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CERTAIN RESULTS OF RICCI-YAMABE SOLITONS ON $(LCS)_N$-MANIFOLDS

Jay Prakash Singh and Chhakchhuak Zosangzuala

Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram, India

Abstract. The goal of this paper is to characterize Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) admitting Ricci-Yamabe solitons. It is shown that an $(LCS)_n$-manifold which admits the Ricci-Yamabe soliton becomes flat when the soliton is steady. Next, we construct a 3-dimensional and 5-dimensional $(LCS)_n$-manifold that satisfy the result. Also, the expression for the scalar λ on an $(LCS)_n$-manifold admitting conformal Ricci-Yamabe soliton is obtained. Lastly, we extend our study to $\eta-$Ricci-Yamabe soliton on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold in which we have shown the condition when the soliton is shrinking, steady and expanding with ξ being a torse forming vector field.
Keywords:Ricci-Yamabe solitons, torse forming, conformally flat, $(LCS)_n-$manifolds.

1. Introduction

Lorentzian manifold provides important tools to tackle problems in mathematical physics, specially to the theory of general relativity and cosmology. LP-Sasakian manifolds introduced by Matsumoto [8] is generalized by Shaikh [13] by introducing the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n-$manifolds). Many geometers studied $(LCS)_n$-manifold since it has been first introduced (see, for details, [1, 12, 14, 15, 16]).

One of the most significant approach to understand the geometric structure in Riemannian geometry is the theory of geometric flows. Hamilton [4] introduced the

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Corresponding Author: Jay Prakash Singh, Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, Mizoram, India | E-mail: jpsmaths@gmail.com
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Ricci flow which plays a crucial role in forming the proof of the well known Poincaré conjecture. The Ricci soliton on a Riemannian manifold M is defined by

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) = 2\lambda g(X,Y),$$

for any vector fields X, Y on M, where $\mathcal{L}_V g$ denotes the Lie derivative of g along a vector field V, S is the Ricci tensor of M and λ, a real constant. The Ricci soliton is a limiting solution to the Ricci flow and is said to be expanding, steady and shrinking depending on the fact whether $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively.

The Yamabe soliton is a limiting solution to the Yamabe flow introduced by Hamilton [4]. A Yamabe soliton on a Riemannian manifold (M, g) is given by

$$(\mathcal{L}_V g)(X,Y) = 2(r - \lambda)g(X,Y),$$

for any vector fields X, Y on M, where r is the scalar curvature of the manifold and λ, a real constant. For dimension $n = 2$, the Yamabe soliton is equivalent to Ricci soliton. However, for dimension $n > 2$, the Yamabe soliton preserves conformal class of the metric but the Ricci soliton does not preserve the same in general and hence the two solitons do not agree anymore.

Güler and Crasmareanu [3] recently introduced a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map. In [3], the authors define the following:

Definition 1.1. [3] A Riemannian flow on M is a smooth map:

$$g : I \subset \mathbb{R} \to Riem(M),$$

where I is a given open interval.

Definition 1.2. [3] The map $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ given by

$$RY^{(\alpha,\beta,g)}(t) := \frac{\partial g}{\partial t}(t) + 2\alpha Ric(t) + \beta R(t)g(t),$$

is called the (α, β) −Ricci-Yamabe map of the Riemannian flow (M, g). If $RY^{(\alpha,\beta,g)} \equiv 0$, then $g(\cdot)$ will be called an (α, β) −Ricci-Yamabe flow.

The (α, β) −Ricci-Yamabe map can be Riemannian or semi-Riemannian or singular Riemannian flow due to the signs of α and β. The Ricci-Yamabe soliton emerges as the limiting solution of Ricci-Yamabe flow. The Ricci-Yamabe soliton is defined as:

Definition 1.3. [2] A Riemannian manifold (M, g), $n > 2$ is said to admit a Ricci-Yamabe soliton (in short, RYS) $(g, V, \lambda, \alpha, \beta)$ if

$$\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r)\, g, \tag{1.1}$$

where $\lambda, \alpha, \beta \in \mathbb{R}$. If V is gradient of some smooth function f on M, then the above notion is called gradient Ricci-Yamabe soliton (in short, GRYS) and then (1.1) reduces to

$$\nabla^2 f + \alpha S = \left(\lambda - \frac{1}{2}\beta r\right) g, \tag{1.2}$$

where $\nabla^2 f$ is the Hessian of f.

Since $\lambda > 0$, $\lambda < 0$ or $\lambda = 0$, then (M, g) is called Ricci-Yamabe shrinker, Ricci-Yamabe expander or Ricci-Yamabe steady soliton respectively. Since the notion of Ricci-Yamabe flow has been first introduced, many geometers studied Ricci-Yamabe soliton to different extent such as Dey [2], Siddiqi and Akyol [17], Khatri and Singh ([7],[18]).

