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ON ESTIMATES FOR THE CANONICAL LINEAR FOURIER-BESSEL TRANSFORM IN THE SPACE $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ $(1 < p \leq 2)$

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Abstract. In this paper, we establish the analog of Abilov's theorems and the analog of Titchmarsh's theorems for the canonical linear Fourier-Bessel transform in a class of functions in the space $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ where $1 < p \leq 2$ and $\alpha > \frac{-1}{2}$. The proof of the theorems is based on the algebraic properties associated with the canonical linear Fourier-Bessel transform.
Keywords:Canonical linear Fourier-Bessel operator, Canonical linear Fourier-Bessel Transform, Translation operators associated with the canonical linear Fourier-Bessel operator, Generalized modulus of continuity.

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1. Introduction and preliminaries

Consider the operator (see [5],[9])

$$\mathrm{B}_{\alpha,m} = \frac{d^2}{dx^2} + \left(\frac{(2\alpha+1)}{x} - 2ix\frac{d}{b}\right)\frac{d}{dx} - \left(x^2\frac{d^2}{b^2} + 2i(\alpha+1)\frac{d}{b}\right) \tag{1.1}$$

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where $\alpha > \frac{-1}{2}$,

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2}$$

is a matrix in $SL_2(\mathbb{R})$ with $b \neq 0$

If

$$m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.3}$$

we obtain the classical Bessel operator

$$B_\alpha = \frac{d^2}{dx^2} + \frac{(2\alpha+1)}{x}\frac{d}{dx}. \tag{1.4}$$

Let $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ the space of measurable functions f on $\mathbb{R}^+$ such that

$$\|f\|_{p,\alpha} = \left(\int_0^{+\infty} |f(x)|^p x^{2\alpha+1} dx\right)^{\frac{1}{p}} < +\infty \tag{1.5}$$

The chirp multiplication operator L_a and the dilatation operator D_a are defined, respectively by $L_a(f)(x) = e^{i\frac{a}{2}x^2} f(x)$; $a \in \mathbb{R}$ and $D_a f(x) = \frac{1}{|a|^{\alpha+1}} f(\frac{x}{a})$; $a \in \mathbb{R}^*$.

The inverse of L_a and the inverse of D_a are given, respectively, by

$$(L_a)^{-1} = L_{-a}; (D_a)^{-1} = D_{-\frac{1}{a}}. \tag{1.6}$$

In case f is even we have $D_a f = D_{|a|} f$.
Let $m \in SL_2(\mathbb{R})$. The canonical Fourier-Bessel transform of a function $f \in L^1(\mathbb{R}^+, x^{2\alpha+1}dx)$ is defined by (see [5],[8])

$$\mathcal{F}_\alpha^m(f)(\lambda) = \frac{c_\alpha}{(ib)^{\alpha+1}} \int_0^{+\infty} \mathcal{K}_\alpha^m(\lambda, x) f(x) x^{2\alpha+1} dx \tag{1.7}$$

where $c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)}$;

$$\mathcal{K}_\alpha^m(\lambda, x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha(\frac{\lambda x}{b}). \tag{1.8}$$

and

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}, z \in \mathbb{C} \tag{1.9}$$

For $\lambda \in \mathbb{R}$, the kernel $\mathcal{K}_\alpha^m(., \lambda)$ of the canonical Fourier-Bessel transform $\mathcal{F}_\alpha^m$ is the unique solution of:

$$\mathcal{B}_{\alpha,m} \mathcal{K}_\alpha^m(., \lambda) = -\frac{\lambda^2}{b^2} \mathcal{K}_\alpha^m(., \lambda) \tag{1.10}$$

with initial conditions $\mathcal{K}_\alpha^m(0,\lambda) = e^{\frac{i}{2}\frac{a}{b}\lambda^2}$ and $\frac{d}{dx}\mathcal{K}_\alpha^m(0,\lambda) = 0$

Let $m \in SL_2(\mathbb{R})$ such that $b \neq 0$. For $f \in \mathcal{C}_{*,c}(\mathbb{R})$, we define the generalized translation operators associated with the operator $B_{\alpha,m}$ by ([9]):

$$\tau_{\alpha,m,h}f(x) = e^{\frac{i}{2}(\frac{d}{b}h^2+\frac{a}{b}x^2)}\tau_{\alpha,h}\left[e^{-\frac{i}{2}\frac{d}{b}s^2}f(s)\right](x); \tag{1.11}$$

where $\tau_{\alpha,h}$ is the translation operator associated with the operator B_α and $\mathcal{C}_{*,c}(\mathbb{R})$ is the space of the even continuous functions with support compact.

