L(a,b) \right| \\
\leq \quad & \frac{\ln b - \ln a}{2} \left(\frac{2 - 2(\frac{1}{2})^{p+1}}{p+1} \right)^{\frac{1}{p}} \left[\left(\frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\frac{a^q - b^q}{q(\ln a - \ln b)} - \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2} \right)^{\frac{1}{q}} \right]; \\
& \left| A(x,x) - L(a,b) \right| \\
\leq \quad & \frac{(2\ln x - \ln a - \ln b - 2)x + a + b}{\ln b - \ln a}; \\
& \left| A(x,x) - L(a,b) \right| \\
\leq \quad & \frac{(\ln x - \ln a)^2}{\ln b - \ln a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(\frac{x^q(\ln x - \ln a)q - x^q + a^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\frac{a^q(\ln a - \ln x)q - a^q + x^q}{q^2(\ln x - \ln a)^2} \right)^{\frac{1}{q}} \right] + \frac{(\ln b - \ln x)^2}{\ln b - \ln a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\
& \cdot \left[\left(\frac{x^q(\ln x - \ln b)q - x^q + b^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} + \left(\frac{b^q(\ln b - \ln x)q - b^q + x^q}{q^2(\ln x - \ln b)^2} \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Proof. Applying Theorems 3.1, 3.2, 3.3, 3.4, for $f(x) = x$ and $\alpha = 1$, one can obtain the results immediately. $\square$

Proposition 4.2. *Let* $a, b \in \mathbb{R}^+ \setminus \{0\}, 0 < a < b, x \in [0, b], n \geq 2$. *Then*

$$\begin{aligned}
&\left|A(a^n, b^n) - \frac{1}{2}L(a,b)L_{n-1}^{n-1}(a,b)\right| \\
\leq\ & \frac{1}{2}(na^{n-1} - 3nb^{n-1})\left(a - \sqrt{ab} + \frac{(\sqrt{a}-\sqrt{b})^2}{\ln a - \ln b}\right) + \frac{1}{2}nb^{n-1}(\sqrt{a}-\sqrt{b})^2 \\
&+(na^{n-1} - nb^{n-1})\left(-a + \frac{\sqrt{ab}}{2} - \frac{2\sqrt{ab}-2a}{\ln a - \ln b} - \frac{2(\sqrt{a}-\sqrt{b})^2}{(\ln a - \ln b)^2}\right) \\
&\left|A(a^n, b^n) - \frac{1}{2}L(a,b)L_{n-1}^{n-1}(a,b)\right|; \\
\leq\ & \frac{\ln b - \ln a}{2}\left(\frac{2 - 2(\frac{1}{2})^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[na^{n-1}\left(\frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2}\right)^{\frac{1}{q}}\right. \\
&\left.+nb^{n-1}\left(\frac{a^q - b^q}{q(\ln a - \ln b)} - \frac{a^q(\ln a - \ln b)q - a^q + b^q}{q^2(\ln a - \ln b)^2}\right)^{\frac{1}{q}}\right]; \\
&\left|A(x^n, x^n) - L(a,b)L_{n-1}^{n-1}(a,b)\right| \\
\leq\ & \frac{nx^n(\ln x - \ln a)^2 - n(\ln x - \ln a)(a^{n-1}x + a^n - 2x^n) + 2n(x-a)(-a^{n-1} + x^{n-1})}{(\ln b - \ln a)(\ln x - \ln a)} \\
&+\frac{nx^n(\ln x - \ln b)^2 - n(\ln x - \ln b)(b^{n-1}x + b^n - 2x^n) + 2n(x-b)(-b^{n-1} + x^{n-1})}{(\ln b - \ln a)(\ln x - \ln b)}; \\
&\left|A(x^n, x^n) - L(a,b)L_{n-1}^{n-1}(a,b)\right| \\
\leq\ & \frac{(\ln x - \ln a)^2}{\ln b - \ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[nx^{n-1}\left(\frac{x^q(\ln x - \ln a)q - x^q + a^q}{q^2(\ln x - \ln a)^2}\right)^{\frac{1}{q}}\right. \\
&\left.+na^{n-1}\left(\frac{a^q(\ln a - \ln x)q - a^q + x^q}{q^2(\ln x - \ln a)^2}\right)^{\frac{1}{q}}\right] + \frac{(\ln b - \ln x)^2}{\ln b - \ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\cdot\left[nx^{n-1}\left(\frac{x^q(\ln x - \ln b)q - x^q + b^q}{q^2(\ln x - \ln b)^2}\right)^{\frac{1}{q}} + nb^{n-1}\left(\frac{b^q(\ln b - \ln x)q - b^q + x^q}{q^2(\ln x - \ln b)^2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Applying Theorems 3.1, 3.2, 3.3, 3.4, for $f(x) = x^n$ and $\alpha = 1$, one can obtain the results immediately. □