An advance extension of Ricci-Yamabe soliton is the concept of $\eta-$Ricci-Yamabe soliton (in short, $\eta-$RYS) defined by Siddiqi and Akyol [17], which is given as

$$\mathcal{L}_V g + 2\alpha S + (2\lambda - \beta r)g + 2\mu\eta \otimes \eta = 0. \tag{1.3}$$

In [17], the authors established the geometrical bearing on Riemannian submersions in terms of $\eta-$Ricci-Yamabe soliton with the potential field and giving the classification of any fiber of Riemannian submersion is an $\eta-$Ricci-Yamabe soliton, $\eta-$Ricci soliton and $\eta-$Yamabe soliton.

Again, from [19], we have the following:

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(QY,Z)X - g(QX,Z)Y], \tag{1.4}$$

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y], \tag{1.5}$$

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.6}$$

which are the projective curvature tensor P [20], the conharmonic curvature tensor H [6], and the Riemannian-Christoffel curvature tensor R [9] respectively in a Riemannian manifold (M^n, g), where Q denotes the Ricci operator.

The paper is organized as follows: After Preliminaries, in Section 3, we generalize the results found in [12]. We have studied the Ricci-Yamabe soliton on $(LCS)_n-$manifolds in which we investigate when such manifold admits a Ricci-Yamabe soliton, then the scalar curvature is constant. Also, we have found that if the manifold admits a Ricci-Yamabe soliton, then $R(\xi, X) \cdot S = 0$. Section 4 is devoted to the construction of an example for a $(LCS)_3-$ and $(LCS)_5-$manifolds on which we can easily verify our results. In Section 5, we give the expression for the scalar λ on an $(LCS)_n$-manifold admitting conformal Ricci-Yamabe soliton. In the last section, we extend our study to $\eta-$Ricci-Yamabe soliton on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold, where we have established the condition when the soliton is shrinking, steady and expanding with ξ being a torse forming vector field on such manifold.

2. Preliminaries

Let an $n-$dimensional Lorentzian manifold M admits the characteristic vector field ξ. Then, we have $g(\xi, \xi) = -1$. Since ξ is a unit concircular vector field, we have a non-zero $1-$form η such that for $g(X,\xi) = \eta(X)$, the following equation holds for all vector fields X, Y on M:

$$(\nabla_X \eta)(Y) = \gamma\{g(X,Y) + \eta(X)\eta(Y)\}, \gamma \neq 0 \tag{2.1}$$

where γ is a scalar function on M which satisfies

$$\nabla_X \gamma = X(\gamma) = d\gamma(X) = \rho\eta(X), \tag{2.2}$$

for $\rho \in \mathbb{C}^\infty(M)$, where ∇ is the Levi-Civita connection of g. Let us take a symmetric $(1,1)$ tensor field ϕ denoted by

$$\phi(X) = X(\gamma) = \frac{1}{\gamma}\nabla_X \xi, \tag{2.3}$$

called the structure tensor of the manifold. Thus, the Lorentzian manifold M equipped with ξ, η and ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n-$manifold) [13]. In fact, if we take $\gamma = 1$, then we obtain the LP-Sasakian structure of Matsumoto [8]. The following relations hold in an $(LCS)_n-$manifold $(n > 2)$ for any X, Y, Z on M ([12],[13]):

$$\phi X = X + \eta(X)\xi, \tag{2.4}$$

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.5}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \quad \text{and} \quad g(\phi X, Y) = g(X, \phi Y), \tag{2.6}$$

$$(\nabla_X \phi)Y = \gamma[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.7}$$

$$\eta(\nabla_X \xi) = 0, \quad \nabla_\xi \xi = 0, \tag{2.8}$$

$$R(X,Y)Z = (\gamma^2 - \rho)[g(Y,Z)X - g(X,Z)Y], \tag{2.9}$$

$$S(X,Y) = (\gamma^2 - \rho)(n-1)g(X,Y), \tag{2.10}$$

$$r = n(n-1)(\gamma^2 - \rho), \tag{2.11}$$

$$\nabla\eta = \gamma(g + \eta \otimes \eta), \quad \nabla_\xi \eta = 0, \tag{2.12}$$

$$\mathcal{L}_\xi \phi = 0, \quad \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi g = 2\nabla\eta = 2\gamma(g + \eta \otimes \eta), \tag{2.13}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor, r is the scalar curvature, ∇ is the Levi-Civita connection associated with g.

In [14], A.A. Shaikh studied a conformally flat $(LCS)_n-$manifold and shown that a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold is $\eta-$Einstein and its Ricci tensor is given by

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.14}$$

where $a = \frac{r}{n-1} - (\gamma^2 - \rho)$ and $b = n(\gamma^2 - \rho) - \frac{r}{n-1}$. Then, from (2.14), we have

$$S(X,\xi) = (a-b)\eta(X), \tag{2.15}$$

and

$$S(\xi,\xi) = -(a-b). \tag{2.16}$$

Definition 2.1. [5] A vector field ξ is called torse forming if it satisfies

$$\nabla_X \xi = fX + \nu(X)\xi, \tag{2.17}$$

for a smooth function $f \in \mathbb{C}^\infty(M)$ and ν is an $1-$form, for all vector field X on M.

