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RANKS OF SUBMATRICES IN THE REFLEXIVE SOLUTIONS OF SOME MATRIX EQUATIONS

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Abstract. Maximal and minimal ranks of the two submatrices X_1 and X_2 in the (skew-) Hermitian reflexive solution $X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ of the matrix equation $AXA^* = C$, in the reflexive solution of the matrix equation $AXB = C$ are derived. Therefore, necessary and sufficient conditions for these reflexive solutions to have special forms, and the general expressions of these reflexive solutions are achieved.
Keywords: matrix equation, rank, reflexive solution.

1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices over $\mathbb{C}$ by $\mathbb{C}^{m\times n}$, the set of all $n \times n$ Hermitian matrices by $\mathbb{C}_H^{n\times n}$, the symbols A^* and $r(A)$ stand for the conjugate transpose and the rank of a given matrix $A \in \mathbb{C}^{n\times m}$ respectively, I_n denotes the identity matrix of order n. The Moore-Penrose inverse of a matrix A, is defined to be the unique matrix A^+ satisfying:

$$AA^+A = A, A^+AA^+ = A^+, \left(AA^+\right)^* = AA^+, \left(A^+A\right)^* = A^+A.$$

Further, the symbols R_A and L_A stand for the two orthogonal projectors $L_A = I_n - A^+A$ and $R_A = I_m - AA^+$ induced by $A \in \mathbb{C}^{m\times n}$. For more informations and basic concepts about the Moore-Penrose generalized inverse see [1], [15].

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A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ and $P^2 = I$. Chen in [2] defined the following subspace of matrices:

$$\mathbb{C}_r^{n\times n}(P) = \left\{A \in \mathbb{C}^{n\times n}, A = PAP\right\}$$

where P is a generalized reflection matrix.

The matrix $A \in \mathbb{C}_r^{n\times n}(P)$ is said to be a generalized reflexive with respect to the generalized reflection matrix P. The generalized reflexive matrices have applications in system and control theory, in engineering, in scientific computations and various other fields (see [2], [3], [7]). In particular the reflexive solutions of the linear matrix equations

$$\begin{aligned} AXA^* &= C \\ AXB &= C \end{aligned}$$

where A, B, C are given matrices, and X is a variable matrix was widely studied by many authors (see [12], [13], [14]), also in [5] Deghan and Hajarian established new necessary and sufficient conditions for the existence of the reflexive (anti-reflexive) solutions to the linear matrix equation $AXB+CYD = E$ and derived representation of the general reflexive (anti-reflexive) solutions to this matrix equation, then in [6] they investigated the solvability of these matrix equations

$$\begin{aligned} A_1XB_1 &= D_1, \\ A_1X &= C_1, XB_2 = C_2, and \\ A_1X &= C_1, XB_2 = C_2, A_3X = C_3, XB_4 = C_4. \end{aligned}$$

over reflexive and anti reflexive matrices, in [4] Cvetković-Ilić studied the existence of a reflexive solution of the matrix equation $AXB = C$, with respect to the generalized reflection matrix P, Liu and Yuan [9] gave some conditions for the existence and the representations for the generalized reflexive and anti-reflexive solutions to matrix equation $AX = B$, In [10], Liu established some conditions for the existence and representations for the common generalized reflexive and anti-reflexive solutions of matrix equations $AX = B$ and $XC = D$, also Liu in [11] discussed the extremal ranks of the matrix expression $A-BXC$ where X is (anti-) reflexive matrix, and in [8] he established some conditions for the existence and the representations for the Hermitian reflexive and Hermitian anti-reflexive, and nonnegative definite reflexive solutions to the matrix equation $AX = B$ with respect to a generalized reflection matrix P by using the Moore-Penrose inverse.

This paper is organized as follows: In Section 2 we derive the extremal ranks of the (skew-) Hermitian reflexive solution of the matrix equation $AXA^* = C$, from these rank formulas we show some forms of the reflexive solution of $AXA^* = C$, also the general expressions of the solution is given. In Section 3, we consider the matrix equation $AXB = C$ over the general reflexive solution and give some forms for this solution.

