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ONE-SIDED GENERALIZED $(\alpha, \beta)-$REVERSE DERIVATIONS OF ASSOCIATIVE RINGS

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Abstract. In this paper, we introduce the notion of the one-sided generalized (α, β)-reverse derivation of a ring R. Let R be a semiprime ring, ϱ be a non-zero ideal of R, α be an epimorphism of ϱ, β be a homomorphism of ϱ (α be a homomorphism of ϱ, β be an epimorphism of ϱ) and $\gamma : \varrho \to R$ be a non-zero (α, β)-reverse derivation. We show that there exists $F : \varrho \to R$, an l-generalized (α, β)-reverse derivation (an r-generalized (α, β)-reverse derivation) associated with γ iff $F(\varrho), \gamma(\varrho) \subset C_R(\varrho)$ and F is an r-generalized (β, α)-derivation (an l-generalized (β, α)-derivation) associated with (β, α)-derivation γ on ϱ. This theorem generalized the results of A. Aboubakr and S. Gonzalez proved in [1, Theorem 3.1, and Theorem 3.2].
Keywords: Semiprime ring, prime ring, one-sided generalized $(\alpha, \beta)-$reverse derivation, $(\alpha, \beta)-$reverse derivation.

1. Introduction

Throughout the paper, R is an associative ring with Z, which the center of R denotes. Recall that a ring R is prime if for any $r_1, r_2 \in R$, $r_1 R r_2 = (0)$ implies $r_1 = 0$ or $r_2 = 0$, and is a semiprime in case $r_1 \in R$, $r_1 R r_1 = (0)$ implies $r_1 = 0$. For $r_1, r_2 \in R$, $[r_1, r_2]$ denotes the element $r_1 r_2 - r_2 r_1$. The symbol $[r_1, r_2]$ stands for Lie commutator of r_1 and r_2 and it satisfies the basic commutator identities: for each $r_1, r_2, r_3 \in R$, $[r_1 + r_2, r_3] = [r_1, r_3] + [r_2, r_3]$, $[r_1, r_2 + r_3] = [r_1, r_2] + [r_1, r_3]$, $[r_1 r_2, r_3] = r_1 [r_2, r_3] + [r_1, r_3] r_2$, $[r_1, r_2 r_3] = [r_1, r_2] r_3 + r_2 [r_1, r_3]$. We denote the

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identity mapping of R by id_R; that is, the mapping $id_R : R \to R$ is defined as $id_R(r_1) = r_1$, for all $r_1 \in R$. For a non-empty subset A of R, $C_R(A)$ is defined as $C_R(A) = \{r \in R : [r, x] = 0, \text{ for all } x \in A\}$.

Let α, β be any two mapping of R. An additive mapping $\delta : R \to R$ is called an (α, β)-derivation if $\delta(r_1r_2) = \delta(r_1)\alpha(r_2) + \beta(r_1)\delta(r_2)$ holds, for all $r_1, r_2 \in R$. An additive mapping $\varphi : R \to R$ is called a right generalized (α, β) $-$derivation (a left generalized (α, β) $-$derivation) of R associated with δ, if $\varphi(r_1r_2) = \delta(r_1)\alpha(r_2) + \beta(r_1)\varphi(r_2)$ $(\varphi(r_1r_2) = \varphi(r_1)\alpha(r_2) + \beta(r_1)\delta(r_2))$, for all $r_1, r_2 \in R$ and φ is said to be a generalized (α, β)-derivation of R with δ if it is both a right and a left generalized (α, β)-derivation of R associated with δ.