2. Main results

Our main results are inspired from the work realised by V.A. Abilov, F. V. Abilova, M.K. Kerimov and E. C. Titchmarsh (see [1], [2], [3],[4], [5], [7], [8], [11]). Briefly, we give new estimates for the canonical Fourier-Bessel transform of a class of function f in a Sobolev space that we will define later.

Lemma 2.1. *(see [6],[10])*
We have the formulas

i) $L_{\frac{-d}{b}} \circ B_{\alpha,m} \circ L_{\frac{d}{b}} = B_\alpha$.

ii) $\mathcal{F}_\alpha^m = e^{-i(\alpha+1)\frac{\pi}{2}sgnb}L_{\frac{d}{b}} \circ D_b \circ \mathcal{F}_\alpha \circ L_{\frac{a}{b}}$ *where* $\mathcal{F}_\alpha$ *is the classical Bessel transform.*

iii) $\tau_{\alpha,m,h} = e^{\frac{i}{2}\frac{d}{b}h^2}L_{\frac{d}{b}}\circ\tau_{\alpha,h}\circ L_{-\frac{d}{b}}$, *where* $\tau_{\alpha,h}$ *is the translation operator associated to the classical Bessel operator.*

Definition 2.1. ([7])
Let $k \in \mathbb{N}$. The k^{th} order modulus of continuity of a function $f \in L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ is defined as

$$\Omega_{k,p,\alpha}(f,\delta) = sup_{0<h\leq\delta}\|\Delta_{\alpha,h}^k f\|_{p,\alpha} \tag{2.1}$$

where,

$$\Delta_{\alpha,h}^0 f = f$$

$$\Delta_{\alpha,h} f = (\tau_{\alpha,h} - \mathcal{I})f$$

$$\Delta_{\alpha,h}^k f = (\tau_{\alpha,h} - \mathcal{I})^k f$$

and $\mathcal{I}$ is the identity operator.

Definition 2.2. Let $k \in \mathbb{N}$. The k^{th} order generalized modulus of continuity of a function $f \in L^p_\alpha(\mathbb{R}^+)$ is defined as

$$\Omega_{k,p,\alpha,m^{-1}}(f,\delta) = sup_{0<h\leq\delta}\|\Delta^k_{\alpha,m^{-1},h}f\|_{p,\alpha}. \tag{2.2}$$

where,

$$\Delta^0_{\alpha,m^{-1},h}f = f, \tag{2.3}$$

$$\Delta_{\alpha,m^{-1},h}f = (\tau_{\alpha,m^{-1},h} - e^{-\frac{i}{2}\frac{a}{b}h^2}\mathcal{I})f \tag{2.4}$$

$$\Delta^k_{\alpha,m^{-1},h}f = (\tau_{\alpha,m^{-1},h} - e^{-\frac{i}{2}\frac{a}{b}h^2}\mathcal{I})^k f. \tag{2.5}$$

We denote in the classical Bessel harmonic analysis by $\mathcal{W}^r_p(B_\alpha)$ the class of functions f belongs to $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ that have generalized derivatives in the sense of Levi (see [9])such that for all $j \in \{1, ..., r\}$ we have $B^j_\alpha f \in L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$.
And by $\mathcal{W}^{r,k}_{p,\Psi}(B_\alpha)$ where, $r \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and Ψ is a nonnegative function on $\mathbb{R}^+$, the class of functions f belongs to $\mathcal{W}^r_p(B_\alpha)$ satisfying the estimate
$\Omega_{k,\alpha}((B_\alpha)^r f, \delta) = \mathcal{O}(\Psi(\delta^k)$ *as* $\delta \to 0$.