First we begin by these lemmas to review some representations of the generalized reflection matrix P and the subspace $\mathbb{C}_r^{n\times n}(P)$ matrices.

Lemma 1.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix, so P can be expressed as*

$$P = U \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} U^*$$

where U is an unitary matrix.

Lemma 1.2. *The matrix $A \in \mathbb{C}_r^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} U^*$$

where $A_1 \in \mathbb{C}^{k\times k}$, $A_2 \in \mathbb{C}^{(n-k)\times(n-k)}$, U is an unitary matrix.

Definition 1.1. Given a generalized reflection matrix $P \in \mathbb{C}^{n\times n}$.
1. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a Hermitian reflexive matrix if $A = A^*$ and $A \in \mathbb{C}_r^{n\times n}(P)$.
2. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a skew-Hermitian reflexive matrix if $A = -A^*$ and $A \in \mathbb{C}_r^{n\times n}(P)$.

The Lemmas 1.3, 1.4 and 1.5 are found in [19] as [Theorem 2.5, Theorem 2.6 and Lemma 2.2] respectively.

Lemma 1.3. *[19] Let $H^{m\times n}$ be the set of all $m \times n$ matrices over the quaternion algebra. Suppose that the matrix equation*

$$AXA^* + BYB^* = C \tag{1.1}$$

where $A \in H^{m\times n}$, $B \in H^{m\times p}$, $C \in H^{m\times m}$, $C = C^*$, $X \in H^{n\times n}$, and $Y \in H^{p\times p}$, $G = \begin{bmatrix} A & B \end{bmatrix}$ has a Hermitian solution. Then,
The maximal and minimal ranks of the general Hermitian solution to (1.1) are given by

$$\max_{\substack{AXA^*+BYB^*=C \\ X=X^*}} r(X) = \min\left\{n, r\begin{bmatrix} B & C \end{bmatrix} + 2n - r(A) - r(G)\right\}$$

$$\min_{\substack{AXA^*+BYB^*=C \\ X=X^*}} r(X) = 2r[B,C] - r\begin{bmatrix} C & B \\ B^* & 0 \end{bmatrix}$$

.

$$\max_{\substack{AXA^*+BYB^*=C \\ Y=Y^*}} r(Y) = \min\left\{p, r\begin{bmatrix} A & C \end{bmatrix} + 2p - r(B) - r(G)\right\}$$

$$\min_{\substack{AXA^*+BYB^*=C \\ Y=Y^*}} r(Y) = 2r\begin{bmatrix} A & C \end{bmatrix} - r\begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}$$

.

Lemma 1.4. *[19] Let $H^{m\times n}$ be the set of all $m\times n$ matrices over the quaternion algebra. Suppose that the matrix equation (1.1), where $A\in H^{m\times n}$, $B\in H^{m\times p}$, $C\in H^{m\times m}$, $C=-C^*$, $X\in H^{n\times n}$, and $Y\in H^{p\times p}$, $G=\begin{bmatrix} A & B\end{bmatrix}$ has a skew Hermitian solution. Then,*
The maximal and minimal ranks of the general skew-Hermitian solution to (1.1) are given by

$$\max_{\substack{AXA^*+BYB^*=C\\ X=-X^*}} r(X)=\min\left\{n, r\begin{bmatrix} B & C\end{bmatrix}+2n-r(A)-r(G)\right\}$$

$$\min_{\substack{AXA^*+BYB^*=C\\ X=-X^*}} r(X)=2r[B,C]-r\begin{bmatrix} C & B\\ -B^* & 0\end{bmatrix}.$$

$$\max_{\substack{AXA^*+BYB^*=C\\ Y=-Y^*}} r(Y)=\min\left\{p, r\begin{bmatrix} A & C\end{bmatrix}+2p-r(B)-r(G)\right\}$$

$$\min_{\substack{AXA^*+BYB^*=C\\ Y=-Y^*}} r(Y)=2r\begin{bmatrix} A & C\end{bmatrix}-r\begin{bmatrix} C & A\\ -A^* & 0\end{bmatrix}.$$