Many authors have investigated the relationship between the commutativity of a ring and the act of derivation ((α, β)-derivation, reverse derivation, (α, β)-reverse derivation, generalized reverse derivation, etc.) defined on the ring. Herstein (1957) was the first to introduce the concept of reverse derivation. An additive mapping $g : R \to R$ is a reverse derivation if $g(r_1r_2) = g(r_2)r_1 + r_2g(r_1)$, for all $r_1, r_2 \in R$. In [4], it is shown that if a prime ring R with a characteristic different from two admits non-zero reverse derivation g, then g is a derivation of R. An additive mapping $d : R \to R$ is an (α, β) $-$reverse derivation if $d(r_1r_2) = d(r_2)\alpha(r_1) + \beta(r_2)d(r_1)$, for all $r_1, r_2 \in R$. In [8], Chaudhry and Thaheem shown that if a semiprime ring R admits non-zero (α, β)-reverse derivation d, then d is (α, β)-reverse derivation of R. Here, α and β are automorphism of R. An additive mapping $H : R \to R$ is called l-generalized reverse derivation (r-generalized reverse derivation) In [1], A. Aboubakr and S. Gonzalez (2015) introduced one-sided generalized reverse derivation. An additive mapping $H : R \to R$ is called an l-generalized reverse derivation (r-generalized reverse derivation) if there exists a reverse derivation $g : R \to R$ such that $H(r_1r_2) = H(r_2)r_1 + r_2g(r_1)$ $(H(r_1r_2) = g(r_2)r_1 + r_2H(r_1))$, for all $r_1, r_2 \in R$. In [1], they have indicated that if a semiprime ring R admits non-zero one-sided generalized reverse derivation H associated with reverse derivation g, then H is a one-sided generalized derivation with associated derivation g. Reverse derivation, generalized reverse derivation, (α, β)-reverse derivation, generalized (α, β)-reverse derivation, multiplicative reverse derivation, multiplicative generalized reverse derivation, multiplicative (α, β)-reverse derivation, and multiplicative generalized (α, β)-reverse derivation of prime or semiprime rings have been studied by a lot of scholars in the literature. (see [2],[3],[4], [9],[10],[12],[13],[14],[15],[16].)

This paper extends the notion of one-sided reverse derivation to one-sided generalized (α, β)-reverse derivation.

Definition 1.1. Let R be a ring, α, β be a mapping of R, and γ be an (α, β)-reverse derivation of R. An additive mapping $F : R \to R$ is said to be an r-generalized (α, β)-reverse derivation of R associated with γ if

$$F(r_1r_2) = \gamma(r_2)\alpha(r_1) + \beta(r_2)F(r_1)$$

for all $r_1, r_2 \in R$, F is said to be an $l-$generalized (α, β)-reverse derivation of R associated with γ if

$$F(r_1r_2) = F(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1)$$

for all $r_1, r_2 \in R$ and F said to be a generalized (α, β) $-$reverse derivation of R associated with γ if it is both an r-generalized and $l-$generalized (α, β)-reverse derivation of R associated with γ.

When $\alpha = \beta = id_R$, an r-generalized (l-generalized) (α, β)-reverse derivation is a r-generalized (l-generalized) reverse derivation. Thus, the one-sided generalized reverse derivation is a special case of one-sided generalized (α, β)-reverse derivation.

This study consists of 2 parts. In the first part, we show that If R is a 2-torsion free semiprime ring, α, β are automorphisms of R, and $\gamma : R \to R$ is a non-zero (α, β)-reverse derivation, then γ is an (α, β)-derivation on R. With this result, we will show that the concepts of (α, β)-reverse derivation and (α, β)-derivation overlap in 2 torsion-free semiprime rings in which α and β are automorphisms of the ring. In the second part, we give a generalization of [1, Theorem 3.1, Theorem 3.2, and Corollary 3.3], which is the main result of the article. In that case, one-sided generalized (α, β)-reverse derivation and one-sided generalized (β, α)-derivation overlap in a semiprime ring where only one of α and β is an epimorphism of the ring. Thus we will show that the intersection of the set of all generalized (α, β)-derivation and the set of all generalized (α, β)-reverse derivation is different from the empty set. At the end of the paper, we showed that in case α is a homomorphism of R and β is an epimorphism of R; there is no non-zero generalized (α, β)-reverse derivation associated with (α, β)-reverse derivation of noncommutative prime ring R.

From now on, R is an associative ring, Z is the center of R, and $\alpha, \beta : R \to R$ are homomorphisms.

2. Preliminary

In this section, we give some auxiliary results that will need later. We begin our discussion with several examples related to (α, β)-reverse derivation and one-sided generalized (α, β)-reverse derivation.