Remark 2.2. Let ξ be a torse forming vector field on an $(LCS)_n$-manifold. We know that in an $(LCS)_n$-manifold, $\nabla_\xi \xi = 0$. Taking $X = \xi$ in (2.17), we get $(f-1)\xi = 0$. Since $\xi \neq 0$, then $f = 1$.

3. Ricci-Yamabe Soliton (RYS) on $(LCS)_n$ manifolds

In this section, we assume that ξ is the Reeb vector field of the Lorentzian concircular structure.

Consider the Ricci-Yamabe soliton (RYS) on an $n-$dimensional $(LCS)_n$ manifold as

$$\mathcal{L}_\xi g + 2\alpha S = (2\lambda - \beta r)g. \tag{3.1}$$

From (2.13), we have

$$2\gamma(g(X,Y) + \eta(X)\eta(Y)) + 2\alpha S(X,Y) = (2\lambda - \beta r)g(X,Y), \tag{3.2}$$

for all vector fields X, Y on M. This implies

$$(2\lambda - \beta r - 2\gamma)g(X,Y) - 2\alpha g(QX,Y) - 2\gamma\eta(X)\eta(Y) = 0. \tag{3.3}$$

Setting $Y = \xi$ in the above equation and using (2.5), we get

$$g((2\lambda - \beta r)X - 2\alpha QX, \xi) = 0. \tag{3.4}$$

Then, we have

$$QX = \frac{2\lambda - \beta r}{2\alpha}X, \alpha \neq 0. \tag{3.5}$$

Contracting the foregoing equation, we obtain

$$r = \frac{2\lambda n}{2\alpha + n\beta}, 2\alpha + n\beta \neq 0. \tag{3.6}$$

Since, α, β, λ are constants, r is also constant. Now, using (2.10), (2.11) and (3.6), we have

$$S(X,Y) = \frac{2\lambda}{2\alpha + n\beta}g(X,Y), 2\alpha + n\beta \neq 0. \tag{3.7}$$

Also, using (3.6) and (3.7) in (3.1), we get

$$\mathcal{L}_\xi g = 0. \tag{3.8}$$

Thus, we can state the following theorem:

Theorem 3.1. *An $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton has constant scalar curvature and the manifold becomes Einstein manifold provided that*

$\alpha \neq \{0, -\frac{n\beta}{2}\}$. *Moreover,* ξ *is the killing vector field.*

Let us assume that the Ricci tensor S of an $(LCS)_n$ manifold is $\eta-$recurrent, we have

$$\nabla S = \eta \otimes S, \tag{3.9}$$

which implies

$$(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z), \tag{3.10}$$

for all vector fields X, Y, Z on M. From (3.7), we get $\nabla S = 0$. Now, from (3.10), we have

$$\eta(X)S(Y,Z) = 0. \tag{3.11}$$

Since $\eta(X) \neq 0$, we obtain $S(Y,Z) = 0$. Thus, from the expression of S in (3.7), we have $\lambda = 0$. Also, from (3.6), we get $r = 0$, which then implies from (2.11) that $(\gamma^2 - \rho) = 0$ provided $n > 1$. Again, in view of (2.9), we have $R(X,Y)Z = 0$ for all vector fields X, Y, Z on M. This results to the following:

Proposition 3.2. If the Ricci tensor S of an $(LCS)_n$ $(n > 1)$ manifold admitting a RYS is $\eta-$recurrent, then the soliton is steady and the manifold becomes flat.

Let us consider a symmetric $(0,2)$ tensor field h such that

$$h = \frac{1}{2\alpha}\mathcal{L}_\xi g - \frac{2\lambda - \beta r}{2\alpha} g, \alpha \neq 0. \tag{3.12}$$

This implies $\nabla h = 0$. Then,

$$\begin{aligned} h(\xi,\xi) &= \mathcal{L}_\xi g(\xi,\xi) - \tfrac{2\lambda-\beta r}{2\alpha} g(\xi,\xi) \\ &= \tfrac{2\lambda-\beta r}{2\alpha}. \end{aligned} \tag{3.13}$$

As $\nabla h = 0$, then using (3.13) and the results obtained in [1], (3.12) becomes

$$\mathcal{L}_\xi g(X,Y) + 2\alpha S(X,Y) = (2\lambda - \beta r) g(X,Y), \tag{3.14}$$

for all vector fields X, Y on M. This leads to the following theorem:

Theorem 3.3. *Let* $(M, g, \xi, \eta, \phi, \gamma)$ *be an* $(LCS)_n-$*manifold such that a symmetric* $(0,2)$ *tensor field* h *given by* $h = \frac{1}{2\alpha}\mathcal{L}_\xi g - \frac{2\lambda-\beta r}{2\alpha} g$ *with* $\alpha \neq 0$ *and* $\nabla h = 0$. *Then,* (g, ξ) *yields a Ricci-Yamabe soliton on* M.