Lemma 1.5. *[19] Consider the linear matrix equation (1.1), where $A\in H^{m\times n}$, $B\in H^{m\times p}$, $C\in H^{m\times m}$ are given, and $X\in H^{n\times n}$, $Y\in H^{p\times p}$ unknown.*
1) If $C=C^*$, and (1.1) has a Hermitian solution, then the general Hermitian solution to (1.1) can be expressed as

$$X=X_0+S_1L_GZL_GS_1^*+L_AV+V^*L_A \tag{1.2}$$

$$Y=Y_0-S_2L_GZL_GS_2^*+L_BW+W^*L_B \tag{1.3}$$

where X_0 and Y_0 are a special pair Hermitian solution of (1.1),

$$S_1=(I_n,0), S_2=(0,I_p), G=\begin{bmatrix} A & B\end{bmatrix} \tag{1.4}$$

Z is an arbitrary Hermitian quaternion matrix with consistent size, and V and W are arbitrary quaternion matrices with suitable sizes.
2) If $C=-C^*$, and (1.1) has a skew-Hermitian solution, then the general skew-Hermitian solution can be expressed as

$$X=X_0+S_1L_GZL_GS_1^*+L_AV-V^*L_A. \tag{1.5}$$

$$Y=Y_0-S_2L_GZL_GS_2^*+L_BW-W^*L_B. \tag{1.6}$$

where X_0 and Y_0 are a special pair skew-Hermitian solution of (1.1), and S_1, S_2, and G are the same as (1.4); Z is an arbitrary skew-Hermitian quaternion matrix with consistent size, and V and W are arbitrary quaternion matrices with suitable sizes.

2. Extremal ranks of submatrices in (skew-)Hermitian reflexive solution of $AXA^* = C$

In this section we will derive the extremal ranks of the (skew-) Hermitian reflexive solution of the matrix equation $AXA^* = C$, as consequences we will show some forms of the reflexive solution of $AXA^* = C$, and some applications on generelized inverses.

Consider the linear matrix equation

$$AXA^* = C \tag{2.1}$$

where A, C are given and X is unknown.

Theorem 2.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^{m\times m}$ be given, suppose that the matrix equation (2.1) has a Hermitian reflexive solution.$X = X^* \in \mathbb{C}_r^{n\times n}(P)$ Then,*
a) The maximal and minimal ranks of the two submatrices X_1 and X_2 in Hermitian reflexive solution $X = U\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ to the matrix equation (2.1) are given by

$$\max_{X_1=X_1^*} r(X_1) = \min\left\{k, r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} + 2k - r(A(I_n+P)) - r(A)\right\}. \tag{2.2}$$

$$\min_{X_1=X_1^*} r(X_1) = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix}. \tag{2.3}$$

$$\max_{X_2=X_2^*} r(X_2) = \min\left\{n-k, r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} + 2(n-k) - r(A(I_n-P)) - r(A)\right\}. \tag{2.4}$$

$$\min_{X_2=X_2^*} r(X_2) = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix}. \tag{2.5}$$

b) The general Hermitian reflexive solution to (2.1) can be expressed as

$$X = U\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^*$$

where

$$X_1 = X_{01} + S_1 L_{AU} Z L_{AU} S_1^* + L_{\left(\frac{1}{2}A(I_n+P)U\right)}V + V^* L_{\left(\frac{1}{2}A(I_n+P)U\right)} \tag{2.6}$$
$$X_2 = X_{02} - S_2 L_{AU} Z L_{AU} S_2^* + L_{\left(\frac{1}{2}A(I_n-P)U\right)}W + W^* L_{\left(\frac{1}{2}A(I_n-P)U\right)} \tag{2.7}$$

where $\begin{bmatrix} X_{01} & 0 \\ 0 & X_{02} \end{bmatrix}$ is a special Hermitian reflexive solution of (2.1), and

$$S_1 = (I_k, 0), S_2 = (0, I_{n-k})$$

V, W and Z are arbitrary matrices with suitable sizes.