Lemma 2.1. [7, Lemma 3] *If the prime ring R contains a commutative non-zero right ideal I, then R is commutative.*

Lemma 2.2. [7, Lemma 4] *Let b and ab be in the center of a prime ring R. If b is not zero, then a is in Z, the center of R.*

Lemma 2.3. [11, Corollary 2.1] *Let R be a 2-torsion free semiprime ring, α, β be automorphisms of R and $L \nsubseteq Z(R)$ be a non-zero square-closed Lie ideal of R. If $\delta : R \to L$ satisfying*

$$\left(a^2\right)^\delta = a^\delta \alpha(a) + \beta(a) a^\delta, \text{ for all } a \in L \tag{2.1}$$

and $a^\delta, \beta(a) \in L$, then δ is a (α, β) $-$derivation on L.

Example 2.1. Consider the ring $R = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ the ring of integers. Let us define $\alpha : R \to R$, $\beta : R \to R$, and $d : R \to R$ as follows:

$$\alpha\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_{11} \end{bmatrix}$$

$$\beta\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{22} & 0 \\ 0 & 0 \end{bmatrix}$$

$$d\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & a_{11} - a_{22} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that d is both an (α, β)-reverse derivation and an (α, β)-derivation.

Example 2.2. Consider the ring $R = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ the ring of integers. Define the mappings $\alpha : R \to R$, $\beta : R \to R$, and $d : R \to R$ as follows:

$$\alpha\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_{11} \end{bmatrix}$$

$$\beta\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{22} & 0 \\ 0 & 0 \end{bmatrix}$$

$$d\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_{12} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that d is an (α, β)-reverse derivation. But d is not an (α, β)-derivation.

Example 2.3. Consider the ring $R = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ the ring of integers. Define the mappings $\alpha : R \to R$, $\beta : R \to R$, and $d : R \to R$ as follows:

$$\alpha\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix}$$

$$\beta\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$d\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that d is an (α, β)-derivation. But d is not an (α, β)-reverse derivation.

Example 2.4. Let $(R_1, +, *)$ be a commutative ring and $(R_2, \oplus, \circledast)$ be a noncommutative ring. Let's consider operation $\otimes : R_2 \times R_2 \to R_2$, $r \otimes s = s \circledast r$. With these operations $(R_2, \oplus, \otimes)$ called opposite ring and it is shown R_2^{op}. α, β are homomorphisms of R_2, $\delta : R_2 \to R_2^{op}$ is an (β, α) −derivation, and $\varphi : R_2 \to R_2^{op}$ is a left generalized (β, α) −derivation with δ. Define the mappings $\widetilde{\alpha}, \widetilde{\beta} : R_2 \times R_1 \to R_2 \times R_1$,and $\widetilde{\delta}, \widetilde{\varphi} : R_2 \times R_1 \to R_2^{op} \times R_1$ as follows:

$$\begin{aligned} \widetilde{\alpha}(r, s) &= (\alpha(r), s) \\ \widetilde{\beta}(r, s) &= (\beta(r), s) \\ \widetilde{\delta}(r, s) &= (\delta(r), s) \\ \widetilde{\varphi}(r, s) &= (\varphi(r), s). \end{aligned}$$

Then it is straightforward to verify that $\widetilde{\varphi}$ is an $l-$generalized (α, β)-reverse derivation with $(\alpha, \beta)-$reverse derivation $\widetilde{\delta}$ of $R_2 \times R_1$. But $\widetilde{\varphi}$ is not a generalized (α, β)-derivation with (α, β)-derivation $\widetilde{\delta}$ of $R_2 \times R_1$.