Let us define a Ricci-Yamabe soliton (RYS) on an $n-$dimensional $(LCS)_n-$manifold M as

$$\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r) g. \tag{3.15}$$

Let $V = t\xi$, where t is a function on M. Then,

$$\mathcal{L}_{t\xi} g(X,Y) + 2\alpha S(X,Y) = (2\lambda - \beta r) g(X,Y), \tag{3.16}$$

for any vector fields X, Y on M. Applying the property of Lie derivative and Levi-Civita connection, we have

$$tg(\nabla_X \xi, Y) + (Xt)\eta(Y) + tg(\nabla_Y \xi, X) + (Yt)\eta(X) + 2\alpha g(QX,Y) = (2\lambda - \beta r) g(X,Y). \tag{3.17}$$

Taking $Y = \xi$ in the above equation and using (2.8), we get

$$-Xt + \left(\xi t + \frac{4\alpha\lambda}{2\alpha + n\beta} - 2\lambda + \beta r\right)\eta(X) = 0, 2\alpha + n\beta \neq 0. \tag{3.18}$$

Taking $X = \xi$ in the foregoing equation, we obtain

$$\xi t = \frac{2\lambda - \beta r}{2} - \frac{2\alpha\lambda}{2\alpha + n\beta}. \tag{3.19}$$

Using (3.19), (3.18) becomes

$$Xt = -\frac{2n\lambda\beta - \beta r(2\alpha + n\beta)}{2(2\alpha + n\beta)}\eta(X). \tag{3.20}$$

Applying exterior differentiation in (3.20), we get

$$\frac{2n\lambda\beta - \beta r(2\alpha + n\beta)}{2(2\alpha + n\beta)} d\eta = 0. \tag{3.21}$$

We know that in an $n-$dimensional $(LCS)_n-$manifold, we have

$$(d\eta)(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]), \tag{3.22}$$

which implies

$$(d\eta)(X, Y) = g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi). \tag{3.23}$$

Using (2.3) and (2.6) in (3.23), we get

$$(d\eta)(X, Y) = 0. \tag{3.24}$$

Hence, the $1-$form η is closed. Then, using the above equation, (3.21) implies either

$$r = \frac{2\lambda n}{2\alpha + n\beta} \quad \text{or} \quad r \neq \frac{2\lambda n}{2\alpha + n\beta}. \tag{3.25}$$

Now, if $r \neq \frac{2\lambda n}{2\alpha + n\beta}$, we have

$$\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r)g. \tag{3.26}$$

Replacing the expression of S from (3.7) in (3.26), we get

$$\mathcal{L}_V g = \left(2\lambda - \frac{4\alpha\lambda}{2\alpha + n\beta} - \beta r\right) g, \tag{3.27}$$

which implies that V is a conformal Killing vector field. Again, if $r = \frac{2\lambda n}{2\alpha + n\beta}$, then from (3.20), we get

$$Xt = 0, \tag{3.28}$$

which implies that t is constant. Therefore, we can state the following theorem:

Theorem 3.4. *If a vector field V on an $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is pointwise collinear with ξ, then either V is a conformal Killing vector field, or V is a constant multiple of ξ, provided that $r \neq \frac{2\lambda n}{2\alpha+n\beta}$ and $2\alpha+n\beta \neq 0$.*

Remark 3.5. The above Theorem 3.4 is a generalization of Theorem 3.8 in Roy *et al.*[12], where they obtained the condition for V to be a conformal Killing vector field is $r \neq \lambda$. It is easy to see that for $\alpha = 0$ and $\beta = 2$, Theorem 3.8 in [12] can be obtained from Theorem 3.4.

As a consequence of Theorem 3.4, substituting $r = \frac{2\lambda n}{2\alpha+n\beta}, \quad 2\alpha + n\beta \neq 0$ in (3.26), we get that

$$\mathcal{L}_V g = 0, \tag{3.29}$$

implying V is a Killing vector field. Then, we have:

Corollary 3.6. If a vector field V on an $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is pointwise collinear with ξ and $r = \frac{2\lambda n}{2\alpha+n\beta}$ with $2\alpha+n\beta \neq 0$, then V becomes a Killing vector field.

Setting $Z = \xi$ in (1.4), we get

$$P(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-1)}[S(Y,\xi)X - S(X,\xi)Y]. \tag{3.30}$$

Taking $Z = \xi$ in (2.9), then using the result and (3.7) in the above equation, we get

$$P(X,Y)\xi = \left((\gamma^2 - \rho) - \frac{2\lambda}{(n-1)(2\alpha+n\beta)}\right)[\eta(Y)X - \eta(X)Y]. \tag{3.31}$$

Using (2.11) and (3.6) in (3.31), we obtain

$$P(X,Y)\xi = 0, \tag{3.32}$$

which results to the following:

Proposition 3.7. An $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is $\xi-$projectively flat.

