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ON THE BIHARMONIC MAPS ON THE GENERALIZED $\mathcal{D}$-HOMOTHETIC DEFORMATION OF ALMOST CONTACT METRIC MANIFOLDS

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Abstract. In this paper, we give some results about the generalized $\mathcal{D}$-homothetic deformation on the almost contact metric manifold and we study the harmonicity and the biharmonicity relative to this type of deformation. In terms of applications, we have constructed several examples of harmonic and biharmonic maps.
Keywords: Generalized $\mathcal{D}$-Homothetic deformation , Almost contact manifolds, Harmonic map, Biharmonic map

1. Introduction

Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map between Riemannian manifolds. Then ϕ is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g$$

with respect to compactly supported variations. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

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$\tau(\phi)$ is called the tension field of ϕ. As the generalizations of harmonic maps, biharmonic maps are defined as follows. The map ϕ is said to be biharmonic if it is a critical point of the bi-energy functional :

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g$$

Equivalently, ϕ is biharmonic if it satisfies the associated Euler-Lagrange equations :

$$\tau_2(\phi) = -Tr_g\left(\nabla^{\phi}\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)\, d\phi = 0,$$

where ∇^{ϕ} is the connection in the pull-back bundle $\phi^{-1}(TN)$ and, if $(e_i)_{1\leq i\leq m}$ is a local orthonormal frame field on M, then

$$Tr_g\left(\nabla^{\phi}\right)^2 \tau(\phi) = \left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^M_{e_i} e_i}\right)\tau(\phi),$$

where we sum over repeated indices. We will call the operator $\tau_2(\phi)$, the bi-tension field of the map ϕ. For more details, we can refer the reader to [1] and [5-7]. A quadruple (φ, ξ, η, g) defined on a Riemannian manifold M^{2m+1} is known as an almost contact metric structure if

$$\varphi^2 X = -X + \eta(X)\,\xi, \eta(\xi) = 1, g(X, \xi) = \eta(X)$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y)$$

hold for all $X, Y \in \Gamma(TM)$, where φ is a $(1,1)$-type vector field, ξ, the structure vector field of type $(1,0)$, η, 1-form and g is the Riemannian metric on M^{2m+1}. The manifold M^{2m+1} equipped with the structure (φ, ξ, η, g) is called an almost contact metric manifold (see [4]). Note that an orthonormal basis on $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is given by

$$\{e_i, \varphi e_i, \xi\}_{i=1}^m .$$

Moreover, if the almost contact metric manifold $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ satisfies the condition

$$(\nabla_X \varphi)(Y) = g(\varphi X, Y)\,\xi - \eta(X)\,\eta(Y)$$

or equivalently

$$\nabla_X \xi = X - \eta(X)\,\xi,$$

for all for all $X, Y \in \Gamma(TM)$, then $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is known as a Kenmotsu manifold (see [8]). It is well known that a Kenmotsu manifold $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ satisfies the following relations :

$$R(X, Y)\,\xi = \eta(X)\,Y - \eta(Y)\,X$$

and

$$R(\xi, X)\,Y = \eta(X)\,Y - g(X, Y)\,\xi.$$

From this las formula, we get

$$Ricci\,(\xi) = Tr_g R\,(\xi, \cdot) \cdot = -2m\xi.$$

In [9],[10] and [11], the authors were interested in the construction of harmonic and biharmonic maps on almost contact metric manifold, in particular on Kenmotsu manifolds. The $\mathcal{D}$-homothetic deformation of almost contact metric manifolds is a transformation that has been studied by several authors, as an example see [5]. Our objective in this paper is to generalize this type of deformation. We consider an almost contact metric manifold $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ and we define a generalized $\mathcal{D}$-homothetic deformation. In the first part of this paper, we give some properties related to this type of deformation (Proposition 1 and Theorem 1). In the second part, we study the harmonicity and the biharmonicity of the identity map in two different cases where we give some special cases and this study allowed us to construct some examples.

2. The main results.

In this section, we consider $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ an almost contact metric manifold. A generalized $\mathcal{D}$-homothetic deformation is defined by

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = \alpha\eta, \qquad \overline{\xi} = \frac{1}{\alpha}\xi, \qquad \overline{g} = \alpha g + \alpha\,(\alpha - 1)\,\eta \otimes \eta,$$

where α is a positive function on M. One can easily check that $\left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is an almost contact metric manifold too. Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections on $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ and $\left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ respectively.

2.1. Some results on the generalized $\mathcal{D}$-homothetic deformation.

As part of our first results, we prove some basic formulas about generalized $\mathcal{D}$-homothetic deformation of almost contact metric manifolds.

Proposition 2.1. *Let $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ be an almost contact metric manifold and let $\left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ be a generalized $\mathcal{D}$-homothetic deformation of $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$. Then, we have*

$$\begin{aligned}
\overline{g}\left(\overline{\nabla}_X Y, Z\right) &= \alpha g\,(\nabla_X Y, Z) + \left(\alpha^2 - \alpha\right)\eta\,(Z)\,\eta\,(\nabla_X Y) + \frac{2\alpha - 1}{2} X\,(\alpha)\,\eta\,(Y)\,\eta\,(Z) \\
&+ \frac{1}{2} g\,(Y, Z)\,X\,(\alpha) + \frac{1}{2} g\,(X, Z)\,Y\,(\alpha) - \frac{1}{2} g\,(X, Y)\,Z\,(\alpha) \\
&+ \frac{2\alpha - 1}{2}\eta\,(X)\,\eta\,(Z)\,Y\,(\alpha) - \frac{2\alpha - 1}{2}\eta\,(X)\,\eta\,(Y)\,Z\,(\alpha) \\
&+ \frac{\alpha^2 - \alpha}{2}\eta\,(X)\,\{g\,(\nabla_Y \xi, Z) - g\,(\nabla_Z \xi, Y)\}
\end{aligned}$$

$$\begin{aligned} &+ \frac{\alpha^2-\alpha}{2}\eta(Y)\{g(\nabla_X\xi,Z)-g(\nabla_Z\xi,X)\} \\ &+ \frac{\alpha^2-\alpha}{2}\eta(Z)\{g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)\}. \end{aligned}$$

Proof of Proposition 2.1. By using the Koszul formula , we have

$$\begin{aligned} 2\overline{g}\left(\overline{\nabla}_X Y,Z\right) &= X(\overline{g}(Y,Z))+Y(\overline{g}(X,Z))-Z(\overline{g}(X,Y)) \\ &+ \overline{g}([X,Y],Z)+\overline{g}([Z,X],Y)-\overline{g}(X,[Y,Z]), \end{aligned}$$

for all $X,Y,Z\in\Gamma(TM)$. As

$$\overline{g}=\alpha g+\alpha(\alpha-1)\eta\otimes\eta,$$

we get

$$\begin{aligned} 2\overline{g}\left(\overline{\nabla}_X Y,Z\right) &= X(\alpha g(Y,Z)+\alpha(\alpha-1)\eta(Y)\eta(Z)) \\ &+ Y(\alpha g(X,Z)+\alpha(\alpha-1)\eta(X)\eta(Z)) \\ &- Z(\alpha g(X,Y)+\alpha(\alpha-1)\eta(X)\eta(Y))+\alpha g([X,Y],Z) \\ &+ \alpha(\alpha-1)\eta([X,Y])\eta(Z)+\alpha(\alpha-1)\eta([Z,X])\eta(Y) \\ &- \alpha(\alpha-1)\eta(X)\eta([Y,Z])+\alpha g([Z,X],Y)-\alpha g(X,[Y,Z]) \end{aligned}$$

For the term $X(\alpha g(Y,Z)+\alpha(\alpha-1)\eta(Y)\eta(Z))$, a long calculation gives

$$\begin{aligned} X(\alpha g(Y,Z)+\alpha(\alpha-1)\eta(Y)\eta(Z)) &= X(\alpha g(Y,Z))+X(\alpha(\alpha-1)\eta(Y)\eta(Z)) \\ &= \alpha g(\nabla_X Y,Z)+\alpha g(Y,\nabla_X Z) \\ &+ (2\alpha-1)\eta(Y)\eta(Z)X(\alpha) \\ &+ g(Y,Z)X(\alpha) \\ &+ \alpha(\alpha-1)\eta(Z)g(\nabla_X\xi,Y) \\ &+ \alpha(\alpha-1)\eta(Z)\eta(\nabla_X Y) \\ &+ \alpha(\alpha-1)\eta(Y)g(\nabla_X\xi,Z) \\ &+ \alpha(\alpha-1)\eta(Y)\eta(\nabla_X Z). \end{aligned}$$

By a similar calculation, we obtain

$$\begin{aligned} Y(\alpha g(X,Z)+\alpha(\alpha-1)\eta(X)\eta(Z)) &= \alpha g(\nabla_Y X,Z)+\alpha g(X,\nabla_Y Z) \\ &+ (2\alpha-1)\eta(X)\eta(Z)Y(\alpha) \\ &+ g(X,Z)Y(\alpha) \\ &+ \alpha(\alpha-1)\eta(Z)g(\nabla_Y\xi,X) \\ &+ \alpha(\alpha-1)\eta(Z)\eta(\nabla_Y X) \\ &+ \alpha(\alpha-1)\eta(X)g(\nabla_Y\xi,Z) \\ &+ \alpha(\alpha-1)\eta(X)\eta(\nabla_Y Z) \end{aligned}$$

and

$$
\begin{array}{rcl}
Z\left(\alpha g\left(X,Y\right)+\alpha\left(\alpha-1\right)\eta\left(X\right)\eta\left(Y\right)\right) & = & \alpha g\left(\nabla_Z X,Y\right)+\alpha g\left(X,\nabla_Z Y\right)\\
& + & \left(2\alpha-1\right)\eta\left(X\right)\eta\left(Y\right)Z\left(\alpha\right)\\
& + & g\left(X,Y\right)Z\left(\alpha\right)\\
& + & \alpha\left(\alpha-1\right)\eta\left(Y\right)g\left(\nabla_Z\xi,X\right)\\
& + & \alpha\left(\alpha-1\right)\eta\left(Y\right)\eta\left(\nabla_Z X\right)\\
& + & \alpha\left(\alpha-1\right)\eta\left(X\right)g\left(\nabla_Z\xi,Y\right)\\
& + & \alpha\left(\alpha-1\right)\eta\left(X\right)\eta\left(\nabla_Z Y\right)
\end{array}
$$