Example 2.5. Let $(R_1, +, *)$ and $(R_2, \oplus, \otimes)$ be a rings as defined in example 2.4. Let α, β be homomorphisms of R_2, $\delta : R_2 \to R_2^{op}$ be an $(\beta, \alpha)-$derivation, and $\varphi : R_2 \to R_2^{op}$ be a right generalized $(\beta, \alpha)-$derivation with δ. Define the mappings $\widetilde{\alpha}, \widetilde{\beta} : R_2 \times R_1 \to R_2 \times R_1$ and $\widetilde{\delta}, \widetilde{\varphi} : R_2 \times R_1 \to R_2^{op} \times R_1$ as follows:

$$\begin{aligned}
\widetilde{\alpha}(r, s) &= (\alpha(x), s) \\
\widetilde{\beta}(r, s) &= (\beta(x), s) \\
\widetilde{\delta}(r, s) &= (\delta(x), s) \\
\widetilde{\varphi}(r, s) &= (\varphi(x), s).
\end{aligned}$$

Then it is straightforward to verify that $\widetilde{\varphi}$ is an $r-$generalized (α, β)-reverse derivation with (α, β)-reverse derivation $\widetilde{\delta}$ of $R_2 \times R_1$. But $\widetilde{\varphi}$ is not a generalized (α, β)-derivation with (α, β)-derivation $\widetilde{\delta}$ of $R_2 \times R_1$.

3. $(\alpha, \beta)-$Reverse Derivation

Theorem 3.1. *Let R be a 2-torsion free semiprime ring, α, β be automorphisms of R. If $\gamma : R \to R$ is a non-zero (α, β)-reverse derivation, then γ is an (α, β)-derivation on R.*

Proof. Suppose that R is non-commutative ring. Let $r_1 \in R$. From the hypothesis, we get

$$\gamma(r_1^2) = \gamma(r_1)\alpha(r_1) + \beta(r_1)\gamma(r_1).$$

This equation ensures equality of (2.1). We know that the ring R is a square closed Lie ideal of R. So, we can think of R instead of L in Lemma 2.3. Thus, γ is an $(\alpha, \beta)-$derivation on R because of Lemma 2.3. While R is a commutative ring, (α, β)-reverse derivation of R is (α, β)-derivation of R. So, the proof ends. □

Theorem 3.2. *Let R be a semiprime ring, ϱ is a non-zero two-sided ideal of R, α be an epimorphism of ϱ and β be a homomorphism of ϱ(or α be a homomorphism of ϱ and β be an epimorphism of ϱ). There exists $\gamma : \varrho \to R$ a non-zero (α, β)-reverse derivation iff $\gamma(\varrho) \subset C_R(\varrho)$ and γ is (β, α)-derivation on ϱ.*

Proof. We only prove case of no parenthesis. The another one has the same argument. Let $x_1, x_2, x_3 \in \varrho$. Since γ is $(\alpha, \beta)-$reverse derivation on ϱ, we have

(3.1)
$$\gamma(x_1x_2x_3) = \gamma(x_1(x_2x_3)) = \gamma(x_3)\alpha(x_2)\alpha(x_1) + \beta(x_3)\gamma(x_2)\alpha(x_1) + \beta(x_2)\beta(x_3)\gamma(x_1)$$

and

(3.2)
$$\gamma(x_1x_2x_3) = \gamma((x_1x_2)x_3) = \gamma(x_3)\alpha(x_1)\alpha(x_2) + \beta(x_3)\gamma(x_2)\alpha(x_1) + \beta(x_3)\beta(x_2)\gamma(x_1).$$

From (3.1) and (3.2),

$$\gamma(x_3)\,[\alpha(x_1),\alpha(x_2)] = [\beta(x_3),\beta(x_2)]\,\gamma(x_1). \tag{3.3}$$

Replacing x_3 by x_2 in (3.3),

$$\gamma(x_2)\,[\alpha(x_1),\alpha(x_2)] = 0$$

for all $x_1, x_2 \in \varrho$. Because α is an epimorphism of ϱ, for each $x_1, x_2 \in \varrho$, we get

$$\gamma(x_2)\,[x_1,\alpha(x_2)] = 0. \tag{3.4}$$

Take $r \in R$. Substituting x_1x_3r for x_1 in (3.4), we obtain $\gamma(x_2)x_1x_3\,[r,\alpha(x_2)] = 0$, for all $x_1, x_2, x_3 \in \varrho, r \in R$. So implies that

$$\gamma(x_2)\varrho\varrho\,[R,\alpha(x_2)] = (0) \tag{3.5}$$

for all $x_2 \in \varrho$. Because ϱ is a semiprime ring, it must contain a family ρ of prime ideals such that $\cap\rho = (0)$. Let ρ_φ be a typical member of this family and $x_2 \in \varrho$; by (3.5),

$$\gamma(x_2)\varrho \subset \rho_\varphi \text{ or } [R,\alpha(x_2)] \subset \rho_\varphi.$$