We denote in the canonical Bessel harmonic analysis by $\mathcal{W}^r_p(B_{\alpha,m^{-1}})$ the class of functions f belongs to $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ that have generalized derivatives in the sense of Levi (see [9]) such that for all $j \in \{1, ..., r\}$ we have $B^j_{\alpha,m^{-1}}f \in L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$.
And by $\mathcal{W}^{r,k}_{p,\Psi}(B_{\alpha,m^{-1}})$ where, $r \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and Ψ is a nonnegative function on $\mathbb{R}^+$, the class of functions f belongs to $\mathcal{W}^r_p(B_{\alpha,m^{-1}})$ satisfying the estimate
$\Omega_{k,p,\alpha,m^{-1}}((B_{\alpha,m^{-1}})^r f, \delta) = \mathcal{O}(\Psi(\delta^k)$ *as* $\delta \to 0$.

Lemma 2.2. *Let* $f \in L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ *we have*

i) $\Omega_{k,p,\alpha,m^{-1}}(f,\delta) = \Omega_{k,p,\alpha}(L_{\frac{a}{b}}f, \delta)$.

ii) $W^r_p(B_{\alpha,m^{-1}}) = L_{-\frac{a}{b}}(W^r_p(B_\alpha))$.

iii) $\mathcal{W}^{r,k}_{p,\Psi}(B_{\alpha,m^{-1}}) = L_{\frac{-a}{b}}\left(\mathcal{W}^{r,k}_{p,\Psi}(B_\alpha)\right)$.

Proof. i) Let $f \in L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$

$$\begin{aligned}
\Omega_{k,p,\alpha,m^{-1}}(f,\delta) &= sup_{0<h\leq\delta}\left\|\Delta^k_{\alpha,m^{-1},h}f\right\|_{p,\alpha} \\
&= sup_{0<h\leq\delta}\left\|\left(\tau_{\alpha,m^{-1},h} - e^{-\frac{i}{2}\frac{a}{b}h^2}\mathcal{I}\right)^k f\right\|_{p,\alpha} \\
&= sup_{0<h\leq\delta}\left\|\left(L_{-\frac{a}{b}}\circ\tau_{\alpha,h}\circ L_{\frac{a}{b}} - \mathcal{I}\right)^k f\right\|_{p,\alpha} \\
&= sup_{0<h\leq\delta}\left\|L_{-\frac{a}{b}}\circ\Delta^k_{\alpha,h}\circ L_{\frac{a}{b}}f\right\|_{p,\alpha} \\
&= sup_{0<h\leq\delta}\left\|\Delta^k_{\alpha,h}\left(L_{\frac{a}{b}}f\right)\right\|_{p,\alpha} \\
&= \Omega_{k,p,\alpha}(L_{\frac{a}{b}}f,\delta).
\end{aligned}$$

ii)

$$\begin{aligned}
&f \in W^r_p(B_{\alpha,m^{-1}}) \Leftrightarrow f \in L^p\left(\mathbb{R}^+, x^{2\alpha+1}dx\right) \quad and \quad \forall j \in \{1,...,r\} \quad B^j_{\alpha,m^{-1}}f \in L^p\left(\mathbb{R}^+, x^{2\alpha+1}dx\right) \\
&\Leftrightarrow f \in L^p\left(\mathbb{R}^+, x^{2\alpha+1}dx\right) \quad and \quad \forall j \in \{1,...,r\} \quad L_{-\frac{a}{b}}\circ B^j_\alpha\circ L_{\frac{a}{b}}(f) \in L^p\left(\mathbb{R}^+, x^{2\alpha+1}dx\right) \\
&\Leftrightarrow f \in L^p\left(\mathbb{R}^+, x^{2\alpha+1}dx\right) \quad and \quad L_{\frac{a}{b}}f \in W^r_p(B_\alpha) \\
&\Leftrightarrow f \in L_{-\frac{a}{b}}\left(W^r_p(B_\alpha)\right).
\end{aligned}$$

iii) $f \in W^{r,k}_{p,\Psi}(B_{\alpha,m^{-1}}) \Leftrightarrow f \in W^r_p(B_{\alpha,m^{-1}})$ and $\Omega_{k,p,\alpha,m^{-1}}(B^r_{\alpha,m^{-1}}f,\delta) = \mathcal{O}(\Psi(\delta^k)$
$\Leftrightarrow L_{\frac{a}{b}}f \in W^r_p(B_\alpha)$ and $\Omega_{k,p,\alpha}(L_{\frac{a}{b}}f,\delta) = \mathcal{O}(\Psi(\delta^k)$ as $\delta \to 0$
$\Leftrightarrow L_{\frac{a}{b}}f \in W^{r,k}_{p,\Psi}(B_\alpha)$
$\Leftrightarrow f \in L_{-\frac{a}{b}}\left(W^{r,k}_{p,\Psi}(B_\alpha)\right)$.
□