Finally, it is clear that

$$
\alpha\left(\alpha-1\right)\eta\left(\left[X,Y\right]\right)\eta\left(Z\right)=\alpha\left(\alpha-1\right)\eta\left(Z\right)\eta\left(\nabla_X Y\right)-\alpha\left(\alpha-1\right)\eta\left(Z\right)\eta\left(\nabla_Y X\right),
$$

$$
\alpha\left(\alpha-1\right)\eta\left(\left[Z,X\right]\right)\eta\left(Y\right)=\alpha\left(\alpha-1\right)\eta\left(Y\right)\eta\left(\nabla_Z X\right)-\alpha\left(\alpha-1\right)\eta\left(Y\right)\eta\left(\nabla_X Z\right)
$$

and

$$
\alpha\left(\alpha-1\right)\eta\left(X\right)\eta\left(\left[Y,Z\right]\right)=\alpha\left(\alpha-1\right)\eta\left(X\right)\eta\left(\nabla_Y Z\right)-\alpha\left(\alpha-1\right)\eta\left(X\right)\eta\left(\nabla_Z Y\right).
$$

It follows that

$$
\begin{array}{rcl}
\overline{g}\left(\overline{\nabla}_X Y,Z\right) & = & \alpha g\left(\nabla_X Y,Z\right)+\left(\alpha^2-\alpha\right)\eta\left(Z\right)\eta\left(\nabla_X Y\right)+\dfrac{2\alpha-1}{2}X\left(\alpha\right)\eta\left(Y\right)\eta\left(Z\right)\\
& + & \dfrac{1}{2}g\left(Y,Z\right)X\left(\alpha\right)+\dfrac{1}{2}g\left(X,Z\right)Y\left(\alpha\right)-\dfrac{1}{2}g\left(X,Y\right)Z\left(\alpha\right)\\
& + & \dfrac{2\alpha-1}{2}\eta\left(X\right)\eta\left(Z\right)Y\left(\alpha\right)-\dfrac{2\alpha-1}{2}\eta\left(X\right)\eta\left(Y\right)Z\left(\alpha\right)\\
& + & \dfrac{\alpha^2-\alpha}{2}\eta\left(X\right)\left\{g\left(\nabla_Y\xi,Z\right)-g\left(\nabla_Z\xi,Y\right)\right\}\\
& + & \dfrac{\alpha^2-\alpha}{2}\eta\left(Y\right)\left\{g\left(\nabla_X\xi,Z\right)-g\left(\nabla_Z\xi,X\right)\right\}\\
& + & \dfrac{\alpha^2-\alpha}{2}\eta\left(Z\right)\left\{g\left(\nabla_X\xi,Y\right)+g\left(\nabla_Y\xi,X\right)\right\}.
\end{array}
$$

Using the proposition 2.1, we get the result which gives the relation between $\overline{\nabla}$ and ∇.

Theorem 2.1. *Let $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ be an almost contact metric manifold and let $\left(M,\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$ be a generalized $\mathcal{D}$-homothetic deformation of $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$. The relation between $\overline{\nabla}$ and ∇ is given by the following formula :*

$$
\begin{array}{rcl}
\overline{\nabla}_X Y & = & \nabla_X Y-\eta\left(X\right)\eta\left(Y\right)grad\alpha+\dfrac{1}{2\alpha}\eta\left(X\right)\eta\left(Y\right)grad\alpha-\dfrac{1}{2\alpha}g\left(X,Y\right)grad\alpha\\
& + & \dfrac{\left(2\alpha-1\right)\left(\alpha-1\right)}{2\alpha^2}\eta\left(X\right)\eta\left(Y\right)\xi\left(\alpha\right)\xi+\dfrac{\alpha-1}{2\alpha^2}g\left(X,Y\right)\xi\left(\alpha\right)\xi+\dfrac{1}{2\alpha}Y\left(\alpha\right)X
\end{array}
$$

$$
\begin{aligned}
(2.1)\quad &+ \frac{1}{2\alpha}X(\alpha)Y + \frac{\alpha-1}{2}\eta(Y)\nabla_X\xi + \frac{\alpha-1}{2}\eta(X)\nabla_Y\xi + \frac{1}{2\alpha}\eta(Y)X(\alpha)\xi \\
&+ \frac{1}{2\alpha}\eta(X)Y(\alpha)\xi + \frac{\alpha-1}{2\alpha}g(\nabla_X\xi, Y)\xi + \frac{\alpha-1}{2\alpha}g(\nabla_Y\xi, X)\xi \\
&- \frac{\alpha-1}{2}\eta(Y)Tr_g g(\nabla_.\xi, X)\cdot - \frac{\alpha-1}{2}\eta(X)Tr_g g(\nabla_.\xi, Y)\cdot \\
&+ \frac{(\alpha-1)^2}{2\alpha}\eta(Y)g(\nabla_\xi\xi, X)\xi + \frac{(\alpha-1)^2}{2\alpha}\eta(X)g(\nabla_\xi\xi, Y)\xi.
\end{aligned}
$$

In particular, if the function α depends only on the direction of ξ, the equation (2.1) becomes

$$
\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y - \frac{2\alpha-1}{2\alpha^2}\eta(X)\eta(Y)\xi(\alpha)\xi - \frac{1}{2\alpha^2}g(X,Y)\xi(\alpha)\xi + \frac{1}{2\alpha}Y(\alpha)X \\
&+ \frac{1}{2\alpha}X(\alpha)Y + \frac{\alpha-1}{2}\eta(Y)\nabla_X\xi + \frac{\alpha-1}{2}\eta(X)\nabla_Y\xi + \frac{1}{2\alpha}\eta(Y)X(\alpha)\xi \\
(2.2)\quad &+ \frac{1}{2\alpha}\eta(X)Y(\alpha)\xi + \frac{\alpha-1}{2\alpha}g(\nabla_X\xi, Y)\xi + \frac{\alpha-1}{2\alpha}g(\nabla_Y\xi, X)\xi \\
&- \frac{\alpha-1}{2}\eta(Y)Tr_g g(\nabla_.\xi, X)\cdot - \frac{\alpha-1}{2}\eta(X)Tr_g g(\nabla_.\xi, Y)\cdot \\
&+ \frac{(\alpha-1)^2}{2\alpha}\eta(Y)g(\nabla_\xi\xi, X)\xi + \frac{(\alpha-1)^2}{2\alpha}\eta(X)g(\nabla_\xi\xi, Y)\xi.
\end{aligned}
$$

Proof of Theorem 2.1. If we consider an orthonormal frame $\{e_i, \varphi e_i, \xi\}_{i=1}^m$ on the almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$, then an orthonormal frame on $(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is given by

$$
\left\{\overline{e}_i = \frac{1}{\sqrt{\alpha}}e_i, \overline{\varphi e}_i = \frac{1}{\sqrt{\alpha}}\varphi e_i, \overline{\xi} = \frac{1}{\alpha}\xi\right\}_{i=1}^m .
$$

For all $X, Y \in \Gamma(TM)$, we have

$$
\overline{\nabla}_X Y = \overline{g}\left(\overline{\nabla}_X Y, \overline{e}_i\right)\overline{e}_i + \overline{g}\left(\overline{\nabla}_X Y, \overline{\varphi e}_i\right)\overline{\varphi e}_i + \overline{g}\left(\overline{\nabla}_X Y, \overline{\xi}\right)\overline{\xi}.
$$

By Proposition 2.1, A rigorous calculation gives us

$$
\begin{aligned}
\overline{g}\left(\overline{\nabla}_X Y, \overline{e}_i\right)\overline{e}_i &= \frac{1}{\alpha}\overline{g}\left(\overline{\nabla}_X Y, e_i\right)e_i \\
&= g(\nabla_X Y, e_i)e_i + \frac{1}{2\alpha}X(\alpha)g(Y, e_i)e_i + \frac{1}{2\alpha}Y(\alpha)g(X, e_i)e_i \\
&- \frac{1}{2\alpha}g(X,Y)e_i(\alpha)e_i - \frac{2\alpha-1}{2\alpha}\eta(X)\eta(Y)e_i(\alpha)e_i \\
&+ \frac{\alpha-1}{2}\eta(X)\{g(\nabla_Y\xi, e_i)e_i - g(\nabla_{e_i}\xi, Y)e_i\} \\
&+ \frac{\alpha-1}{2}\eta(Y)\{g(\nabla_X\xi, e_i)e_i - g(\nabla_{e_i}\xi, X)e_i\},
\end{aligned}
$$