Let $M = \{x_2 \in \varrho : \gamma(x_2)\varrho \subset \rho_\varphi\}$ and $N = \{x_2 \in \varrho : [R,\alpha(x_2)] \subset \rho_\varphi\}$. Clearly, each group M and N is additive subgroup of ϱ such that $\varrho = M \cup N$. But a group cannot be a set union of two proper subgroups. Hence, $M = \varrho$ or $N = \varrho$. Since ρ_φ is ideal of ϱ, it holds that $\gamma(\varrho)\varrho\,[R,\alpha(\varrho)] \subset \rho_\varphi$. Thus $\gamma(\varrho)\varrho\,[R,\alpha(\varrho)] \subset \cap\rho = (0)$. Because α is an epimorphism of ϱ, it provides that $\gamma(\varrho)\varrho\,[R,\varrho] = 0$. Let $x_1, x_2, x_3 \in \varrho, r \in R$. Means that,

$$\gamma(x_1)x_2\,[r,x_3] = 0. \tag{3.6}$$

Let $x_4 \in \varrho$. In (3.6), replacing r by $x_4\gamma(x_1)$ and x_3 by x_2,we get

$$\gamma(x_1)x_2x_4\,[\gamma(x_1),x_2] = 0. \tag{3.7}$$

In (3.6), substituting x_2 by x_4, we get $\gamma(x_1)x_4\,[r,x_3] = 0$. In this equation replacing x_3 by x_2, r by $\gamma(x_1)$ and multiply from the left by x_2, it holds

$$x_2\gamma(x_1)x_4\,[\gamma(x_1),x_2] = 0. \tag{3.8}$$

From (3.7) and (3.8),

$$[\gamma(x_1),x_2]\,x_4\,[\gamma(x_1),x_2] = 0, \text{ for all } x_1, x_2, x_4 \in \varrho. \tag{3.9}$$

Since ϱ is a semiprime ring,

$$[\gamma(x_1),x_2] = 0, \text{ for all } x_1, x_2 \in \varrho.$$

That is $\gamma(\varrho) \subset C_R(\varrho)$. We get

$$\begin{aligned}\gamma(x_1x_2) &= \gamma(x_2)\alpha(x_1) + \beta(x_2)\gamma(x_1)\\ &= \gamma(x_1)\beta(x_2) + \alpha(x_1)\gamma(x_2)\end{aligned}$$

for all $x_1, x_2 \in \varrho$. This means that γ is (β,α)-derivation on ϱ. The converse is trivial. $\square$

If consider R instead of ϱ in Theorem 3.2, we get

Corollary 3.1. *Let R be a semiprime ring, α be an epimorphism of R and β be a homomorphism of R(or α be a homomorphism of R and β be an epimorphism of R). There exists $\gamma : R \to R$ a non-zero (α, β)-reverse derivation iff central γ is (β, α)-derivation on R.*

Corollary 3.2. *Let R be a prime ring, α be an epimorphism of R and β be a homomorphism of R(or α be a homomorphism of R and β be an epimorphism of R). There exists $\gamma : R \to R$ a non-zero (α, β)-reverse derivation iff R is commutative and γ is an (α, β)-derivation of R.*