Again, taking $Z = \xi$ in (1.5), we get

$$\begin{aligned} H(X,Y)\xi = R(X,Y)\xi - \frac{1}{(n-2)} \quad & [g(Y,\xi)QX - g(X,\xi)QY \\ & +S(Y,\xi)X - S(X,\xi)Y]. \end{aligned} \tag{3.33}$$

Putting $Z = \xi$ in (2.9), then using the result and (3.7) in (3.33), we obtain

$$H(X,Y)\xi = \left((\gamma^2 - \rho) - \frac{4\lambda}{(n-2)(2\alpha+n\beta)}\right)[\eta(Y)X - \eta(X)Y]. \tag{3.34}$$

Using (2.11) and (3.6) in the above equation, we get

$$H(X,Y)\xi = -\frac{2n\lambda}{(n-1)(n-2)(2\alpha+n\beta)}[\eta(Y)X - \eta(X)Y]. \tag{3.35}$$

This implies that $H(X,Y)\xi = 0$ if and only if $\lambda = 0$. Hence, we can state the following:

Proposition 3.8. An $(LCS)_n-$manifold admitting a Ricci-Yamabe soliton is $\xi-$conharmonically flat if and only if the soliton is steady.

Now, we know that

$$R(\xi,X)\cdot S = S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z), \tag{3.36}$$

for all vector fields X, Y, Z on M. Interchanging Y and Z, then putting $Z = \xi$ in (2.9), then using the result and (3.7) in (3.36), the above equation becomes

$$\begin{aligned} R(\xi,X)\cdot S &= \tfrac{2\lambda}{2\alpha+n\beta}(\gamma^2-\rho)[g(X,Y)\eta(Z) - g(X,Z)\eta(Y) \\ &\quad + g(X,Z)\eta(Y) - g(Y,X)\eta(Z)], \end{aligned} \tag{3.37}$$

which implies that $R(\xi,X)\cdot S = 0$. This leads to the following theorem:

Theorem 3.9. *If an $(LCS)_n-$manifold admits a Ricci-Yamabe soliton, then the manifold is $\xi-$semi symmetric.*

Let us assume that $S(\xi,X)\cdot R = 0$, which implies

$$\begin{aligned} &S(X,R(Y,Z)W)\xi - S(\xi,R(Y,Z)W)X + S(X,Y)R(\xi,Z)W \\ &-S(\xi,Y)R(X,Z)W + S(X,Z)R(Y,\xi)W - S(\xi,Z)R(Y,X)W \\ &+S(X,W)R(Y,Z)\xi - S(\xi,W)R(Y,Z)X = 0, \end{aligned} \tag{3.38}$$

for any vector fields X, Y, Z, W on M. Taking the inner product with ξ, (3.38) becomes

$$\begin{aligned} &-S(X,R(Y,Z)W) - S(\xi,R(Y,Z)W)\eta(X) + S(X,Y)\eta(R(\xi,Z)W) \\ &-S(\xi,Y)\eta(R(X,Z)W) + S(X,Z)\eta(R(Y,\xi)W) - S(\xi,Z)\eta(R(Y,X)W) \\ &+S(X,W)\eta(R(Y,Z)\xi) - S(\xi,W)\eta(R(Y,Z)X) = 0. \end{aligned} \tag{3.39}$$

Taking $Z = W = \xi$ in (3.39) and replacing the expression of S from (3.7), we get

$$\begin{aligned} &\tfrac{2\lambda}{2\alpha+n\beta}[-g(X,R(Y,\xi)\xi) - \eta(R(Y,\xi)\xi)\eta(X) + g(X,Y)\eta(R(\xi,\xi)\xi) \\ &-\eta(Y)\eta(R(X,\xi)\xi) + \eta(X)\eta(R(Y,\xi)\xi) + \eta(R(Y,X)\xi) + \eta(X)\eta(R(Y,\xi)\xi) \\ &+\eta(R(Y,\xi)X)] = 0. \end{aligned} \tag{3.40}$$

In view of (2.9) and on simplification, the above equation becomes

$$\frac{4\lambda}{2\alpha+n\beta}(\gamma^2-\rho)[g(X,Y) + \eta(X)\eta(Y)] = 0. \tag{3.41}$$

Using (2.6), we get

$$\frac{4\lambda}{2\alpha+n\beta}(\gamma^2-\rho)g(\phi X,\phi Y)=0, \tag{3.42}$$

for all vector fields X, Y on M, which implies that

$$\frac{4\lambda}{2\alpha+n\beta}(\gamma^2-\rho)=0. \tag{3.43}$$

Using (2.11) and (3.6) in (3.43), we get

$$\frac{8\lambda^2}{(n-1)(2\alpha+n\beta)^2}=0. \tag{3.44}$$

This implies that $\lambda = 0$, then using (3.6), $r = 0$. From (2.11), $r = 0$ implies $(\gamma^2 - \rho) = 0$ provided $n > 1$. Again, in view of (2.9), we have $R(X, Y)Z = 0$ for all vector fields X, Y, Z on M. Hence, we can state the following theorem:

Theorem 3.10. *If an $(LCS)_n$ $(n > 1)$ manifold admitting a Ricci-Yamabe soliton satisfies $S(\xi, X)\cdot R = 0$, then the manifold becomes flat and the soliton is steady.*

4. Examples of $(LCS)_3$ and $(LCS)_5-$ manifold satisfying RYS

In this section, we constructed examples for the 3-dimensional and 5-dimensional $(LCS)_n$-manifold in which we verify our results.

Example 4.1. Consider the 3−dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let E_1, E_2, E_3 be a linearly independent system of vector fields on M given by

$$E_1 = y\frac{\partial}{\partial x}, \quad E_2 = y\frac{\partial}{\partial y}, \quad E_3 = y\frac{\partial}{\partial z}.$$

Let g be the Lorentzian metric defined by

$$g(E_1, E_1) = g(E_2, E_2) = 1, g(E_3, E_3) - 1,$$
$$g(E_i, E_j) = 0 \quad \forall i \neq j; i, j = 1, 2, 3.$$

Let η be the 1−form defined by $\eta(Z) = g(Z, e_2)$ and ϕ be the $(1, 1)$−tensor field defined by

$$\phi E_1 = E_1, \quad \phi E_2 = 0, \quad \phi E_3 = E_3.$$

Then, using the linearity of ϕ and g, we have

$$\eta(E_2) = -1, \quad \phi^2(Z) = Z + \eta(Z)E_2$$
$$\text{and} \quad g(\phi Z, \phi V) = g(Z, V) + \eta(Z)\eta(V),$$

for all $Z, V \in \chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, we have

$$[E_1, E_2] = -E_1, \quad [E_2, E_3] = E_3, \quad [E_1, E_3] = 0.$$

Using Koszuls formula and taking $\xi = E_2$, we can easily calculate

$$\nabla_{E_1}E_3 = 0, \quad \nabla_{E_2}E_3 = 0, \quad \nabla_{E_3}E_3 = \tfrac{1}{2}(1 + 2E_2),$$
$$\nabla_{E_1}E_1 = \tfrac{1}{2}(1 + 2E_2), \quad \nabla_{E_2}E_1 = 0, \quad \nabla_{E_3}E_1 = 0,$$
$$\nabla_{E_1}E_2 = -E_1, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_3}E_2 = -E_3.$$

Hence, in this case, the data $(g, \xi, \eta, \phi, \gamma)$ is an $(LCS)_3-$structure on M, where $\gamma = -1$. Also, as $\gamma = -1$, then $\rho = 0$ and consequently $(M, g, \xi, \eta, \phi, \gamma)$ is an $(LCS)_3-$manifold.
Now, from (2.11), we have $r = 6$. Let us consider that g defines a RYS on M. Putting the value of r and γ in (3.2), we have

$$\left(2\lambda - 6\beta + 2 - \frac{4\alpha\lambda}{2\alpha + 3\beta}\right) g(X, Y) + 2\eta(X)\eta(Y) = 0.$$

Taking $X = Y = \xi$, we get $\lambda = 2\alpha + 3\beta$ which satisfies (3.6). Again, from (3.7), we have $S(X, Y) = 2g(X, Y)$ and therefore, this example verifies Theorem 3.1 in 3-dimension.

Example 4.2. Consider the $5-$dimensional manifold $M = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_5 \neq 0\}$, where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinates in $\mathbb{R}^5$. Let E_1, E_2, E_3, E_4, E_5 be a linearly independent global frame on M given by

$$E_1 = x_5\left(\tfrac{\partial}{\partial x_1} + x_2\tfrac{\partial}{\partial x_2}\right), \quad E_2 = x_5\tfrac{\partial}{\partial x_2}, \quad E_3 = x_5\left(\tfrac{\partial}{\partial x_3} + \tfrac{\partial}{\partial x_4}\right),$$
$$E_4 = x_5\tfrac{\partial}{\partial x_4}, \quad E_5 = (x_5)^4\tfrac{\partial}{\partial x_5}.$$

Let us define ϕ, ξ, η, g by

$$\phi E_1 = E_1, \quad \phi E_2 = E_2, \quad \phi E_3 = E_3, \quad \phi E_4 = E_4, \quad \phi E_5 = 0, \quad \xi = E_5,$$
$$\eta(X) = g(X, E_5) \quad \text{for any} \quad X \in \chi(M), \quad g(E_i, E_i) = 1 \quad \forall i = 1, 2, 3, 4,$$
$$\text{and} \quad g(E_5, E_5) = -1, \quad g(E_i, E_j) = 0 \quad \forall i \neq j, i, j = 1, 2, 3, 4, 5.$$