Theorem 2.1. *Let $1 < p \leq 2$, $m \in SL_2(\mathbb{R})$ and ψ be a nonnegative function on $\mathbb{R}^+$. For all $f \in \mathcal{W}^{r,k}_{p,\Psi}(B_{\alpha,m^{-1}})$ we have,*

$$\int_{|\lambda|\geq N}|\mathcal{F}^m_\alpha(f)(\lambda)|^q\lambda^{2\alpha+1}d\lambda = \mathcal{O}\left(N^{-rq}(\Psi)^q\left((\frac{c}{N})^k\right)\right) \quad as\ N \to \ +\infty$$

where c is a positive constant, $r \in \mathbb{N}$; $k \in \mathbb{N}^*$; q is the conjugate exponent of p.

Proof. We have $f \in \mathcal{W}^{r,k}_{p,\Psi}(B_{\alpha,m^{-1}}) \Leftrightarrow L_{\frac{a}{b}}f \in \mathcal{W}^{r,k}_{p,\Psi}(B_\alpha)$. Then by [7] we have

$$\int_{|\lambda|\geq N}|\mathcal{F}_\alpha(L_{\frac{a}{b}}f)(\lambda)|^q\lambda^{2\alpha+1}d\lambda = \mathcal{O}\left(N^{-rq}(\Psi)^q\left((\frac{c}{N})^k\right)\right) \quad as\ N \to \ +\infty$$

where c is a positive constant, $r \in \mathbb{N}$; $k \in \mathbb{N}^*$; q is the conjugate exponent of p. By relations (see[6],[10])

$$D_b f = D_{|b|}f \quad for \ all \ \ b \in \mathbb{R}^* \ \ and \ \ f \ \ is \ is \ \ even \ \ function. \tag{2.6}$$

$$\mathcal{F}_{\alpha}^{m}=e^{-i(\alpha+1)\frac{\pi}{2}sgnb}L_{\frac{d}{b}}\circ D_b\circ\mathcal{F}_{\alpha}\circ L_{\frac{a}{b}} \tag{2.7}$$

we deduce the result. □

Corollary 2.1. *Let $\Psi(t)=t^{\beta}$ and $f\in\mathcal{W}_{p,t^{\beta}}^{r,k}(B_{\alpha,m^{-1}})$ where $\beta>0$ and $1<p\leq 2$. Then,*

$$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha}^{m}(f)(\lambda)|^{q}\lambda^{2\alpha+1}d\lambda=\mathcal{O}\left(N^{-rq-qk\beta}\right)\ \ as\ N\to\ \ +\infty$$

where, q is the conjugate exponent of p.

Theorem 2.2. *Let $\Psi(t)=t^{\beta}$ and $0<\beta<2$ this conditions are equivalents*

i) $f\in\mathcal{W}_{2,t^{\beta}}^{r,k}(B_{\alpha,m^{-1}})$

ii) $\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha}^{m}(f)(\lambda)|^{2}\lambda^{2\alpha+1}d\lambda=\mathcal{O}\left(N^{-2r-2k\beta}\right)$ *as* $N\to\ +\infty$

Proof. *i)* ⇒ *ii)*. By corollary (2.1) we deduce easily the result.
ii) ⇒ *i)*. Let $f\in L^{2}\left(\mathbb{R}^{+},x^{2\alpha+1}\right)$ such that

$$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha}^{m}(f)(\lambda)|^{2}\lambda^{2\alpha+1}d\lambda=\mathcal{O}\left(N^{-2r-2k\beta}\right)\ \ as\ N\to\ \ +\infty$$

by lemma (2.1) we have the formula

$$\mathcal{F}_{\alpha}^{m}=e^{-i(\alpha+1)\frac{\pi}{2}sgnb}L_{\frac{d}{b}}\circ D_b\circ\mathcal{F}_{\alpha}\circ L_{\frac{a}{b}}$$