Proof. We only prove a case in which α is an epimorphism of R and β is a homomorphism of R. Another case has the similar argument. By Corollary 3.1, γ is a central (β, α) $-$derivation of R. Let $r_1, r_2 \in R$. It is clear that

$$[\gamma(r_1 r_2), \beta(r_2)] = 0.$$

Applying Lie commutator features, we get

$$\begin{aligned} [\gamma(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1), \beta(r_2)] &= [\gamma(r_2)\alpha(r_1), \beta(r_2)] + [\beta(r_2)\gamma(r_1), \beta(r_2)] \\ &= \gamma(r_2)\,[\alpha(r_1), \beta(r_2)] + [\gamma(r_2), \beta(r_2)]\,\alpha(r_1) \\ &\quad + \beta(r_2)\,[\gamma(r_1), \beta(r_2)] + [\beta(r_2), \beta(r_2)]\,\gamma(r_1) \end{aligned}$$

for all $r_1, r_2 \in R$. In the last equation, since $\gamma(r_1), \gamma(r_2) \in Z$, we have

$$\gamma(r_2)\,[\alpha(r_1), \beta(r_2)] = 0$$

for all $r_1, r_2 \in R$. Let $r_3 \in R$. Since α is an epimorphism of R, we get

$$\gamma(r_2) r_3\,[r_1, \beta(r_2)] = 0.$$

Thus, for each $r_2 \in R$, we write

$$\gamma(r_2) R\,[R, \beta(r_2)] = (0).$$

By the primeness of R, for each $r_2 \in R$, we get

$$\gamma(r_2) = 0 \text{ or } \beta(r_2) \in Z.$$

Let $M = \{r_2 \in R : \gamma(r_2) = 0\}$ and $N = \{r_2 \in R : \beta(r_2) \in Z\}$. Clearly, each group M and N is additive subgroup of R such that $R = M \cup N$. But a subgroup cannot be a set union of two proper subgroups. Hence, $M = R$ or $N = R$. Since γ is a non-zero $(\alpha, \beta)-$reverse derivation of R, it happens $\beta(R) \subset Z$. Since $\gamma(r_1 r_2) \in Z$ and Z is a subring of R, we have

$$\gamma(r_2)\alpha(r_1) \in Z, \text{ for all } r_1, r_2 \in R.$$

In view of Lemma 2.2, for each $r_1 \in R$, we have $\alpha(r_1) \in Z$. In addition, since α is an epimorphism of R, we have R is commutative. Therefore, we conclude that

$$\gamma(r_1 r_2) = \gamma(r_2 r_1) = \gamma(r_1)\alpha(r_2) + \beta(r_1)\gamma(r_2)$$

for all $r_1, r_2 \in R$. This implies γ is an (α, β)-derivation of R. $\square$

4. One-Sided Generalized (α,β) −Reverse Derivation

Theorem 4.1. *Let R be a semiprime ring, ϱ is a non-zero two-sided ideal of R, α be an epimorphism of ϱ, β be homomorphism of ϱ and $\gamma:\varrho\to R$ be a non-zero (α,β)-reverse derivation. There exists $F:\varrho\to R$, a l-generalized (α,β)-reverse derivation associated with γ iff $F(\varrho),\gamma(\varrho)\subset C_R(\varrho)$ and F is $r-$generalized (β,α)-derivation associated with (β,α)-derivation γ on ϱ.*

Proof. Let $x_1,x_2,x_3\in\varrho$. Using the definition of l-generalized (α,β)-reverse derivation one can easily see that

$$F(x_1(x_2x_3))=F(x_3)\alpha(x_2)\alpha(x_1)+\beta(x_3)\gamma(x_2)\alpha(x_1)+\beta(x_2)\beta(x_3)\gamma(x_1) \tag{4.1}$$

and

$$F((x_1x_2)x_3)=F(x_3)\alpha(x_1)\alpha(x_2)+\beta(x_3)\gamma(x_2)\alpha(x_1)+\beta(x_3)\beta(x_2)\gamma(x_1) \tag{4.2}$$

Combining (4.1) and (4.2),

$$F(x_3)\left[\alpha(x_2),\alpha(x_1)\right]=\left[\beta(x_3),\beta(x_2)\right]\gamma(x_1). \tag{4.3}$$