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g, then we have

$$[E_1, E_5] = -(x_5)^3E_1, \quad [E_2, E_5] = -(x_5)^3E_2, \quad [E_3, E_5] = -(x_5)^3E_3,$$
$$[E_4, E_5] = -(x_5)^3E_4, \quad [E_1, E_2] = -(x_5)E_2,$$

and all other remaining $[E_i, E_j]$ vanishes. Taking $\xi = E_5$ and using Koszul formula for the Lorentzian metric g, we obtain

$$\nabla_{E_1}E_5 = -(x_5)^3E_1, \quad \nabla_{E_2}E_5 = -(x_5)^3E_2, \quad \nabla_{E_3}E_5 = -(x_5)^3E_3,$$
$$\nabla_{E_4}E_5 = -(x_5)^3E_4, \quad \nabla_{E_2}E_1 = (x_5)E_2,$$
$$\nabla_{E_1}E_1 = \nabla_{E_3}E_3 = \nabla_{E_4}E_4 = -\tfrac{1}{2}(1 - 2(x_5)^3E_5),$$
$$\nabla_{E_2}E_2 = -\tfrac{1}{2}(1 - 2(x_5)^3E_5 + 2(x_5)E_1).$$

Hence, $(g, \phi, \xi, \eta, \gamma)$ is an $(LCS)_5-$structure on M. Consequently, $M^5(g, \xi, \eta, \phi, \gamma)$ is an $(LCS)_5-$manifold with $\gamma = -(x_5)^3 \neq 0$, where $\rho = 3(x_5)^6$.
Now, from (2.11), we get $r = -40(x_5)^6$. Assume g defines a RYS on M and from (3.2), we obtain

$$\left(2\lambda + 40(x_5)^6\beta + 2(x_5)^3 - \frac{4\alpha\lambda}{2\alpha + 5\beta}\right) g(X, Y) + 2(x_5)^3\eta(X)\eta(Y) = 0.$$

Setting $X = Y = \xi$, we get $\lambda = -4(x_5)^6(2\alpha + 5\beta)$ which satisfies (3.6). Again, from (3.7), we have $S(X, Y) = -8(x_5)^6 g(X, Y)$ and thus confirms Theorem 3.1 in 5-dimension.

5. Conformal Ricci-Yamabe soliton on $(LCS)_n$-manifolds

In this section, we obtained the expression for the scalar λ on an $(LCS)_n$-manifold admitting a conformal Ricci-Yamabe soliton, where the notion of the soliton was introduced in [11] while studying a perfect fluid spacetime, the soliton is given by

$$\mathcal{L}_V g + 2\alpha S = \left[2\lambda - \beta r - \left(p + \frac{2}{n}\right)\right] g, \tag{5.1}$$

where p is a conformal pressure.

Taking $V = \xi$ in (5.1), we get

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\alpha S(X, Y) = \left[2\lambda - \beta r - \left(p + \frac{2}{n}\right)\right] g(X, Y), \tag{5.2}$$

for all X, Y. Now, using (2.3), (2.4) and (2.6) in the above equation and on simplification, we obtain

$$S(X, Y) = \frac{1}{\alpha}\left[\lambda - \frac{\beta r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \gamma\right] g(X, Y) - \frac{\gamma}{\alpha}\eta(X)\eta(Y), \alpha \neq 0. \tag{5.3}$$

Now, contracting (5.3), we have

$$\lambda = \frac{p}{2} + \frac{(2\alpha + n\beta)}{2n} r + \frac{(n-1)\gamma + 1}{n}. \tag{5.4}$$

Thus, we can state the following:

Theorem 5.1. *If an $(LCS)_n$-manifold admits a conformal Ricci-Yamabe soliton, then the manifold becomes η-Einstein and the scalar λ is given by $\lambda = \frac{p}{2} + \frac{(2\alpha+n\beta)}{2n} r + \frac{(n-1)\gamma+1}{n}$ provided $\alpha \neq 0$.*

6. $\eta-$Ricci-Yamabe Soliton ($\eta-$RYS) on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold

Here, we study a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold which admit $\eta-$Ricci-Yamabe soliton.

Lemma 6.1. *If a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admits a $\eta-RYS$, then*

$$\alpha(a-b)+\lambda-\frac{\beta r}{2}-\mu=0.$$

Proof. From (1.3), we have

$$g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)+2\alpha S(X,Y)+(2\lambda-\beta r)g(X,Y) + 2\mu\eta(X)\eta(Y)=0. \tag{6.1}$$

Substituting (2.14) in the foregoing equation, we get

$$g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi) + 2\left[\left(\lambda-\tfrac{\beta r}{2}+a\alpha\right)g(X,Y)+(b\alpha+\mu)\eta(X)\eta(Y)\right]=0. \tag{6.2}$$

Putting $X=Y=\xi$ in (6.2), we get

$$g(\nabla_\xi\xi,\xi)=\alpha(a-b)+\lambda-\frac{\beta r}{2}-\mu. \tag{6.3}$$

Using (2.8), $g(\nabla_\xi\xi,\xi)=0$. Then,

$$\alpha(a-b)+\lambda-\frac{\beta r}{2}-\mu=0. \tag{6.4}$$

Hence, we get the result.