$$
\begin{aligned}
\overline{g}\left(\overline{\nabla}_X Y, \overline{\varphi e_i}\right)\overline{\varphi e_i} &= \frac{1}{\alpha}\overline{g}\left(\overline{\nabla}_X Y, \varphi e_i\right)\varphi e_i \\
&= g\left(\nabla_X Y, \varphi e_i\right)\varphi e_i + \frac{1}{2\alpha}X\left(\alpha\right)g\left(Y, \varphi e_i\right)\varphi e_i \\
&+ \frac{1}{2\alpha}Y\left(\alpha\right)g\left(X, \varphi e_i\right)\varphi e_i - \frac{1}{2\alpha}g\left(X, Y\right)\left(\varphi e_i\right)\left(\alpha\right)\varphi e_i \\
&- \frac{2\alpha-1}{2\alpha}\eta\left(X\right)\eta\left(Y\right)\left(\varphi e_i\right)\left(\alpha\right)\varphi e_i \\
&+ \frac{\alpha-1}{2}\eta\left(X\right)\left\{g\left(\nabla_Y \xi, \varphi e_i\right)\varphi e_i - g\left(\nabla_{\varphi e_i}\xi, Y\right)\varphi e_i\right\} \\
&+ \frac{\alpha-1}{2}\eta\left(Y\right)\left\{g\left(\nabla_X \xi, \varphi e_i\right)\varphi e_i - g\left(\nabla_{\varphi e_i}\xi, X\right)\varphi e_i\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{g}\left(\overline{\nabla}_X Y, \overline{\xi}\right)\overline{\xi} &= \frac{1}{\alpha^2}\overline{g}\left(\overline{\nabla}_X Y, \xi\right)\xi \\
&= \frac{1}{\alpha}g\left(\nabla_X Y, \xi\right) + \frac{\alpha-1}{\alpha}\eta\left(\nabla_X Y\right) + \frac{2\alpha-1}{2\alpha^2}X\left(\alpha\right)\eta\left(Y\right) \\
&+ \frac{1}{2\alpha^2}g\left(Y, \xi\right)X\left(\alpha\right) + \frac{1}{2\alpha^2}g\left(X, \xi\right)Y\left(\alpha\right) - \frac{1}{2\alpha^2}g\left(X, Y\right)\xi\left(\alpha\right) \\
&+ \frac{2\alpha-1}{2\alpha^2}\eta\left(X\right)Y\left(\alpha\right) - \frac{2\alpha-1}{2\alpha^2}\eta\left(X\right)\eta\left(Y\right)\xi\left(\alpha\right) \\
&- \frac{\alpha-1}{2\alpha}\left\{\eta\left(X\right)g\left(\nabla_\xi \xi, Y\right) + \eta\left(Y\right)g\left(\nabla_\xi \xi, X\right)\right\} \\
&+ \frac{\alpha-1}{2\alpha}\left\{g\left(\nabla_X \xi, Y\right) + g\left(\nabla_Y \xi, X\right)\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y - \eta\left(X\right)\eta\left(Y\right)grad\alpha + \frac{1}{2\alpha}\eta\left(X\right)\eta\left(Y\right)grad\alpha - \frac{1}{2\alpha}g\left(X, Y\right)grad\alpha \\
&+ \frac{\left(2\alpha-1\right)\left(\alpha-1\right)}{2\alpha^2}\eta\left(X\right)\eta\left(Y\right)\xi\left(\alpha\right)\xi + \frac{\alpha-1}{2\alpha^2}g\left(X, Y\right)\xi\left(\alpha\right)\xi + \frac{1}{2\alpha}Y\left(\alpha\right)X \\
&+ \frac{1}{2\alpha}X\left(\alpha\right)Y + \frac{\alpha-1}{2}\eta\left(Y\right)\nabla_X \xi + \frac{\alpha-1}{2}\eta\left(X\right)\nabla_Y \xi + \frac{1}{2\alpha}\eta\left(Y\right)X\left(\alpha\right)\xi \\
&+ \frac{1}{2\alpha}\eta\left(X\right)Y\left(\alpha\right)\xi + \frac{\alpha-1}{2\alpha}g\left(\nabla_X \xi, Y\right)\xi + \frac{\alpha-1}{2\alpha}g\left(\nabla_Y \xi, X\right)\xi \\
&- \frac{\alpha-1}{2}\eta\left(Y\right)Tr_g g\left(\nabla_{.}\xi, X\right)\cdot - \frac{\alpha-1}{2}\eta\left(X\right)Tr_g g\left(\nabla_{.}\xi, Y\right)\cdot \\
&+ \frac{\left(\alpha-1\right)^2}{2\alpha}\eta\left(Y\right)g\left(\nabla_\xi \xi, X\right)\xi + \frac{\left(\alpha-1\right)^2}{2\alpha}\eta\left(X\right)g\left(\nabla_\xi \xi, Y\right)\xi.
\end{aligned}
$$

If we assume that the function α depends only on the direction of ξ, we obtain

$$e_i\left(\alpha\right)e_i = \left(\varphi e_i\right)\left(\alpha\right)\varphi e_i = 0,$$

then

$$grad\alpha = \xi\left(\alpha\right)\xi.$$

By replacing this last formula in (2.1), obtain equation (2.1). From Theorem 2.1, if $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ is a Kenmotsu manifold, we obtain the following Corollary.

Corollary 2.1. *Let $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ be a Kenmotsu manifold, by using the fact that*

$$\nabla_X\xi = X-\eta(X)\xi, \qquad \nabla_Y\xi = Y-\eta(Y)\xi, \qquad \nabla_\xi\xi = 0$$

and

$$Tr_g g(X,\nabla.\xi)\cdot = X-\eta(X)\xi, \qquad Tr_g g(Y,\nabla.\xi)\cdot = Y-\eta(Y)\xi,$$

the equation (2.1) becomes

$$\begin{aligned}\overline{\nabla}_XY &= \nabla_XY-\frac{2\alpha-1}{2\alpha}\eta(X)\eta(Y)\,grad\alpha-\frac{1}{2\alpha}g(X,Y)\,grad\alpha+\frac{1}{2\alpha}Y(\alpha)X\\ &+\frac{1}{2\alpha}X(\alpha)Y+\frac{(2\alpha-1)(\alpha-1)}{2\alpha^2}\eta(X)\eta(Y)\xi(\alpha)\xi+\frac{\alpha-1}{2\alpha^2}g(X,Y)\xi(\alpha)\xi\\ &+\frac{1}{2\alpha}\eta(Y)X(\alpha)\xi+\frac{1}{2\alpha}\eta(X)Y(\alpha)\xi+\frac{\alpha-1}{\alpha}g(X,Y)\xi\\ &-\frac{\alpha-1}{\alpha}\eta(X)\eta(Y)\xi.\end{aligned}\tag{2.3}$$

If the function α depends only on the direction of ξ, then

$$\begin{aligned}\overline{\nabla}_XY &= \nabla_XY-\frac{2\alpha-1}{2\alpha^2}\eta(X)\eta(Y)\xi(\alpha)\xi-\frac{1}{2\alpha^2}g(X,Y)\xi(\alpha)\xi\\ &+\frac{1}{2\alpha}\eta(Y)X(\alpha)\xi+\frac{1}{2\alpha}\eta(X)Y(\alpha)\xi+\frac{1}{2\alpha}Y(\alpha)X\\ &+\frac{1}{2\alpha}X(\alpha)Y+\frac{\alpha-1}{\alpha}g(X,Y)\xi-\frac{\alpha-1}{\alpha}\eta(X)\eta(Y)\xi.\end{aligned}\tag{2.4}$$

As the first application, we will study the hamonicity of the identity map $Id:\left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)\longrightarrow\left(M^{2m+1},\varphi,\xi,\eta,g\right)$.

2.2. The biharmonicity of the identity map $Id:\left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)\longrightarrow\left(M^{2m+1},\varphi,\xi,\eta,g\right)$.

In this part, we consider an orthonormal frame $\{e_i,\varphi e_i,\xi\}_{i=1}^m$ on the almost contact metric manifold $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$, then an orthonormal frame on $\left(M^{2m+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)$ is given by

$$\left\{\overline{e}_i=\frac{1}{\sqrt{\alpha}}e_i, \quad \overline{\varphi e}_i=\frac{1}{\sqrt{\alpha}}\varphi e_i, \quad \overline{\xi}=\frac{1}{\alpha}\xi\right\}_{i=1}^m.$$

Thanks to Equation (2.1) of Theorem 2.1, we deduce that

$$\begin{aligned}\overline{\nabla}_{e_i}e_i &= \nabla_{e_i}e_i-\frac{m}{2\alpha}grad\alpha+\frac{m(\alpha-1)}{2\alpha^2}\xi(\alpha)\xi\\ &+\frac{1}{\alpha}e_i(\alpha)e_i+\frac{\alpha-1}{\alpha}g(\nabla_{e_i}\xi,e_i)\xi\end{aligned}\tag{2.5}$$

and

$$\overline{\nabla}_{\varphi e_i}\varphi e_i = \nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\alpha}grad\alpha + \frac{m(\alpha-1)}{2\alpha^2}\xi(\alpha)\xi + \frac{1}{\alpha}(\varphi e_i)(\alpha)\varphi e_i + \frac{\alpha-1}{\alpha}g(\nabla_{\varphi e_i}\xi, \varphi e_i)\xi. \tag{2.6}$$

For the term $\overline{\nabla}_\xi \xi$, by noting that

$$Tr_g g(\nabla_\cdot \xi, \xi)\cdot = g(\nabla_\xi \xi, \xi)\xi = 0,$$

we deduce that

$$\overline{\nabla}_\xi \xi = \alpha \nabla_\xi \xi - grad\alpha + \frac{\alpha+1}{\alpha}\xi(\alpha)\xi. \tag{2.7}$$

By using the fact that

$$\overline{e}_i = \frac{1}{\sqrt{\alpha}}e_i, \qquad \overline{\varphi e}_i = \frac{1}{\sqrt{\alpha}}\varphi e_i, \qquad \overline{\xi} = \frac{1}{\alpha}\xi,$$

we obtain the following formulas

$$\overline{\nabla}_{\overline{e_i}}\overline{e_i} = \frac{1}{\alpha}\overline{\nabla}_{e_i}e_i - \frac{1}{2\alpha^2}e_i(\alpha)e_i,$$

$$\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\alpha}\overline{\nabla}_{\varphi e_i}\varphi e_i - \frac{1}{2\alpha^2}(\varphi e_i)(\alpha)\varphi e_i$$

and

$$\overline{\nabla}_{\overline{\xi}}\overline{\xi} = \frac{1}{\alpha^2}\overline{\nabla}_\xi \xi - \frac{1}{\alpha^3}\xi(\alpha)\xi.$$