Substituting x_3 by x_2 in (4.3),

$$F(x_2)\left[\alpha(x_2),\alpha(x_1)\right]=0$$

for all $x_1,x_2\in\varrho$. Since α is an epimorphism of ϱ, for each $x_1,x_2\in\varrho$, we have

$$F(x_2)\left[\alpha(x_2),x_1\right]=0. \tag{4.4}$$

Taking $x_3\in\varrho, r\in R$. Replacing x_1 by x_1x_3r in (4.4), $F(x_2)x_1x_3\left[\alpha(x_2),r\right]=0$. For each $x_2\in\varrho$, we have $F(x_2)\varrho\varrho\left[\alpha(x_2),R\right]=(0)$. Now, when similar steps are applied to the steps from equality (3.5) to equality (3.9), for each $x_1,x_2,x_3\in\varrho$, we have $\left[F(x_1),x_2\right]x_3\left[F(x_1),x_2\right]=0$. Since ϱ is a semiprime ring,

$$\left[F(x_1),x_2\right]=0$$

for all $x_1,x_2\in\varrho$. That is $F(\varrho)\subset C_R(\varrho)$. Moreover, if γ is (α,β) −reverse derivation of R, then by Theorem 3.2, $\gamma(\varrho)\subset C_R(\varrho)$ and γ is an (β,α) −derivation on ϱ. Hence,

$$\begin{aligned}F(x_1x_2)&=F(x_2)\alpha(x_1)+\beta(x_2)\gamma(x_1)\\&=\gamma(x_1)\beta(x_2)+\alpha(x_1)F(x_2)\end{aligned}$$

for all $x_1,x_2\in\varrho$ and F is a $r-$generalized (β,α) −derivation associated with $(\beta,\alpha)-$ derivation γ on ϱ. The converse is a trivial. □

Corollary 4.1. *Let R be a semiprime ring, α be an epimorphism of R, β be homomorphism of R and $\gamma:R\to R$ be a non-zero (α,β)-reverse derivation. There exists $F:R\to R$, a l-generalized (α,β)-reverse derivation associated with γ iff $F(I),\gamma(I)\subset Z$ and F is r-generalized (β,α)-derivation associated with (β,α)-derivation γ of R.*

Theorem 4.2. *Let R be a semiprime ring, ϱ is a non-zero two-sided ideal of R, α be a homomorphism of ϱ, β be an epimorphism of ϱ and $\gamma : \varrho \to R$ be a non-zero (α, β)-reverse derivation. There exists $F : \varrho \to R$, a r-generalized (α, β)-reverse derivation associated with γ iff $F(\varrho), \gamma(\varrho) \subset C_R(\varrho)$ and F is $l-$generalized (β, α)-derivation associated with (β, α)-derivation γ on ϱ.*

Proof. By a similar proof in Theorem 4.1, desired is achieved. □

Corollary 4.2. *Let R be a semiprime ring, α be an homomorphism of R, β be an epimorphism of R and $\gamma : R \to R$ be a non-zero (α, β)-reverse derivation. There exists $F : R \to R$, a r-generalized (α, β)-reverse derivation associated with γ iff $F(R), \gamma(R) \subset Z$ and F is $l-$generalized (β, α)-derivation associated with (β, α)-derivation γ of R.*

Theorem 4.3. *Let R be a semiprime ring, α and β be an epimorphisms of R and $\gamma : R \to R$ be a non-zero $(\alpha, \beta)-$reverse derivation. If there exists $F : R \to R$, a non-zero l-generalized (α, β)-reverse derivation (r-generalized (α, β)-reverse derivation) associated with γ then R contains a non-zero central ideal.*

Proof. Assume that $F : R \to R$ is a l-generalized (α, β)-reverse derivation associated with non-zero (α, β)-reverse derivation γ of R. From Corollary 4.1, it holds $\gamma(R), F(R) \subset Z$. For all $r_1, r_2 \in R$,

$$[F(r_1 r_2), \beta(r_2)] = 0$$

is obtained. This means

$$F(r_2)\,[\alpha(r_1), \beta(r_2)] = 0$$

for all $r_1, r_2 \in R$. Because α is an epimorphism of R, for each $r_1, r_2 \in R$, we get $F(r_2)\,[r_1, \beta(r_2)] = 0$. Let r_3. Replacing r_1 by $r_1 r_3$ in $F(r_2)\,[r_1, \beta(r_2)] = 0$, we get