Proof. a) From lemma 1.2 the Hermitian reflexive solution to $AXA^* = C$ can be written as

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$$

where $X_1 = X_1^* \in \mathbb{C}^{k\times k}$, $X_2 = X_2^* \in \mathbb{C}^{(n-k)\times(n-k)}$, and arbitrary unitary matrix $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, with $U_1 \in \mathbb{C}^{n\times k}$, $U_2 \in \mathbb{C}^{n\times(n-k)}$.
We denote $AU = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, where $A_1 \in \mathbb{C}^{m\times k}$, $A_2 \in \mathbb{C}^{m\times(n-k)}$, we have

$$\begin{aligned} AXA^* = C \iff & \quad AU \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*A^* = C \\ \iff & \quad [A_1 \;\; A_2] \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \end{bmatrix} = C \\ \iff & \quad A_1X_1A_1^* + A_2X_2A_2^* = C. \end{aligned} \tag{2.8}$$

Then, the two equations (2.1) and (2.8) are equivalent, so from Lemma 1.3 we have

$$\begin{aligned} &\max_{\substack{A_1X_1A_1^* + A_2X_2A_2^* = C \\ X_1 = X_1^*}} r(X_1) \\ &\quad = \min\left\{k,\; r[A_2 \;\; C] + 2k - r(A_1) - r\begin{bmatrix} A_1 & A_2 \end{bmatrix}\right\} \end{aligned} \tag{2.9}$$

$$\min_{\substack{A_1X_1A_1^* + A_2X_2A_2^* = C \\ X_1 = X_1^*}} r(X_1) = 2r\begin{bmatrix} A_2 & C \end{bmatrix} - r\begin{bmatrix} C & A_2 \\ A_2^* & 0 \end{bmatrix}. \tag{2.10}$$

$$\begin{aligned} &\max_{\substack{A_1X_1A_1^* + A_2X_2A_2^* = C \\ X_2 = X_2^*}} r(X_2) \\ &\quad = \min\left\{n-k,\; r\begin{bmatrix} A_1 & C \end{bmatrix} + 2(n-k) - r(A_2) - r\begin{bmatrix} A_1 & A_2 \end{bmatrix}\right\} \end{aligned} \tag{2.11}$$

$$\min_{\substack{A_1X_1A_1^* + A_2X_2A_2^* = C \\ X_2 = X_2^*}} r(X_2) = 2r\begin{bmatrix} A_1 & C \end{bmatrix} - r\begin{bmatrix} C & A_1 \\ A_1^* & 0 \end{bmatrix}. \tag{2.12}$$

From Lemmas 1.1 and 1.2 we can simplify:

$$\begin{aligned} r\begin{bmatrix} A_1 & C \end{bmatrix} &= r\begin{bmatrix} A_1 & 0 & C \end{bmatrix} \\ &= r\begin{bmatrix} \frac{1}{2}A(I_n + P)U & C \end{bmatrix} \\ &= r\begin{bmatrix} A(I_n + P) & C \end{bmatrix}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} r\begin{bmatrix} A_2 & C \end{bmatrix} &= r\begin{bmatrix} 0 & A_2 & C \end{bmatrix} \\ &= r\begin{bmatrix} \frac{1}{2}A(I_n - P)U & C \end{bmatrix} \\ &= r\begin{bmatrix} A(I_n - P) & C \end{bmatrix}, \end{aligned} \tag{2.14}$$

$$r\begin{bmatrix} C & A_1 \\ A_1^* & 0 \end{bmatrix} = r\begin{bmatrix} C & A_1 & 0 \\ A_1^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n+P)U \\ \frac{1}{2}U(I_n+P)A^* & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix}, \tag{2.15}$$

$$r\begin{bmatrix} C & A_2 \\ A_2^* & 0 \end{bmatrix} = r\begin{bmatrix} C & 0 & A_2 \\ 0 & 0 & 0 \\ A_2^* & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & \frac{1}{2}A(I_n-P)U \\ \frac{1}{2}U(I_n-P)A^* & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix}. \tag{2.16}$$

Substituting (2.13)-(2.16) into (2.9)-(2.12) yields (2.2)-(2.5).
b) Necessary substitutions from (2.13)-(2.16) into (1.2)-(1.3) yields (2.6) and (2.7). □