Proposition 4.3. *Let* $a, b \in \mathbb{R}^+ \setminus \{0\}, 0 < a < b, x \in [0, b], n \geq 2$. *Then*

$$\begin{aligned}
&\left|A\left(\frac{1}{a}, \frac{1}{b}\right) - \frac{1}{2}L\left(\frac{1}{a}, \frac{1}{b}\right)\right| \\
\leq\ & \frac{1}{2}\left(\frac{1}{a^2} - \frac{3}{b^2}\right)\left(a - \sqrt{ab} - \frac{(\sqrt{a}-\sqrt{b})^2}{\ln a - \ln b}\right) + \frac{1}{2b^2}(\sqrt{a}-\sqrt{b})^2
\end{aligned}$$

$$+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(-a+\frac{1}{2}\sqrt{ab}-\frac{2\sqrt{ab}-2a}{\ln a-\ln b}-\frac{2(\sqrt{a}-\sqrt{b})^2}{(\ln a-\ln b)^2}\right);$$

$$\left|A\left(\frac{1}{a},\frac{1}{b}\right)-\frac{1}{2}L\left(\frac{1}{a},\frac{1}{b}\right)\right|$$
$$\leq \frac{\ln b-\ln a}{2}\left(\frac{2-2(\frac{1}{2})^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{a^2}\left(\frac{a^q(\ln a-\ln b)q-a^q+b^q}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{b^2}\left(\frac{a^q-b^q}{q(\ln a-\ln b)}-\frac{a^q(\ln a-\ln b)q-a^q+b^q}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right];$$

$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-L\left(\frac{1}{a},\frac{1}{b}\right)\right|$$
$$\leq \frac{1}{\ln b-\ln a}\left[\left(\frac{1}{x^2}-\frac{1}{a^2}\right)\frac{x(\ln x-\ln a)^2-2x(\ln x-\ln a)+2x-2a}{\ln x-\ln a}\right.$$
$$\left.+\frac{x(\ln x-\ln a)-x+a}{a^2}\right]+\frac{1}{\ln b-\ln a}\left[\left(\frac{1}{x^2}-\frac{1}{b^2}\right)\right.$$
$$\left.\cdot\frac{x(\ln x-\ln b)^2-2x(\ln x-\ln b)+2x-2b}{\ln x-\ln b}+\frac{x(\ln x-\ln b)-x+b}{b^2}\right];$$

$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-L\left(\frac{1}{a},\frac{1}{b}\right)\right|$$
$$\leq \frac{(\ln x-\ln a)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{x^2}\left(\frac{x^q(\ln x-\ln a)q-x^q+a^q}{q^2(\ln x-\ln a)^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{a^2}\left(\frac{a^q(\ln a-\ln x)q-a^q+x^q}{q^2(\ln x-\ln a)^2}\right)^{\frac{1}{q}}\right]+\frac{(\ln b-\ln x)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\cdot\left[\frac{1}{x^2}\left(\frac{x^q(\ln x-\ln b)q-x^q+b^q}{q^2(\ln x-\ln b)^2}\right)^{\frac{1}{q}}+\frac{1}{b^2}\left(\frac{b^q(\ln b-\ln x)q-b^q+x^q}{q^2(\ln x-\ln b)^2}\right)^{\frac{1}{q}}\right].$$

Proof. Applying Theorems 3.1, 3.2, 3.3, 3.4, for $f(x)=\frac{1}{x}$ and $\alpha=1$, one can obtain the results immediately. □