Returning to equations (2.5), (2.6) and (2.7), we conclude that

$$\overline{\nabla}_{\overline{e_i}}\overline{e_i} = \frac{1}{\alpha}\nabla_{e_i}e_i - \frac{m}{2\alpha^2}grad\alpha + \frac{m(\alpha-1)}{2\alpha^3}\xi(\alpha)\xi + \frac{1}{2\alpha^2}e_i(\alpha)e_i + \frac{\alpha-1}{\alpha^2}g(\nabla_{e_i}\xi, e_i)\xi, \tag{2.8}$$

$$\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\alpha}\nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\alpha^2}grad\alpha + \frac{m(\alpha-1)}{2\alpha^3}\xi(\alpha)\xi + \frac{1}{2\alpha^2}(\varphi e_i)(\alpha)\varphi e_i + \frac{\alpha-1}{\alpha^2}g(\nabla_{\varphi e_i}\xi, \varphi e_i)\xi \tag{2.9}$$

and

$$\overline{\nabla}_{\overline{\xi}}\overline{\xi} = \frac{1}{\alpha}\nabla_\xi \xi - \frac{1}{\alpha^2}grad\alpha + \frac{1}{\alpha^2}\xi(\alpha)\xi. \tag{2.10}$$

Theorem 2.2. *The identity map* $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ *is harmonic if and only if*

$$\frac{m}{\alpha^2}grad\alpha - \left(\frac{m\alpha - m + 1}{\alpha^3}\right)\xi(\alpha)\xi - \frac{\alpha-1}{\alpha^2}\left((div\xi)\xi + \nabla_\xi \xi\right) = 0. \tag{2.11}$$

In particular, if the function α depends only on the direction of ξ, the equation (2.11) becomes

$$\frac{m-1}{\alpha^3}\xi(\alpha)\xi - \frac{\alpha-1}{\alpha^2}((div\xi)\xi + \nabla_\xi\xi) = 0. \tag{2.12}$$

Proof of Theorem 2.2. By definition of the tension field of the identity map $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$, we have

$$\overline{\tau}(Id) = \nabla_{\overline{e_i}}\overline{e_i} - \overline{\nabla}_{\overline{e_i}}\overline{e_i} + \nabla_{\overline{\varphi e_i}}\overline{\varphi e_i} - \overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} + \nabla_{\overline{\xi}}\overline{\xi} - \overline{\nabla}_{\overline{\xi}}\overline{\xi}.$$

Using the fact that

$$\overline{e}_i = \frac{1}{\sqrt{\alpha}}e_i, \qquad \overline{\varphi e}_i = \frac{1}{\sqrt{\alpha}}\varphi e_i, \qquad \overline{\xi} = \frac{1}{\alpha}\xi,$$

we obtain

$$\nabla_{\overline{e_i}}\overline{e_i} = \frac{1}{\alpha}\nabla_{e_i}e_i - \frac{1}{2\alpha^2}e_i(\alpha)e_i,$$

$$\nabla_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\alpha}\nabla_{\varphi e_i}\varphi e_i - \frac{1}{2\alpha^2}(\varphi e_i)(\alpha)\varphi e_i$$

and

$$\nabla_{\overline{\xi}}\overline{\xi} = \frac{1}{\alpha^2}\nabla_\xi\xi - \frac{1}{\alpha^3}\xi(\alpha)\xi.$$

Thanks to equations , (2.9) and (2.10), we deduce that

$$\begin{aligned}\nabla_{\overline{e_i}}\overline{e_i} - \overline{\nabla}_{\overline{e_i}}\overline{e_i} &= \frac{m}{2\alpha^2}grad\alpha - \frac{m(\alpha-1)}{2\alpha^3}\xi(\alpha)\xi - \frac{1}{\alpha^2}e_i(\alpha)e_i \\ &- \frac{\alpha-1}{\alpha^2}g(\nabla_{e_i}\xi, e_i)\xi,\end{aligned}$$

$$\begin{aligned}\nabla_{\overline{\varphi e_i}}\overline{\varphi e_i} - \overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} &= \frac{m}{2\alpha^2}grad\alpha - \frac{m(\alpha-1)}{2\alpha^3}\xi(\alpha)\xi - \frac{1}{\alpha^2}(\varphi e_i)(\alpha)\varphi e_i \\ &- \frac{\alpha-1}{\alpha^2}g(\nabla_{\varphi e_i}\xi, \varphi e_i)\xi\end{aligned}$$

and

$$\nabla_{\overline{\xi}}\overline{\xi} - \overline{\nabla}_{\overline{\xi}}\overline{\xi} = \frac{1-\alpha}{\alpha^2}\nabla_\xi\xi + \frac{1}{\alpha^2}grad\alpha - \frac{\alpha+1}{\alpha^3}\xi(\alpha)\xi.$$

It follows that

$$\overline{\tau}(Id) = \frac{m}{\alpha^2}grad\alpha - \frac{m\alpha-m+1}{\alpha^3}\xi(\alpha)\xi - \frac{\alpha-1}{\alpha^2}\{(div\xi)\xi + \nabla_\xi\xi\}, \tag{2.13}$$

where

$$div\xi = g(\nabla_{e_i}\xi, e_i) + g(\nabla_{\varphi e_i}\xi, \varphi e_i).$$

Then, the identity map $Id : (M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \longrightarrow (M^{2m+1}, \varphi, \xi, \eta, g)$ is harmonic if and only if

$$\frac{m}{\alpha^2}grad\alpha - \left(\frac{m\alpha-m+1}{\alpha^3}\right)\xi(\alpha)\xi - \frac{\alpha-1}{\alpha^2}((div\xi)\xi + \nabla_\xi\xi) = 0.$$

In particular, if the function α depends only on the direction of ξ $(grad\alpha = \xi(\alpha)\xi)$, the harmonicity condition of $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is given by

$$\frac{m-1}{\alpha^3}\xi(\alpha)\xi - \frac{\alpha-1}{\alpha^2}((div\xi)\xi + \nabla_\xi\xi) = 0.$$

In the case where $\left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is a Kenmotsu manifold, we obtain the following Corollary.

Corollary 2.2. *Let $\left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ be a Kenmotsu manifold. Then the identity map $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is harmonic if and only if*

$$m\alpha grad\alpha - (m\alpha - m + 1)\xi(\alpha)\xi - 2m\alpha(\alpha-1)\xi = 0. \tag{2.14}$$

Moreover, if α depends only on the direction of ξ, the equation (2.14) becomes

$$(m-1)\xi(\alpha) - 2m\alpha(\alpha-1) = 0. \tag{2.15}$$

To study the biharmonicity of the identity map $Id : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$, we need the following Lemma.

Lemma 2.1. *Let $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ be a Kenmotsu manifold and let*

$$\left(\overline{\varphi} = \varphi, \overline{\xi} = \frac{1}{\alpha}\xi, \overline{\eta} = \alpha\eta, \overline{g} = \alpha g + \alpha(\alpha-1)\eta\otimes\eta\right)$$

a generalized $\mathcal{D}$-homothetic deformation defined on M, where α depends only on the direction of ξ. Then for any smooth function f on M which depends only on the direction of ξ, we have

$$Tr_{\overline{g}}\nabla^2 f\xi = \frac{1}{\alpha^2}\xi(\xi(f))\xi + \frac{m-1}{\alpha^3}\xi(\alpha)\xi(f)\xi + \frac{2m}{\alpha^2}\xi(f)\xi - \frac{2mf}{\alpha}\xi \tag{2.16}$$

Proof of Lemma 2.1. Let $\{e_i, \varphi e_i, \xi\}_{i=1}^m$ be an orthonormal frame on the Kenmotsu manifold $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$. By definition, we have

$$\begin{aligned} Tr_{\overline{g}}\nabla^2 f\xi &= \nabla_{\overline{e_i}}\nabla_{\overline{e_i}}f\xi - \nabla_{\overline{\nabla}_{\overline{e_i}}\overline{e_i}}f\xi + \nabla_{\overline{\varphi e_i}}\nabla_{\overline{\varphi e_i}}f\xi \\ &- \nabla_{\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i}}f\xi + \nabla_{\overline{\xi}}\nabla_{\overline{\xi}}f\xi - \nabla_{\overline{\nabla}_{\overline{\xi}}\overline{\xi}}f\xi, \end{aligned} \tag{2.17}$$

where

$$\overline{e}_i = \frac{1}{\sqrt{\alpha}}e_i, \quad \overline{\varphi e}_i = \frac{1}{\sqrt{\alpha}}\varphi e_i, \quad \overline{\xi} = \frac{1}{\alpha}\xi.$$

Using the fact that $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ be a Kenmotsu manifold and the fact that α depends only on the direction of ξ, we obtain

$$\nabla_{e_i}\xi = e_i, \quad \nabla_{\varphi e_i}\xi = \varphi e_i, \quad \nabla_\xi\xi = 0.$$

Then equations (2.8), (2.9) and (2.10) give us

$$\overline{\nabla}_{\overline{e_i}}\overline{e_i} = \frac{1}{\alpha}\nabla_{e_i}e_i - \frac{m}{2\alpha^3}\xi(\alpha)\xi + \frac{m(\alpha-1)}{\alpha^2}\xi,$$

$$\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i} = \frac{1}{\alpha}\nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\alpha^3}\xi(\alpha)\xi + \frac{m(\alpha-1)}{\alpha^2}\xi$$

and

$$\overline{\nabla}_{\overline{\xi}}\overline{\xi} = 0.$$

For the term $\nabla_{\overline{e_i}}\nabla_{\overline{e_i}}f\xi - \nabla_{\overline{\nabla}_{\overline{e_i}}\overline{e_i}}f\xi$, we obtain

$$\nabla_{\overline{e_i}}\nabla_{\overline{e_i}}f\xi - \nabla_{\overline{\nabla}_{\overline{e_i}}\overline{e_i}}f\xi = \frac{f}{\alpha}\nabla_{e_i}\nabla_{e_i}\xi - \frac{1}{\alpha}\nabla_{\nabla_{e_i}e_i}f\xi + \frac{m}{2\alpha^3}\xi(\alpha)\nabla_\xi f\xi - \frac{m(\alpha-1)}{\alpha^2}\nabla_\xi f\xi.$$