Corollary 2.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^{m\times m}$ be given, suppose that the matrix equation (2.1) has a Hermitian reflexive solution.$X = X^* \in \mathbb{C}_r^{n\times n}(P)$ Then.*

a) Equation (2.1) has a Hermitian reflexive solution of the form $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} C & A(I_n+P) \\ (I_n+P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix}.$$

b) All Hermitian reflexive solutions of equation (2.1) have the form $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} = r(A(I_n-P)) + r(A) - 2(n-k).$$

c) Equation (2.1) has a Hermitian reflexive solution of the form $X = U\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} C & A(I_n-P) \\ (I_n-P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix}.$$

d) All Hermitian reflexive solutions of equation (2.1) have the form $X = U\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} = r(A(I_n+P)) + r(A) - 2k.$$

e) Equation (2.1) has a null solution if and only if

$$r\begin{bmatrix} C & A\left(I_n+P\right) \\ \left(I_n+P\right)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A\left(I_n+P\right) & C \end{bmatrix},$$
$$r\begin{bmatrix} C & A\left(I_n-P\right) \\ \left(I_n-P\right)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A\left(I_n-P\right) & C \end{bmatrix}.$$

f) All Hermitian reflexive solutions of equation (2.1) are nulls if and only if

$$\begin{aligned} r\begin{bmatrix} A\left(I_n+P\right) & C \end{bmatrix} &= r\left(A\left(I_n-P\right)\right)+r\left(A\right)-2\left(n-k\right), \\ r\begin{bmatrix} A\left(I_n-P\right) & C \end{bmatrix} &= r\left(A\left(I_n+P\right)\right)+r\left(A\right)-2k. \end{aligned}$$

It is well known that, the generalized inverse A^- for a given matrix A is a solution of the matrix equation $AXA = A$, so we apply Corollary 2.1 to the equation $AXA = A$ we obtain this result.

Corollary 2.2. *Let $A \in \mathbb{C}^{n\times n}$, for some unitary matrix U. Then,*
a) A has a generalized inverse A^- of the form $A^- = U\begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} A & A\left(I_n+P\right) \\ \left(I_n+P\right)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A\left(I_n+P\right) & A \end{bmatrix}.$$

b) A has a generalized inverse A^- of the form $A^- = U\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix}U^*$ if and only if

$$r\begin{bmatrix} A & A\left(I_n-P\right) \\ \left(I_n-P\right)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A\left(I_n-P\right) & A \end{bmatrix}.$$

A square complex matrix A is defined as EP (Equal-Range Projection) or (range-Hermitian) when both the matrix A and its conjugate transpose A^* have identical ranges. Tian in [18] compiled established characterizations for EP matrices and provided additional new characterizations for this class of matrices, hence if the two matrices N_1 and N_2 in Corollary (2.2) satisfy some conditions we have the result

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$, If N_1 and N_2 in Corollary (2.2) are nonsingular, for some unitary matrix U, we have:*
A is an EP matrix if and only if

$$r\begin{bmatrix} A & A\left(I_n+P\right) \\ \left(I_n+P\right)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A\left(I_n+P\right) & A \end{bmatrix}.$$

$$\begin{bmatrix} A & A\left(I_n-P\right) \\ \left(I_n-P\right)A & 0 \end{bmatrix} = 2r\begin{bmatrix} A\left(I_n-P\right) & A \end{bmatrix}.$$

Proof. From ([18] Theorem 2.1) for a given matrix $A \in \mathbb{C}^{n\times n}$, the following statements are equivalent
i) A is EP
ii) A^{-} is EP
iii) There exists an unitary matrix U such that $UAU^* = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where A_1 is nonsingular.
By applying i), ii) and iii) to a) and b) of Corollary (2.2) leads to result in Corollary (2.3). □