Proposition 4.4. *Let* $a,b\in\mathbb{R}^+\setminus\{0\}, 0<a<b, x\in[0,b], n\geq 2$. *Then*

$$\left|A(x,x)-L(b^{-1},a^{-1})\right|$$
$$\leq \frac{(2\ln x+\ln a+\ln b-2)x+a^{-1}+b^{-1}}{\ln b-\ln a};$$

$$\left|A(x,x)-L(b^{-1},a^{-1})\right|$$
$$\leq \frac{(\ln x+\ln b)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{x^q(\ln x+\ln b)q-x^q+b^{-q}}{q^2(\ln x+\ln b)^2}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{b^{-q}(-\ln b-\ln x)q-b^{-q}+x^q}{q^2(\ln x+\ln b)^2}\right)^{\frac{1}{q}}\Bigg]+\frac{(\ln a+\ln x)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\cdot\left[\left(\frac{x^q(\ln x+\ln a)q-x^q+a^{-q}}{q^2(\ln x+\ln a)^2}\right)^{\frac{1}{q}}+\left(\frac{a^{-q}(-\ln a-\ln x)q-a^{-q}+x^q}{q^2(\ln x+\ln a)^2}\right)^{\frac{1}{q}}\right];$$

$$\left|A(b^{-1},a^{-1})-L(b^{-1},a^{-1})\right|$$

$$\leq \frac{b^{-1}(\ln a-\ln b)-(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2}{\ln b-\ln a}+\frac{b^{-1}+a^{-1}}{2};$$

$$\left|A(b^{-1},a^{-1})-L(b^{-1},a^{-1})\right|$$

$$\leq \frac{\ln b-\ln a}{2}\left(\frac{2-2(\frac{1}{2})^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{b^{-q}(\ln a-\ln b)q-b^{-q}+a^{-q}}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{b^{-q}-a^{-q}}{q(\ln a-\ln b)}-\frac{b^{-q}(\ln a-\ln b)q-b^{-q}+a^{-q}}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right];$$

$$\left|A(x^n,x^n)-L(b^{-1},a^{-1})L_{n-1}^{n-1}(b^{-1},a^{-1})\right|$$

$$\leq \frac{nx^n(\ln x+\ln b)^2-n(\ln x+\ln b)(b^{1-n}x+b^{-n}-2x^n)+2n(x-b^{-1})(-b^{1-n}+x^{n-1})}{(\ln b-\ln a)(\ln x+\ln b)}$$

$$+\frac{nx^n(\ln x+\ln a)^2-n(\ln x+\ln a)(a^{1-n}x+a^{-n}-2x^n)+2n(x-a^{-1})(-a^{1-n}+x^{n-1})}{(\ln b-\ln a)(\ln x+\ln a)};$$

$$\left|A(x^n,x^n)-L(b^{-1},a^{-1})L_{n-1}^{n-1}(b^{-1},a^{-1})\right|$$

$$\leq \frac{(\ln x+\ln b)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[nx^{n-1}\left(\frac{x^q(\ln x+\ln b)q-x^q+b^{-q}}{q^2(\ln x+\ln b)^2}\right)^{\frac{1}{q}}\right.$$

$$\left.+nb^{1-n}\left(\frac{b^{-q}(-\ln b-\ln x)q-b^{-q}+x^q}{q^2(\ln x+\ln b)^2}\right)^{\frac{1}{q}}\right]+\frac{(\ln x+\ln a)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\cdot\left[nx^{n-1}\left(\frac{x^q(\ln x+\ln a)q-x^q+a^{-q}}{q^2(\ln x+\ln a)^2}\right)^{\frac{1}{q}}+na^{1-n}\left(\frac{a^{-q}(-\ln a-\ln x)q-a^{-q}+x^q}{q^2(\ln x+\ln a)^2}\right)^{\frac{1}{q}}\right];$$

$$\left|A(b^{-n},a^{-n})-\frac{1}{2}L(b^{-1},a^{-1})L_{n-1}^{n-1}(b^{-1},a^{-1})\right|$$

$$\leq \frac{1}{2}(nb^{1-n}-3na^{1-n})\left(b^{-1}-\sqrt{b^{-1}a^{-1}}+\frac{(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2}{\ln a-\ln b}\right)+\frac{1}{2}na^{1-n}(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2$$