$$F(r_2) r_1\,[r_3, \beta(r_2)] = 0.$$

Now, when similar steps are applied to the steps from equality (3.5) to equality (3.9), for each $r_1, r_2, r_3 \in R$, we have $F(r_1)\,[r_2, \beta(r_3)] = 0$. Because β is an epimorphism of R, we get $F(r_1)\,[r_2, r_3] = 0$. That is

$$[F(r_1) r_2, r_3] = 0$$

for all $r_1, r_2, r_3 \in R$. This means $F(R)R \subset Z$. Since F is non-zero $l-$generalized $(\alpha, \beta)-$reverse derivation and R is semiprime, $F(R)R \neq (0)$. $F(R)R$ is obviously central ideal of R. The proof has a similar argument if F is $r-$generalized $(\alpha, \beta)-$reverse derivation of R. □

Corollary 4.3. *Let R be a semiprime ring, α and β be an epimorphisms of R and $\gamma : R \to R$ be a non-zero (α, β)-reverse derivation. If there exists $F : R \to R$, a non-zero generalized (α, β)-reverse derivation associated with γ then R contains a non-zero central ideal.*

Corollary 4.4. *Let R be a prime ring, α and β be an epimorphisms of R and $\gamma : R \to R$ be a non-zero (α, β)-reverse derivation. If there exists $F : R \to R$, a non-zero generalized (α, β)-reverse derivation associated with d then R is commutative ring and F is a generalized (α, β)-derivation associated with an (α, β)-derivation γ of R.*

Theorem 4.4. *Let R be a noncommutative prime ring, α be a homomorphism of R and β be an epimorphism of R. If $F : R \to R$ is a generalized (α, β)-reverse derivation associated with non-zero (α, β)-reverse derivation γ of R then $F = \gamma$.*

Proof. Assume that $F : R \to R$ is a generalized (α, β)-reverse derivation associated with non-zero (α, β)-reverse derivation γ of R. Let $r_1, r_2 \in R$. Then,

$$F(r_1r_2) = F(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1) = \gamma(r_2)\alpha(r_1) + \beta(r_2)F(r_1).$$

That is,

$$(F - \gamma)(r_2)\alpha(r_1) - \beta(r_2)(F - \gamma)(r_1) = 0.$$

Let us introduce mapping $\varphi : R \to R$, $\varphi(r_1) = (F - \gamma)(r_1)$. Moreover, the last equation implies that

$$\varphi(r_2)\alpha(r_1) = \beta(r_2)\varphi(r_1). \tag{4.5}$$

Let $r_1, r_2 \in R$. Since F is an r-generalized (α, β)-reverse derivation (l- of generalized (α, β)-reverse derivation) R and γ is an (α, β)-reverse derivation of R, the mapping φ respectively ensures:

$$\begin{aligned} \varphi(r_1r_2) &= (F - \gamma)(r_1r_2) = \gamma(r_2)\alpha(r_1) + \beta(r_2)F(r_1) - \gamma(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1) \\ &= \beta(r_2)\varphi(r_1) \end{aligned}$$

and

$$\begin{aligned} \varphi(r_1r_2) &= (F - \gamma)(r_1r_2) = F(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1) - \gamma(r_2)\alpha(r_1) + \beta(r_2)\gamma(r_1) \\ &= \varphi(r_2)\alpha(r_1). \end{aligned}$$

That is,

$$\varphi(r_1r_2) = \beta(r_2)\varphi(r_1). \tag{4.6}$$

$$\varphi(r_1r_2) = \varphi(r_2)\alpha(r_1). \tag{4.7}$$

Let $r_3 \in R$. Writing r_2r_3 by r_2 in (4.5), we get

$$\varphi(r_2r_3)\alpha(r_1) - \beta(r_2r_3)\varphi(r_1) = 0.$$

In the last equality, using (4.6) and (4.7), we get

$$\beta([r_3, r_2])\varphi(r_1) = 0$$

for all $r_1, r_2, r_3 \in R$. Because β is an epimorphism, for each $r_1, r_2, r_3 \in R$, we have $[r_3, r_2]\,\varphi(r_1) = 0$. Given that R is a noncommutative prime ring, we get $\varphi = 0$. That is, $F = \gamma$. $\square$

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