A simple calculation gives us

$$\nabla_{e_i}\nabla_{e_i}\xi = \nabla_{e_i}e_i,$$

$$\nabla_{\nabla_{e_i}e_i}f\xi = f\nabla_{e_i}e_i + mf\xi - m\xi(f)\xi$$

and

$$\nabla_\xi f\xi = \xi(f)\xi,$$

it follows that

$$\nabla_{\overline{e_i}}\nabla_{\overline{e_i}}f\xi - \nabla_{\overline{\nabla}_{\overline{e_i}}\overline{e_i}}f\xi = \frac{m}{2\alpha^3}\xi(\alpha)\xi(f)\xi + \frac{m}{\alpha^2}\xi(f)\xi - \frac{mf}{\alpha}\xi. \tag{2.18}$$

The same method of calculation gives

$$\nabla_{\overline{\varphi e_i}}\nabla_{\overline{\varphi e_i}}f\xi - \nabla_{\overline{\nabla}_{\overline{\varphi e_i}}\overline{\varphi e_i}}f\xi = \frac{m}{2\alpha^3}\xi(\alpha)\xi(f)\xi + \frac{m}{\alpha^2}\xi(f)\xi - \frac{mf}{\alpha}\xi. \tag{2.19}$$

To complete the proof, note that

$$\nabla_{\overline{\xi}}\nabla_{\overline{\xi}}f\xi = \frac{1}{\alpha^2}\xi(\xi(f))\xi - \frac{1}{\alpha^3}\xi(\alpha)\xi(f)\xi. \tag{2.20}$$

By replacing (2.18), (2.19) and (2.20) in (2.17), we deduce that

$$Tr_{\overline{g}}\nabla^2 f\xi = \frac{1}{\alpha^2}\xi(\xi(f))\xi + \frac{m-1}{\alpha^3}\xi(\alpha)\xi(f)\xi + \frac{2m}{\alpha^2}\xi(f)\xi - \frac{2mf}{\alpha}\xi.$$

Thanks to Lemma 2.1, we will characterize the biharmonicity of the identity map $Id_M : \left(M^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2n+1}, \varphi, \xi, \eta, g\right)$.

Theorem 2.3. *Let* $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ *be a Kenmotsu manifold and let*

$$\left(\overline{\varphi} = \varphi, \overline{\xi} = \frac{1}{\alpha}\xi, \overline{\eta} = \alpha\eta, \overline{g} = \alpha g + \alpha(\alpha-1)\eta\otimes\eta\right)$$

a generalized $\mathcal{D}$-homothetic deformation defined on M where we suppose that α depends only on the direction of ξ. The identity map $Id_M : \left(M^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2n+1}, \varphi, \xi, \eta, g\right)$ is biharmonic if and only if

$$\begin{array}{l} \frac{(m-1)}{\alpha^5}\xi^{(3)}(\alpha) + \frac{(m-1)(m-10)}{\alpha^6}\xi(\alpha)\xi^{(2)}(\alpha) + \frac{2m(m-3)}{\alpha^5}\xi^{(2)}(\alpha) \\ +\frac{2m}{\alpha^4}\xi^{(2)}(\alpha) - \frac{3(m-1)(m-5)}{\alpha^7}(\xi(\alpha))^3 - \frac{2m(5m-11)}{\alpha^6}(\xi(\alpha))^2 \\ +\frac{2m(m-3)}{\alpha^5}(\xi(\alpha))^2 - \frac{8m^2}{\alpha^5}\xi(\alpha) + \frac{4m}{\alpha^4}\xi(\alpha) + \frac{8m^2(\alpha-1)}{\alpha^3} = 0. \end{array}$$

Proof of Theorem 2.3. The tension field of the identity map

$$Id_M : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$$

is given by

$$\overline{\tau}(Id_M) = \frac{m-1}{\alpha^3}\xi(\alpha)\xi - \frac{2m(\alpha-1)}{\alpha^2}\xi,$$

then, $Id_M : \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow \left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ is biharmonic if and only if

$$\begin{array}{l} (m-1)\left\{Tr_{\overline{g}}\nabla^2\frac{1}{\alpha^3}\xi(\alpha)\xi + \frac{1}{\alpha^3}\xi(\alpha)Tr_{\overline{g}}R(\xi,\cdot)\cdot\right\} \\ -2m\left\{Tr_{\overline{g}}\nabla^2\frac{\alpha-1}{\alpha^2}\xi + \frac{\alpha-1}{\alpha^2}Tr_{\overline{g}}R(\xi,\cdot)\cdot\right\} = 0. \end{array} \tag{2.21}$$

As α depends only on the direction of ξ, it is easy to prove that $\xi(\alpha)$ depends only on the direction of ξ. Using Lemma 2.1, we obtain

$$\begin{aligned} Tr_{\overline{g}}\nabla^2\frac{1}{\alpha^3}\xi(\alpha)\xi &= \frac{1}{\alpha^2}\xi^{(2)}\left(\frac{1}{\alpha^3}\xi(\alpha)\right)\xi + \frac{m-1}{\alpha^3}\xi(\alpha)\xi\left(\frac{1}{\alpha^3}\xi(\alpha)\right)\xi \\ &+ \frac{2m}{\alpha^2}\xi\left(\frac{1}{\alpha^3}\xi(\alpha)\right)\xi - \frac{2m}{\alpha^4}\xi(\alpha)\xi. \end{aligned}$$

A simple calculation gives

$$\xi\left(\frac{1}{\alpha^3}\xi(\alpha)\right) = \frac{1}{\alpha^3}\xi^{(2)}(\alpha) - \frac{3}{\alpha^4}(\xi(\alpha))^2$$

and

$$\xi^{(2)}\left(\frac{1}{\alpha^3}\xi(\alpha)\right) = \frac{1}{\alpha^3}\xi^{(3)}(\alpha) - \frac{9}{\alpha^4}\xi(\alpha)\xi^{(2)}(\alpha) + \frac{12}{\alpha^5}(\xi(\alpha))^3,$$

it follows that

$$\begin{aligned} Tr_{\overline{g}}\nabla^2\frac{1}{\alpha^3}\xi(\alpha)\xi &= \frac{1}{\alpha^5}\xi^{(3)}(\alpha)\xi + \frac{m-10}{\alpha^6}\xi(\alpha)\xi^{(2)}(\alpha)\xi + \frac{2m}{\alpha^5}\xi^{(2)}(\alpha)\xi \\ &- \frac{3(m-5)}{\alpha^7}(\xi(\alpha))^3\xi - \frac{6m}{\alpha^6}(\xi(\alpha))^2\xi - \frac{2m}{\alpha^4}\xi(\alpha)\xi. \end{aligned} \tag{2.22}$$

A similar calculation gives us

$$\xi\left(\frac{\alpha-1}{\alpha^2}\right) = \frac{2}{\alpha^3}\xi(\alpha) - \frac{1}{\alpha^2}\xi(\alpha)$$

and

$$\xi\left(\xi\left(\frac{\alpha-1}{\alpha^2}\right)\right)=\frac{2}{\alpha^3}\xi^{(2)}(\alpha)-\frac{1}{\alpha^2}\xi^{(2)}(\alpha)+\frac{2}{\alpha^3}(\xi(\alpha))^2-\frac{6}{\alpha^4}(\xi(\alpha))^2,$$

which gives

$$\begin{aligned} Tr_{\overline{g}}\nabla^2\frac{\alpha-1}{\alpha^2}\xi &= \frac{2}{\alpha^5}\xi^{(2)}(\alpha)\xi-\frac{1}{\alpha^4}\xi^{(2)}(\alpha)\xi+\frac{2(m-4)}{\alpha^6}(\xi(\alpha))^2\xi \\ &- \frac{m-3}{\alpha^5}(\xi(\alpha))^2\xi+\frac{4m}{\alpha^5}\xi(\alpha)\xi-\frac{2m}{\alpha^4}\xi(\alpha)\xi-\frac{2m(\alpha-1)}{\alpha^3}\xi. \end{aligned} \tag{2.23}$$

Finally, it is easy to see that

$$Tr_{\overline{g}}R(\xi,\cdot)\cdot=-\frac{2m}{\alpha}\xi. \tag{2.24}$$

By replacing (2.24), (2.23) and (2.22) in (2.21), we deduce that the identity map $Id_M:\left(M^{2n+1},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)\longrightarrow\left(M^{2n+1},\varphi,\xi,\eta,g\right)$ is biharmonic if and only if

$$\begin{aligned} &\tfrac{(m-1)}{\alpha^5}\xi^{(3)}(\alpha)+\tfrac{(m-1)(m-10)}{\alpha^6}\xi(\alpha)\xi^{(2)}(\alpha)+\tfrac{2m(m-3)}{\alpha^5}\xi^{(2)}(\alpha) \\ &+\tfrac{2m}{\alpha^4}\xi^{(2)}(\alpha)-\tfrac{3(m-1)(m-5)}{\alpha^7}(\xi(\alpha))^3-\tfrac{2m(5m-11)}{\alpha^6}(\xi(\alpha))^2 \\ &+\tfrac{2m(m-3)}{\alpha^5}(\xi(\alpha))^2-\tfrac{8m^2}{\alpha^5}\xi(\alpha)+\tfrac{4m}{\alpha^4}\xi(\alpha)+\tfrac{8m^2(\alpha-1)}{\alpha^3}=0. \end{aligned}$$

Example 2.1. We consider the manifold $M=\left\{(x,y,z,u,v)\in\mathbb{R}^5\right\}$, where (x,y,z,u,v) are the standard coordinate in $\mathbb{R}^5$. An orthonormal frame is given by $e_1=e^{-v}\frac{\partial}{\partial x}$, $e_2=e^{-v}\frac{\partial}{\partial y}$ and $e_3=e^{-v}\frac{\partial}{\partial z}$, $e_4=e^{-v}\frac{\partial}{\partial u}$ and $e_5=e^{-v}\frac{\partial}{\partial v}$. taking $e_5=\xi$ and using Koszul's formula we get the following

$$\begin{array}{lllll} \nabla_{e_1}e_1=-e_5, & \nabla_{e_1}e_2=0, & \nabla_{e_1}e_3=0, & \nabla_{e_1}e_4=0, & \nabla_{e_1}e_5=e_1 \\ \nabla_{e_2}e_1=0, & \nabla_{e_2}e_2=-e_5, & \nabla_{e_2}e_3=0, & \nabla_{e_2}e_4=0, & \nabla_{e_2}e_5=e_2 \\ \nabla_{e_3}e_1=0, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_3=-e_5, & \nabla_{e_3}e_4=0, & \nabla_{e_3}e_5=e_3 \\ \nabla_{e_4}e_1=0, & \nabla_{e_4}e_2=0, & \nabla_{e_4}e_3=0, & \nabla_{e_4}e_4=-e_5, & \nabla_{e_4}e_5=e_4 \\ \nabla_{e_5}e_1=0, & \nabla_{e_5}e_2=0, & \nabla_{e_5}e_3=0, & \nabla_{e_5}e_4=0, & \nabla_{e_5}e_5=0. \end{array}$$