$$+(nb^{1-n}-na^{1-n})\left(-b^{-1}+\frac{\sqrt{b^{-1}a^{-1}}}{2}-\frac{2\sqrt{b^{-1}a^{-1}}-2a^{-1}}{\ln a-\ln b}-\frac{2(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2}{(\ln a-\ln b)^2}\right);$$

$$\left|A(b^{-n},a^{-n})-\frac{1}{2}L(b^{-1},a^{-1})L_{n-1}^{n-1}(b^{-1},a^{-1})\right|$$

$$\leq \frac{\ln b-\ln a}{2}\left(\frac{2-2(\frac{1}{2})^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[nb^{1-n}\left(\frac{b^{-q}(\ln a-\ln b)q-b^{-q}+a^{-q}}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right.$$

$$\left.+na^{1-n}\left(\frac{b^{-q}-a^{-q}}{q(\ln a-\ln b)}-\frac{b^{-q}(\ln a-\ln b)q-b^{-q}+a^{-q}}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right];$$

$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-L(b,a)\right|$$

$$\leq \frac{1}{\ln b-\ln a}\Big[(\frac{1}{x^2}-b^2)\frac{x(\ln x+\ln b)^2-2x(\ln x+\ln b)+2x-2b^{-1}}{\ln x+\ln b}$$

$$+b^2\left(x(\ln x+\ln b)-x+b^{-1}\right)\Big]+\frac{1}{\ln b-\ln a}\Big[(\frac{1}{x^2}-a^2)$$

$$\cdot\frac{x(\ln x+\ln a)^2-2x(\ln x+\ln a)+2x-2a^{-1}}{\ln x+\ln a}+a^2\left(x(\ln x+\ln a)-x+a^{-1}\right)\Big];$$

$$\left|A\left(\frac{1}{x},\frac{1}{x}\right)-L(b,a)\right|$$

$$\leq \frac{(\ln x+\ln b)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{x^2}\left(\frac{x^q(\ln x+\ln b)q-x^q+b^{-q}}{q^2(\ln x+\ln b)^2}\right)^{\frac{1}{q}}\right.$$

$$\left.+b^2\left(\frac{b^{-q}(-\ln b-\ln x)q-b^{-q}+x^q}{q^2(\ln x+\ln b)^2}\right)^{\frac{1}{q}}\right]+\frac{(\ln a+\ln x)^2}{\ln b-\ln a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\cdot\left[\frac{1}{x^2}\left(\frac{x^q(\ln x+\ln a)q-x^q+a^{-q}}{q^2(\ln x+\ln a)^2}\right)^{\frac{1}{q}}+a^2\left(\frac{a^{-q}(-\ln a-\ln x)q-a^{-q}+x^q}{q^2(\ln x+\ln a)^2}\right)^{\frac{1}{q}}\right];$$

$$\left|A(b,a)-\frac{1}{2}L(b,a)\right|$$

$$\leq \frac{1}{2}(b^2-3a^2)\bigg(b^{-1}-\sqrt{b^{-1}a^{-1}}-\frac{(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2}{\ln a-\ln b}\bigg)+\frac{1}{2}a^2(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2$$

$$+(b^2-a^2)\bigg(-b^{-1}+\frac{1}{2}\sqrt{b^{-1}a^{-1}}-\frac{2\sqrt{b^{-1}a^{-1}}-2b^{-1}}{\ln a-\ln b}-\frac{2(\sqrt{b^{-1}}-\sqrt{a^{-1}})^2}{(\ln a-\ln b)^2}\bigg);$$

$$\left|A(b,a)-\frac{1}{2}L(b,a)\right|$$

$$\leq \frac{\ln b-\ln a}{2}\left(\frac{2-2(\frac{1}{2})^{p+1}}{p+1}\right)^{\frac{1}{p}}\left[b^2\left(\frac{b^{-q}(\ln a-\ln b)q-b^{-q}+a^{-q}}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right.$$

$$\left.+a^2\left(\frac{b^{-q}-a^{-q}}{q(\ln a-\ln b)}-\frac{b^{-q}(\ln a-\ln b)q-b^{-q}+a^{-q}}{q^2(\ln a-\ln b)^2}\right)^{\frac{1}{q}}\right].$$

Proof. Making the substitutions $a\to b^{-1}$, $b\to a^{-1}$ in Proposition 4.1, Proposition 4.2 and Proposition 4.3, one can obtain the desired inequalities respectively. □

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