Remark 6.2. For a particular case such that $\alpha=1$ and $\beta=0$ in the above Lemma 6.1, the relation becomes $a-b+\lambda-\mu=0$. This result is obtained in [5].

Theorem 6.3. *If ξ is a torse forming vector field on a conformally flat $(LCS)_n-$manifold admitting $\eta-RYS$, then $\lambda=\frac{\beta r}{2}-a\alpha-1$, η is closed and*

$$b=a-(n-1) \quad \textit{and} \quad \mu=\alpha(n-1)+\lambda-\frac{\beta r}{2}.$$

Proof. Let ξ be a torse forming vector field on a conformally flat $(LCS)_n-$manifold which admits $\eta-$RYS. Then, taking inner product with ξ in (2.17), we get

$$f\eta(X)=\nu(X). \tag{6.5}$$

In view of the above relation, (2.17) becomes

$$\nabla_X\xi=f[X+\eta(X)\xi]. \tag{6.6}$$

Using (6.6) in (6.2) and in view of Lemma 6.1, we get

$$\left(f+\lambda-\frac{\beta r}{2}+a\alpha\right)[g(X,Y)+\eta(X)\eta(Y)]=0, \tag{6.7}$$

for all vector fields X and Y and hence it follows that

$$f = -\left(\lambda - \frac{\beta r}{2} + a\alpha\right). \tag{6.8}$$

Using the fact that $f = 1$ from Remark 2.2 in (6.8), we get

$$\lambda = \frac{\beta r}{2} - a\alpha - 1. \tag{6.9}$$

Now, using (6.8) in (6.6), we get

$$\nabla_X \xi = -\left(\lambda - \frac{\beta r}{2} + a\alpha\right)[X + \eta(X)\xi], \tag{6.10}$$

which means that $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X and hence we get $d\eta = 0$, i.e., η is closed. From (1.6), we know that

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{6.11}$$

In view of (6.10), (6.11) yields

$$R(X,Y)\xi = \left(\lambda - \frac{\beta r}{2} + a\alpha\right)^2 [\eta(Y)X - \eta(X)Y]. \tag{6.12}$$

Again, in view of (2.9), (2.10), (6.9) and (6.12), we get

$$S(X,\xi) = (n-1)\eta(X). \tag{6.13}$$

Comparing (6.13) with (2.15), we obtain

$$b = a - (n-1), \tag{6.14}$$

$$\text{and} \quad \mu = \alpha(n-1) + \lambda - \tfrac{\beta r}{2}. \tag{6.15}$$

Hence, we get the theorem. The following result follows immediately from Theorem 6.3:

Corollary 6.4. If ξ is a torse forming vector field on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting $\eta-$RYS, then the soliton is expanding, steady and shrinking according as $\beta r < 2(a\alpha + 1)$, $\beta r = 2(a\alpha + 1)$ and $\beta r > 2(a\alpha + 1)$ respectively.

Again, as a consequence of Theorem 6.3, in particular, if $\mu = 0$, then we obtain $\lambda = \alpha(b-a) + \frac{\beta r}{2}$ which results in the following corollary:

Corollary 6.5. If ξ is a torse forming vector field on a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting Ricci-Yamabe soliton with $\alpha \neq 0$, then the soliton is shrinking, steady and expanding according as $\beta r > 2\alpha(a-b)$, $\beta r = 2\alpha(a-b)$, and $\beta r < 2\alpha(a-b)$ respectively.

7. Conclusion

This work is an extension of the work done on $(LCS)_n$-manifolds by Roy *et al.* [12]. We generalized their results in Section 3 and obtained more general value for the scalar curvature tensor on an $(LCS)_n$-manifold admitting the Ricci-Yamabe soliton and have shown that it is constant. The prominence of this result is that it holds for a larger group of solitons. In Section 4, we verified our result by constructing a 3-dimensional and 5-dimensional $(LCS)_n$-manifold. We also obtained the expression for the scalar λ when the manifold admits a conformal Ricci-Yamabe soliton in Section 5. In Section 6, the conditions under which a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold admitting a torse forming vector field ξ is expanding, steady and shrinking η-Ricci-Yamabe soliton is obtained. Also, we give the expression for λ in a conformally flat $(LCS)_n$ $(n \geq 4)$ manifold which admits a torse forming η-Ricci-Yamabe soliton. Further, it is shown that the results obtained in [12] are particular results.

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