We suppose that the function α depends only on v, by Theorem 2.2, we deduce that the identity map $Id_M:\left(M,\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)\longrightarrow(M,\varphi,\xi,\eta,g)$ is harmonic if and only if

$$e^{-v}\alpha'(v)-4\alpha(\alpha-1)=0.$$

By solving this last equation, we conclude that $Id_M:\left(M,\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)\longrightarrow(M,\varphi,\xi,\eta,g)$ is harmonic if and only if

$$\alpha(v)=\frac{1}{Ce^{4e^v}+1}.$$

By Theorem 2.3, the identity map $Id_M:\left(M,\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g}\right)\longrightarrow(M,\varphi,\xi,\eta,g)$ is biharmonic if and only if

$$\begin{aligned} &\tfrac{e^{-v}}{\alpha^5}\alpha^{(3)}-\tfrac{8e^{-2v}}{\alpha^6}\alpha'\alpha''+\tfrac{e^{-v}(4\alpha-6)}{\alpha^5}\alpha'' \\ &+\tfrac{e^{-v}(4\alpha+27)}{\alpha^5}\alpha'-\tfrac{4e^{-2v}(\alpha-3)}{\alpha^6}(\alpha')^2 \\ &+\tfrac{9e^{-3v}}{\alpha^7}(\alpha')^3+\tfrac{32(\alpha-1)}{\alpha^3}=0. \end{aligned}$$

Example 2.2. We consider the manifold $M = \left\{(x, y, z) \in \mathbb{R}^3, z \neq 0\right\}$. The Riemannian metric on M is defined by

$$g = \frac{1}{z^2}dx^2 + \frac{1}{z^2}dy^2 + \frac{1}{z^2}dz^2,$$

and the orthonormal frame is given by $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$ and $e_3 = -z\frac{\partial}{\partial z}$. The vector fields e_1, e_2 and e_3 satisfies

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

If we suppose that the function α depends only on z, we deduce that identity map $Id : \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M, \varphi, \xi, \eta, g)$ is biharmonic if and only if α is solution of the following differential equation

$$\begin{array}{c} \frac{2z}{\alpha^5}\alpha'' - \frac{z}{\alpha^4}\alpha'' + \frac{2}{\alpha^5}\alpha' - \frac{1}{\alpha^4}\alpha' + \frac{6z^2}{\alpha^6}\left(\alpha'\right)^2 \\ -\frac{2z^2}{\alpha^5}\left(\alpha'\right)^2 + \frac{4z}{\alpha^5}\alpha' - \frac{2z}{\alpha^4}\alpha' + \frac{4(\alpha-1)}{\alpha^3} = 0. \end{array}$$

Example 2.3. We consider the manifold $\left(M^3, \varphi, \xi, \eta, g\right)$ be a Kenmotsu manifold with a φ-basis

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{-z}\frac{\partial}{\partial y}, \qquad e_3 = \xi = \frac{\partial}{\partial z}.$$

The Riemannian metric on M is defined by

$$g = e^{2z}dx^2 + e^{2z}dy^2 + dz^2,$$

The vector fields e_1, e_2 and e_3 satisfies

$$\begin{array}{lll} \nabla_{e_1} e_1 = -\xi, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -\xi, & \nabla_{e_2} e_3 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

If we suppose that the function α depends only on z, we deduce that identity map $Id : \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right) \longrightarrow (M, \varphi, \xi, \eta, g)$ is biharmonic if and only if α is solution of the following differential equation

$$\alpha\left(\alpha - 2\right)\alpha'' - 2\left(\alpha - 3\right)\left(\alpha'\right)^2 + 2\alpha\left(\alpha - 2\right)\alpha' + 4\alpha^3\left(\alpha - 1\right) = 0.$$

2.3. The biharmonicity of the identity map $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$.

Theorem 2.4. *The identity map* $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is harmonic if and only if*

$$\begin{array}{c} -\left(\frac{\alpha+m-1}{\alpha}\right) grad\alpha + \left(\frac{\alpha^2+m\alpha-m}{\alpha^2}\right)\xi\left(\alpha\right)\xi \\ +\frac{\alpha-1}{\alpha}\left(div\xi\right)\xi + \left(\alpha - 1\right)\nabla_\xi \xi = 0. \end{array} \tag{2.25}$$

In particular, if the function α *depends only on the direction of* ξ*, the equation (2.25) becomes*

$$\left(\frac{\alpha - m}{\alpha^2}\xi\left(\alpha\right) + \frac{\alpha - 1}{\alpha}\left(div\xi\right)\right)\xi + \left(\alpha - 1\right)\nabla_\xi \xi = 0. \tag{2.26}$$

Proof of Theorem 2.4. By definition of the tension field of the identity map $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$, we have

$$\overline{\tau}\,(Id) = \overline{\nabla}_{e_i} e_i - \nabla_{e_i} e_i + \overline{\nabla}_{\varphi e_i}\varphi e_i - \nabla_{\varphi e_i}\varphi e_i + \overline{\nabla}_\xi \xi - \nabla_\xi \xi.$$

Using Theorem 2.1, we obtain

$$\begin{aligned} \overline{\nabla}_{e_i} e_i &= \nabla_{e_i} e_i - \frac{m}{2\alpha} grad\alpha + \frac{m\,(\alpha-1)}{2\alpha^2}\xi\,(\alpha)\,\xi \\ &+ \frac{1}{\alpha} e_i\,(\alpha)\,e_i + \frac{\alpha-1}{\alpha} g\,(\nabla_{e_i}\xi, e_i)\,\xi, \end{aligned}$$

$$\begin{aligned} \overline{\nabla}_{\varphi e_i}\varphi e_i &= \nabla_{\varphi e_i}\varphi e_i - \frac{m}{2\alpha} grad\alpha + \frac{m\,(\alpha-1)}{2\alpha^2}\xi\,(\alpha)\,\xi \\ &+ \frac{1}{\alpha}\,(\varphi e_i)\,(\alpha)\,\varphi e_i + \frac{\alpha-1}{\alpha} g\,(\nabla_{\varphi e_i}\xi, \varphi e_i)\,\xi \end{aligned}$$

and

$$\overline{\nabla}_\xi \xi = \alpha \nabla_\xi \xi - grad\alpha + \frac{\alpha+1}{\alpha}\xi\,(\alpha)\,\xi.$$

It follows that

$$\begin{aligned} \overline{\tau}\,(Id) &= -\frac{\alpha+m}{\alpha} grad\alpha + \frac{\alpha^2+(m+1)\,\alpha-m}{\alpha^2}\xi\,(\alpha)\,\xi + \frac{1}{\alpha} e_i\,(\alpha)\,e_i \\ &+ \frac{\alpha-1}{\alpha} g\,(\nabla_{e_i}\xi, e_i)\,\xi + \frac{1}{\alpha}\,(\varphi e_i)\,(\alpha)\,\varphi e_i \\ &+ \frac{\alpha-1}{\alpha} g\,(\nabla_{\varphi e_i}\xi, \varphi e_i)\,\xi + (\alpha-1)\,\nabla_\xi \xi. \end{aligned}$$

Finally, using the fact that

$$grad\alpha = e_i\,(\alpha)\,e_i + (\varphi e_i)\,(\alpha)\,\varphi e_i + \xi\,(\alpha)\,\xi$$

and

$$div\xi = g\,(\nabla_{e_i}\xi, e_i) + g\,(\nabla_{\varphi e_i}\xi, \varphi e_i)\,,$$

we deduce that

$$\begin{aligned} \overline{\tau}\,(Id) &= -\frac{\alpha+m-1}{\alpha} grad\alpha + \frac{\alpha^2+m\alpha-m}{\alpha^2}\xi\,(\alpha)\,\xi \\ &+ \frac{\alpha-1}{\alpha}\,(div\xi)\,\xi + (\alpha-1)\,\nabla_\xi \xi. \end{aligned}$$

Then, the identity map $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is harmonic if and only if

$$-\frac{\alpha+m-1}{\alpha} grad\alpha + \frac{\alpha^2+m\alpha-m}{\alpha^2}\xi\,(\alpha)\,\xi + \frac{\alpha-1}{\alpha}\,(div\xi)\,\xi + (\alpha-1)\,\nabla_\xi \xi = 0.$$

In particular, if the function α depends only on the direction of ξ, the harmonicity condition of $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is given by

$$\left(\frac{\alpha-m}{\alpha^2}\xi\,(\alpha) + \frac{\alpha-1}{\alpha}\,(div\xi)\right)\xi + (\alpha-1)\,\nabla_\xi \xi = 0.$$

Corollary 2.3. *The identity map* $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is harmonic if and only if*

$$\left(-\alpha^2 - m\alpha + \alpha\right) grad\alpha + \left(\alpha^2 + m\alpha - m\right) \xi(\alpha) \xi + 2m\alpha(\alpha - 1) \xi = 0. \tag{2.27}$$

In particular if α *depends only on the direction of* ξ*, the equation (2.27) becomes*

$$(\alpha - m) \xi(\alpha) + 2m\alpha(\alpha - 1) = 0. \tag{2.28}$$

Now, we study the biharmonicity of the identity map $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$.