where $\mathcal{F}_{\alpha}$ is the classical Bessel transform. Then

$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha}^{m}(f)(\lambda)|^{2}\lambda^{2\alpha+1}d\lambda=\int_{|\lambda|\geq N}|e^{-i(\alpha+1)\frac{\Pi}{2}sgnb}L_{\frac{d}{b}}\circ D_{|b|}\circ\mathcal{F}_{\alpha}\circ L_{\frac{a}{b}}(f)(\lambda)|^{2}\lambda^{2\alpha+1}d\lambda$
$=\int_{|\lambda|\geq N}|D_{|b|}\circ\mathcal{F}_{\alpha}(L_{\frac{a}{b}}f)(\lambda)|^{2}\lambda^{2\alpha+1}d\lambda$
$=\int_{|\lambda|\geq N}|\frac{1}{|b|^{\alpha+1}}\mathcal{F}_{\alpha}(L_{\frac{a}{b}}f)(\frac{\lambda}{|b|})|^{2}\lambda^{2\alpha+1}d\lambda$
by change of variables we put $\lambda=|b|\mu$. we have $d\lambda=|b|d\mu$ and $|\lambda|\geq N$ equivalent to $|\mu|\geq\frac{N}{|b|}$. Then,
$\int_{|\lambda|\geq N}|\mathcal{F}_{\alpha}^{m}(f)(\lambda)|^{2}\lambda^{2\alpha+1}d\lambda=\frac{1}{|b|^{2\alpha+2}}\int_{|\mu|\geq\frac{N}{|b|}}|\mathcal{F}_{\alpha}(L_{\frac{a}{b}}f)(\mu)|^{2}(|b|\mu)^{2\alpha+1}|b|d\mu$
$=\int_{|\mu|\geq\frac{N}{|b|}}|\mathcal{F}_{\alpha}(L_{\frac{a}{b}}f)(\mu)|^{2}\mu^{2\alpha+1}d\mu.$
Since,
$\int_{|\mu|\geq\frac{N}{|b|}}|\mathcal{F}_{\alpha}(L_{\frac{a}{b}}f)(\mu)|^{2}\mu^{2\alpha+1}d\mu=\mathcal{O}\left(N^{-2r-2k\beta}\right)$ as $N\to+\infty$
and by [4] we have $L_{\frac{a}{b}}f\in W_{2,t^{\beta}}^{r,k}(B_{\alpha})$. Thus
$f\in L_{-\frac{a}{b}}\left(W_{2,t^{\beta}}^{r,k}(B_{\alpha})\right)=W_{2,t^{\beta}}^{r,k}(B_{\alpha,m^{-1}})$.
□

Theorem 2.3. *Let* $0 < \gamma \leq k$ *and* $f \in \mathcal{W}_p^r(B_{\alpha,m^{-1}})$ *such that*

$$\|\Delta^k_{\alpha,m^{-1},h}(B_{\alpha,m^{-1}})^r f\|_{p,\alpha} = \mathcal{O}(h^\gamma) \text{ as } h \to 0,$$

then

$$\mathcal{F}^m_\alpha(f) \in L^\beta(\mathbb{R}^+, x^{2\alpha+1}dx)$$

for

$$\frac{2\alpha p + 2p}{2p + 2\alpha(p-1) - 2 + \gamma p + rp} < \beta \leq \frac{p}{p-1}$$

.

Proof. Let $f \in \mathcal{W}_p^r(B_{\alpha,m^{-1}})$ and $1 < p \leq 2$ we have
$\|\Delta^k_{\alpha,m^{-1},h}(B_{\alpha,m^{-1}})^r f\|_{p,\alpha} = \| \left(T_{\alpha,m^{-1},h} - e^{-\frac{i}{2}\frac{a}{b}h^2}\mathcal{I}\right)^k \left((B_{\alpha,m^{-1}})^r f\right) \|_{p,\alpha}$
$= \|L_{-\frac{a}{b}} \circ \Delta^k_{\alpha,h} \circ L_{\frac{a}{b}} \left((B_{\alpha,m^{-1}})^r f\right) \|_{p,\alpha}$
$= \|\Delta^k_{\alpha,h} \left((B_\alpha)^r L_{\frac{a}{b}} f\right) \|_{p,\alpha}$
$= \mathcal{O}(h^\gamma) \;\; as \;\; h \to 0$
Therefore by the result in [5] and lemma (2.1) and lemma (2.2) we deduce $\mathcal{F}^m_\alpha(f) \in L^\beta_\alpha(\mathbb{R}^+)$

$$\frac{2\alpha p + 2p}{2p + 2\alpha(p-1) - 2 + \gamma p + rp} < \beta \leq \frac{p}{p-1}$$

.