Theorem 2.5. *Let* $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ *be a Kenmotsu manifold and let*

$$\left(\overline{\varphi} = \varphi, \overline{\xi} = \frac{1}{\alpha}\xi, \overline{\eta} = \alpha\eta, \overline{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta\right)$$

a generalized $\mathcal{D}$*-homothetic deformation defined on* M*, where we suppose that the function* α *depend only on the direction of* ξ*. Then, the identity map* $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ *is biharmonic if and only if*

$$\begin{aligned}\tfrac{\alpha-m}{\alpha^2}\xi^{(3)}(\alpha) + \tfrac{2m(\alpha+m)}{\alpha^4}\xi(\alpha)\xi^{(2)}(\alpha) + \tfrac{2m\left(2\alpha^2-2m\alpha+m\right)}{\alpha^3}\xi^{(2)}(\alpha) - \tfrac{m(\alpha+m)}{\alpha^5}(\xi(\alpha))^3 \\ + \tfrac{2m(2m-1)}{\alpha^3}(\xi(\alpha))^2 + \tfrac{4m\left(m\alpha^2-(m+1)\alpha+2m\right)}{\alpha^3}\xi(\alpha) - \tfrac{8m^2(\alpha-1)}{\alpha^2} = 0.\end{aligned}$$

To prove Theorem 2.5, we will need two lemmas. In the first lemma, we calculate the two terms $Tr_g\overline{\nabla}^2\xi$ and $Tr_g\overline{\nabla}^2 f\xi$, where the function f depends only on the direction of ξ

Lemma 2.2. *Let* $\left(M^{2m+1}, \varphi, \xi, \eta, g\right)$ *be a Kenmotsu manifold and let*

$$\left(\overline{\varphi} = \varphi, \overline{\xi} = \frac{1}{\alpha}\xi, \overline{\eta} = \alpha\eta, \overline{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta\right)$$

a generalized $\mathcal{D}$*-homothetic deformation defined on* M*, where* α *depend only on the direction of* ξ*. If we consider a smooth function* f *on* M *where we suppose that* f *depends only on the direction of* ξ*, we have*

$$\begin{aligned} Tr_g\overline{\nabla}^2\xi &= \frac{1}{\alpha}\xi^2(\alpha)\xi - \frac{m}{2\alpha^3}(\xi(\alpha))^2\xi \\ &+ \frac{2m(\alpha-1)}{\alpha^2}\xi(\alpha)\xi - \frac{2m}{\alpha}\xi \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} Tr_g\overline{\nabla}^2 f\xi &= \xi^{(2)}(f)\xi + \frac{f}{\alpha}\xi^{(2)}(\alpha)\xi + \frac{2}{\alpha}\xi(\alpha)\xi(f)\xi + 2m\xi(f)\xi \\ &+ \frac{2m(\alpha-1)f}{\alpha^2}\xi(\alpha)\xi - \frac{mf}{2\alpha^3}(\xi(\alpha))^2\xi - \frac{2mf}{\alpha}\xi. \end{aligned} \tag{2.30}$$

Proof of Lemma 2.2. First, we will simplify the term $Tr_g\overline{\nabla}^2\xi$. By definition, we have

$$Tr_g\overline{\nabla}^2\xi = \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi + \overline{\nabla}_{\varphi e_i}\overline{\nabla}_{\varphi e_i}\xi - \overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi + \overline{\nabla}_{\xi}\overline{\nabla}_{\xi}\xi - \overline{\nabla}_{\nabla_{\xi}\xi}\xi.$$

Using equation (2.4), we obtain

$$\overline{\nabla}_{e_i}\xi = e_i + \frac{1}{2\alpha}\xi(\alpha)\, e_i,$$

then

$$\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi = \overline{\nabla}_{e_i}e_i + \overline{\nabla}_{e_i}\frac{1}{2\alpha}\xi(\alpha)\, e_i.$$

Using the fact that

$$\overline{\nabla}_{e_i}e_i = \nabla_{e_i}e_i - \frac{m}{2\alpha^2}\xi(\alpha)\,\xi + \frac{m(\alpha-1)}{\alpha}\xi$$

and

$$e_i(\alpha) = e_i(\xi(\alpha)) = 0,$$

we deduce that

$$\begin{aligned}\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi &= \nabla_{e_i}e_i + \frac{1}{2\alpha}\xi(\alpha)\nabla_{e_i}e_i - \frac{m}{4\alpha^3}(\xi(\alpha))^2\xi \\ &+ \frac{m(\alpha-2)}{2\alpha^2}\xi(\alpha)\,\xi + \frac{m(\alpha-1)}{\alpha}\xi.\end{aligned}$$

For the term $\overline{\nabla}_{\nabla_{e_i}e_i}\xi$, noting that

$$\eta(\nabla_{e_i}e_i) = g(\nabla_{e_i}e_i, \xi) = -m$$

and

$$(\nabla_{e_i}e_i)(\alpha) = -m\xi(\alpha),$$

a similar calculation gives us

$$\overline{\nabla}_{\nabla_{e_i}e_i}\xi = \nabla_{e_i}e_i + \frac{1}{2\alpha}\xi(\alpha)\nabla_{e_i}e_i - \frac{m}{2\alpha}\xi(\alpha)\,\xi + m\xi,$$

it follows that

$$\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\xi - \overline{\nabla}_{\nabla_{e_i}e_i}\xi = -\frac{m}{4\alpha^3}(\xi(\alpha))^2\xi + \frac{m(\alpha-1)}{\alpha^2}\xi(\alpha)\,\xi - \frac{m}{\alpha}\xi.$$

The same calculation method gives

$$\overline{\nabla}_{\varphi e_i}\overline{\nabla}_{\varphi e_i}\xi - \overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi = -\frac{m}{4\alpha^3}(\xi(\alpha))^2\xi + \frac{m(\alpha-1)}{\alpha^2}\xi(\alpha)\,\xi - \frac{m}{\alpha}\xi.$$

Finally, by a simple calculation, we prove that

$$\overline{\nabla}_{\xi}\overline{\nabla}_{\xi}\xi = \frac{1}{\alpha}\xi^2(\alpha)\,\xi,$$

where

$$\xi^2(\alpha) = \xi(\xi(\alpha)).$$

We conclude that

$$Tr_g\overline{\nabla}^2\xi = \frac{1}{\alpha}\xi^2(\alpha)\xi - \frac{m}{2\alpha^3}(\xi(\alpha))^2\xi + \frac{2m(\alpha-1)}{\alpha^2}\xi(\alpha)\xi - \frac{2m}{\alpha}\xi.$$

For the term $Tr_g\overline{\nabla}^2 f\xi$, a rigorous calculation gives us

$$\overline{\nabla}_{\nabla_{e_i}e_i} f\xi = f\overline{\nabla}_{\nabla_{e_i}e_i}\xi - m\xi(f)\xi,$$

$$\overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i} f\xi = f\overline{\nabla}_{\nabla_{\varphi e_i}\varphi e_i}\xi - m\xi(f)\xi$$

and

$$\overline{\nabla}_\xi\overline{\nabla}_\xi f\xi = f\overline{\nabla}_\xi\overline{\nabla}_\xi\xi + \frac{2}{\alpha}\xi(f)\xi(\alpha)\xi + \xi^2(f)\xi.$$

It follows that

$$Tr_g\overline{\nabla}^2 f\xi = fTr_g\overline{\nabla}^2\xi + \xi^2(f)\xi + \frac{2}{\alpha}\xi(f)\xi(\alpha)\xi + 2m\xi(f)\xi.$$

Coming back to the equation (2.29), we deduce that

$$\begin{aligned} Tr_g\overline{\nabla}^2 f\xi &= \xi^{(2)}(f)\xi + \frac{f}{\alpha}\xi^{(2)}(\alpha)\xi + \frac{2}{\alpha}\xi(\alpha)\xi(f)\xi + 2m\xi(f)\xi \\ &+ \frac{2m(\alpha-1)f}{\alpha^2}\xi(\alpha)\xi - \frac{mf}{2\alpha^3}(\xi(\alpha))^2\xi - \frac{2mf}{\alpha}\xi \end{aligned}$$

In the following lemma, we will look at the term $Tr_g\overline{R}(\xi,\cdot)\cdot$.

Lemma 2.3. *Let $\left(M^{2m+1},\varphi,\xi,\eta,g\right)$ be a Kenmotsu manifold and let*

$$\left(\overline{\varphi} = \varphi, \overline{\xi} = \frac{1}{\alpha}\xi, \overline{\eta} = \alpha\eta, \overline{g} = \alpha g + \alpha(\alpha-1)\eta\otimes\eta\right)$$

a generalized $\mathcal{D}$-homothetic deformation defined on M, where α depend only on the direction of ξ. If we consider a smooth any function f on M where we suppose that the f depend only on the direction of ξ, we have

$$Tr_g\overline{R}(\xi,\cdot)\cdot = -\frac{m}{\alpha^2}\xi^{(2)}(\alpha)\xi + \frac{3m}{2\alpha^3}(\xi(\alpha))^2\xi - \frac{2m}{\alpha}\xi. \tag{2.31}$$

Proof of Lemma 2.3. By definition, we have

$$Tr_g\overline{R}(\xi,\cdot)\cdot = \overline{R}(\xi,e_i)e_i + \overline{R}(\xi,\varphi e_i)\varphi e_i.$$

The first term $\overline{R}(\xi,e_i)e_i$ is given by

$$\overline{R}(\xi,e_i)e_i = \overline{\nabla}_\xi\overline{\nabla}_{e_i}e_i - \overline{\nabla}_{e_i}\overline{\nabla}_\xi e_i - \overline{\nabla}_{[\xi,e_i]}e_i.$$

A simple calculation gives us

$$\begin{aligned}\overline{\nabla}_{\xi}\overline{\nabla}_{e_i}e_i &= \overline{\nabla}_{\xi}\nabla_{e_i}e_i - \frac{m}{2}\overline{\nabla}_{\xi}\frac{1}{\alpha^2}\xi(\alpha)\xi + m\overline{\nabla}_{\xi}\frac{(\alpha-1)}{\alpha}\xi \\ &= \nabla_{\xi}\nabla_{e_i}e_i - \frac{m}{2\alpha}\xi(\alpha)\xi + \frac{1}{2\alpha}\xi(\alpha)\nabla_{e_i}e_i \\ &- \frac{m}{2\alpha^3}(\xi(\alpha))^2\xi - \frac{m}{2\alpha^2}\xi^{(2)}(\alpha)\xi \\ &+ \frac{m}{\alpha^3}(\xi(\alpha))^2\xi + \frac{m(\alpha-1)}{\alpha^2}\xi(\alpha)\xi \\ &+ \frac{m}{\alpha^2}\xi(\alpha)\xi \\ &= \nabla_{\xi}\nabla_{e_i}e_i + \frac{1}{2\alpha}\xi(\alpha)\nabla_{e_i}e_i - \frac{m}{2\alpha^2}\xi^{(2)}(\alpha)\xi \\ &+ \frac{m}{2\alpha^3}(\xi(\alpha))^2\xi + \frac{m}{2\alpha}\xi(\alpha)\xi,\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_{e_i}\overline{\nabla}_{\xi}e_i &= \overline{\nabla}_{e_i}\nabla_{\xi}e_i + \frac{1}{2\alpha}\xi(\alpha)\overline{\nabla}_{e_i}e_i \\ &= \nabla_{e_i}\nabla_{\xi}e_i + \frac{1}{2\alpha}\xi(\alpha)\nabla_{e_i}e_i \\ &- \frac{m}{4\alpha^3}(\xi(\alpha))^2\xi + \frac{m(\alpha-1)}{2\alpha^2}\xi(\alpha)\xi\end{aligned}$$

and

$$\overline{\nabla}_{[\xi,e_i]}e_i = \nabla_{[\xi,e_i]}e_i + \frac{m}{2\alpha^2}\xi(\alpha)\xi - \frac{m(\alpha-1)}{\alpha}\xi.$$

Which gives us

$$\overline{R}(\xi,e_i)e_i = -\frac{m}{2\alpha^2}\xi^{(2)}(\alpha)\xi + \frac{3m}{4\alpha^3}(\xi(\alpha))^2\xi - \frac{m}{\alpha}\xi.$$

The same method gives

$$\overline{R}(\xi,\varphi e_i)\varphi e_i = -\frac{m}{2\alpha^2}\xi^{(2)}(\alpha)\xi + \frac{3m}{4\alpha^3}(\xi(\alpha))^2\xi - \frac{m}{\alpha}\xi.$$

We deduce that

$$Tr_g\overline{R}(\xi,\cdot)\cdot = -\frac{m}{\alpha^2}\xi^{(2)}(\alpha)\xi + \frac{3m}{2\alpha^3}(\xi(\alpha))^2\xi - \frac{2m}{\alpha}\xi.$$

Remark 2.1. Using Lemmas 2.2 and 2.3, we obtain the following formula :

$$\begin{aligned}Tr_g\overline{\nabla}^2 f\xi + Tr_g\overline{R}(f\xi,\cdot)\cdot &= \xi^{(2)}(f)\xi + \frac{f}{\alpha}\xi^{(2)}(\alpha)\xi - \frac{mf}{\alpha^2}\xi^{(2)}(\alpha)\xi + \frac{2}{\alpha}\xi(\alpha)\xi(f)\xi \\ &+ 2m\xi(f)\xi + \frac{2m(\alpha-1)f}{\alpha^2}\xi(\alpha)\xi + \frac{mf}{\alpha^3}(\xi(\alpha))^2\xi - \frac{4mf}{\alpha}\xi. \qquad (2.32)\end{aligned}$$

Proof of Theorem 2.5. The tension field of the identity map

$$Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$$

is given by

$$\overline{\tau}\left(Id_M\right) = \frac{\alpha - m}{\alpha^2}\xi\left(\alpha\right)\xi + \frac{2m\left(\alpha - 1\right)}{\alpha}\xi,$$

then Id_M is biharmonic if and only if

$$\begin{array}{c} Tr_g\overline{\nabla}^2 \frac{\alpha-m}{\alpha^2}\xi\left(\alpha\right)\xi + Tr_g\overline{R}\left(\frac{\alpha-m}{\alpha^2}\xi\left(\alpha\right)\xi, \cdot\right)\cdot \\ +2m\left(Tr_g\overline{\nabla}^2 \frac{\alpha-1}{\alpha}\xi + Tr_g\overline{R}\left(\frac{\alpha-1}{\alpha}\xi, \cdot\right)\cdot\right) = 0. \end{array}$$

Using the following formulas

$$\xi\left(\frac{\alpha - m}{\alpha^2}\xi\left(\alpha\right)\right) = \frac{\alpha - m}{\alpha^2}\xi^{(2)}\left(\alpha\right) - \frac{\alpha - 2m}{\alpha^3}\left(\xi\left(\alpha\right)\right)^2,$$

$$\xi^{(2)}\left(\frac{\alpha - m}{\alpha^2}\xi\left(\alpha\right)\right) = \frac{\alpha - m}{\alpha^2}\xi^{(3)}\left(\alpha\right) - \frac{3\left(\alpha - 2m\right)}{\alpha^3}\xi\left(\alpha\right)\xi^{(2)}\left(\alpha\right) + \frac{2\left(\alpha - 3m\right)}{\alpha^4}\left(\xi\left(\alpha\right)\right)^3,$$

$$\xi\left(\frac{\alpha - 1}{\alpha}\right) = \frac{1}{\alpha^2}\xi\left(\alpha\right)$$

and

$$\xi^{(2)}\left(\frac{\alpha - 1}{\alpha}\right) = \frac{1}{\alpha^2}\xi^{(2)}\left(\alpha\right) - \frac{2}{\alpha^3}\left(\xi\left(\alpha\right)\right)^2,$$

we obtain

$$\begin{array}{c} Tr_g\overline{\nabla}^2 \frac{\alpha-m}{\alpha^2}\xi\left(\alpha\right)\xi + Tr_g\overline{R}\left(\frac{\alpha-m}{\alpha^2}\xi\left(\alpha\right)\xi, \cdot\right)\cdot \\ = \frac{\alpha-m}{\alpha^2}\xi^{(3)}\left(\alpha\right)\xi + \frac{2\alpha m+m^2}{\alpha^4}\xi\left(\alpha\right)\xi^{(2)}\left(\alpha\right)\xi \\ +\frac{2m(\alpha-m)}{\alpha^2}\xi^{(2)}\left(\alpha\right)\xi - \frac{m(\alpha+m)}{\alpha^5}\left(\xi\left(\alpha\right)\right)^3\xi \\ +\frac{2m(m\alpha-\alpha+m)}{\alpha^4}\left(\xi\left(\alpha\right)\right)^2\xi - \frac{4m(\alpha-m)}{\alpha^3}\xi\left(\alpha\right)\xi \end{array}$$

and

$$\begin{array}{rcl} Tr_g\overline{\nabla}^2\frac{\alpha - 1}{\alpha}\xi + Tr_g\overline{R}\left(\frac{\alpha - 1}{\alpha}\xi, \cdot\right)\cdot & = & \frac{\alpha^2 - m\alpha + m}{\alpha^3}\xi^{(2)}\left(\alpha\right)\xi + \frac{m\left(\alpha - 1\right)}{\alpha^4}\left(\xi\left(\alpha\right)\right)^2\xi \\ & + & \frac{2m\left(\alpha^2 - \alpha + 1\right)}{\alpha^3}\xi\left(\alpha\right)\xi - \frac{4m\left(\alpha - 1\right)}{\alpha^2}\xi. \end{array}$$

We conclude that the identity map $Id_M : \left(M^{2m+1}, \varphi, \xi, \eta, g\right) \longrightarrow \left(M^{2m+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is biharmonic if and only if

$$\begin{array}{c} \frac{\alpha-m}{\alpha^2}\xi^{(3)}\left(\alpha\right) + \frac{m(2\alpha+m)}{\alpha^4}\xi\left(\alpha\right)\xi^{(2)}\left(\alpha\right) + \frac{2m\left(2\alpha^2-2m\alpha+m\right)}{\alpha^3}\xi^{(2)}\left(\alpha\right) - \frac{m(\alpha+m)}{\alpha^5}\left(\xi\left(\alpha\right)\right)^3 \\ +\frac{2m(2m-1)}{\alpha^3}\left(\xi\left(\alpha\right)\right)^2 + \frac{4m\left(m\alpha^2-(m+1)\alpha+2m\right)}{\alpha^3}\xi\left(\alpha\right) - \frac{8m^2(\alpha-1)}{\alpha^2} = 0. \end{array}$$

Example 2.4. We consider the manifold $\left(M^3, \varphi, \xi, \eta, g\right)$ be a Kenmotsu manifold with a φ-basis

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \qquad e_2 = e^{-z}\frac{\partial}{\partial y}, \qquad e_3 = \xi = \frac{\partial}{\partial z}.$$

The Riemannian metric on M is defined by

$$g = e^{2z}dx^2 + e^{2z}dy^2 + dz^2,$$

The vector fields e_1, e_2 and e_3 satisfies

$$\begin{array}{lll} \nabla_{e_1}e_1 = -\xi, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = -\xi, & \nabla_{e_2}e_3 = e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0. \end{array}$$

If we suppose that the non-constant function α depends only on z, we deduce that identity map $Id : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is harmonic if and only if α is solution of the following differential equation

$$(\alpha - 1)\,\alpha' + 2\alpha\,(\alpha - 1) = 0,$$

then $Id : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is harmonic if and only if $\alpha = Ce^{-2z}$. By Theorem 2.5, the identity map $Id : (M, \varphi, \xi, \eta, g) \longrightarrow \left(M, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ is biharmonic if and only if α is solution of the following differential equation

$$\alpha^3\,(\alpha - 1)\,\alpha^{(3)} + \alpha\,(2\alpha + 1)\,\alpha'\alpha'' + 2\alpha^2\left(2\alpha^2 - 2\alpha + 1\right)\alpha'' - (\alpha + 1)\,(\alpha')^3$$
$$+ 2\alpha^2\,(\alpha')^2 + 4\alpha^2\left(\alpha^2 - 2\alpha + 2\right)\alpha' - 8\alpha^3\,(\alpha - 1) = 0.$$

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