□

Definition 2.3. Let $0 < \gamma \leq 1$. A function $f \in \mathcal{W}_2^r(B_{\alpha,m^{-1}})$ is said to be in the k-m^{-1}-Bessel Lipschitz class, denoted by $DLip(\gamma, 2, k, m^{-1})$, if

(2.8) $$\|\Delta^k_{\alpha,m^{-1},h}(B_{\alpha,m^{-1}})^r f\|_{2,\alpha} = \mathcal{O}(h^\gamma) \text{ as } h \to 0.$$

Lemma 2.3. *Let* $0 < \gamma \leq 1$ *and* $k \in \mathbb{N}$ *we have*

(2.9) $$DLip(\gamma, 2, k, m^{-1}) = L_{-\frac{a}{b}}\left(DLip(\gamma, 2, k)\right)$$

Proof. We have

$f \in DLip(\gamma, 2, k, m^{-1}) \Leftrightarrow f \in \mathcal{W}_2^r(B_{\alpha,m^{-1}})$ and $\|\Delta^k_{\alpha,m^{-1},h}(B_{\alpha,m^{-1}})^r f\|_{2,\alpha} = \mathcal{O}(h^\gamma)$ *as* $h \to 0$
$\Leftrightarrow L_{\frac{a}{b}} f \in W_2^r(B_\alpha)$ and $\|\Delta^k_{\alpha,m^{-1},h}(B_{\alpha,m^{-1}})^r f\|_{2,\alpha} = \mathcal{O}(h^\gamma)$ *as* $h \to 0$
$\Leftrightarrow L_{\frac{a}{b}} f \in W_2^r(B_\alpha)$ and $\|\Delta^k_{\alpha,h}(B_\alpha)^r L_{\frac{a}{b}} f\|_{2,\alpha} = \mathcal{O}(h^\gamma)$ *as* $h \to 0$
$\Leftrightarrow L_{\frac{a}{b}} f \in DLip(\gamma, 2, k)$
$\Leftrightarrow f \in L_{-\frac{a}{b}}\left(DLip(\gamma, 2, k)\right)$.
□

Theorem 2.4. *Let* $f \in \mathcal{W}_2^r(B_{\alpha,m^{-1}})$. *The following are equivalents*

1) $f \in DLip(\gamma, 2, k, m^{-1})$

2) $\int_{|\lambda|\geq s} |\lambda|^{2r} |\mathcal{F}_\alpha^m(f)(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = \mathcal{O}(s^{-2\gamma})$ *as* $s \to +\infty$

Proof. By lemma (2.3) we have
$f \in Dlip(\gamma, 2, k, m^{-1}) \Leftrightarrow L_{\frac{a}{b}} f \in DLip(\gamma, 2, k)$

□

Corollary 2.2. *Let* $f \in DLip(\gamma, 2, k, m^{-1})$*. Then*

$$\int_{|\lambda|\geq s} |\mathcal{F}_\alpha^m(f)(\lambda)|^2 \lambda^{2\alpha+1} d\lambda = \mathcal{O}(s^{-2\gamma-2r}) \quad as \quad s \to +\infty \tag{2.10}$$

3. Conclusion

In this work, via the chirp multiplication from $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ into $L^p(\mathbb{R}^+, x^{2\alpha+1}dx)$ and by the algebraic relations:

i) $L_{\frac{-d}{b}} \circ B_\alpha^m \circ L_{\frac{d}{b}} = B_\alpha$.

ii) $\mathcal{F}_\alpha^m = e^{-i(\alpha+1)\frac{\pi}{2} sgn} L_{\frac{d}{b}} \circ D_b \circ \mathcal{F}_\alpha \circ L_{\frac{a}{b}}$ where $\mathcal{F}_\alpha$ is the classical Bessel transform.

iii) $\tau_{\alpha,m,h} = e^{i\frac{d}{2b}h^2} L_{\frac{d}{b}} \circ \tau_{\alpha,h} \circ L_{-\frac{d}{b}}$, where $\tau_{\alpha,h}$ is the translation operator associated to the classical Bessel operator.

we were able to establish, without using calculations, the first and the second generalized Abilov's theorems and generalized Titchmarsh's theorems.

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