Theorem 2.2. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times m}$, $C = -C^*$, and assume that the matrix equation (2.1) has a skew-Hermitian reflexive solution $X = -X^* \in \mathbb{C}_r^{n\times n}(P)$ Then,*
(a) The maximal and minimal ranks of the two submatrices X_1 and X_2 in skew-Hermitian reflexive solution $X = U\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^*$ to the matrix equation (2.1) are given by

$$\max_{X_1=-X_1^*} r(X_1) = \min\left\{k, r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} + 2k - r(A(I_n+P)) - r(A)\right\}.$$

$$\min_{X_1=-X_1^*} r(X_1) = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n-P) \\ -(I_n-P)A^* & 0 \end{bmatrix}.$$

$$\max_{X_2=-X_2^*} r(X_2) = \min\left\{n-k, r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} + 2(n-k) - r(A(I_n-P)) - r(A)\right\}.$$

$$\min_{X_2=-X_2^*} r(X_2) = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} - r\begin{bmatrix} C & A(I_n+P) \\ -(I_n+P)A^* & 0 \end{bmatrix}.$$

b) The general skew- Hermitian reflexive solution of (2.1) can be expressed as

$$X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}U^*$$

where

$$\begin{aligned} X_1 &= X_{01} + S_1 L_{AU} Z L_{AU} S_1^* + L_{\left(\frac{1}{2}A(I_n+P)U\right)}V - V^* L_{\left(\frac{1}{2}A(I_n+P)U\right)}, \\ X_2 &= X_{02} - S_2 L_{AU} Z L_{AU} S_2^* + L_{\left(\frac{1}{2}A(I_n-P)U\right)}W - W^* L_{\left(\frac{1}{2}A(I_n-P)U\right)} \end{aligned}$$

where $\begin{bmatrix} X_{01} & 0 \\ 0 & X_{02} \end{bmatrix}$ is a special skew-Hermitian reflexive solution of (2.1), and

$$S_1 = (I_k, 0), S_2 = (0, I_{n-k})$$

Z, V and W are arbitrary matrices with suitable sizes.

Proof. The poof is similar to that of Theorem 2.1. □

Corollary 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}_H^{m\times m}$ be given and assume that the matrix equation (2.1) has a skew-Hermitian reflexive solution $X = -X^* \in \mathbb{C}_r^{n\times n}(P)$ Then.*
a) Equation (2.1) has a skew-Hermitian reflexive solution of the form $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$r\begin{bmatrix} C & A(I_n+P) \\ -(I_n+P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix}.$$

b) All skew-Hermitian reflexive solutions of equation (2.1) have the form $X = U\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} = r(A(I_n-P)) + r(A) - 2(n-k).$$

c) Equation (2.1) has a skew-Hermitian reflexive solution of the form $X = U\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$r\begin{bmatrix} C & A(I_n-P) \\ -(I_n-P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix}.$$

d) All skew-Hermitian reflexive solutions of equation (2.1) have the form $X = U\begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} = r(A(I_n+P)) + r(A) - 2k.$$

e) Equation (2.1) has a null solution if and only if

$$r\begin{bmatrix} C & A(I_n+P) \\ -(I_n+P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n+P) & C \end{bmatrix},$$
$$r\begin{bmatrix} C & A(I_n-P) \\ -(I_n-P)A^* & 0 \end{bmatrix} = 2r\begin{bmatrix} A(I_n-P) & C \end{bmatrix}.$$

f) All skew-Hermitian reflexive solutions of equation (2.1) are null solutions if and only if

$$r\begin{bmatrix} A(I_n+P) & C \end{bmatrix} = r(A(I_n-P)) + r(A) - 2(n-k),$$
$$r\begin{bmatrix} A(I_n-P) & C \end{bmatrix} = r(A(I_n+P)) + r(A) - 2k.$$

3. Extremal ranks of submatrices in generalized reflexive solution of $AXB = C$

In this section we will review special forms of the reflexive solution of the equation $AXB = C$ with respect to the generalized reflexion matrix P.
Consider the linear matrix equation

$$AXB = C \tag{3.1}$$

where A, B and C are given, and X is unknown.

The following Lemma is the same that corollary 3.5 in [20], (also it is the same that Theorem 2.2 in [17]).

Lemma 3.1. *[20]We adopt the following notations:*

$$\begin{aligned} J_3 &= \{X_1 \in H^{p_1\times q_1} \mid A_3X_1B_1 + A_4X_2B_2 = C_3\} \\ J_4 &= \{X_2 \in H^{p_2\times q_2}|, A_3X_1B_1 + A_4X_2B_2 = C_3\}. \end{aligned}$$

Assume that $A_3 \in H^{s\times p_1}$, $A_4 \in H^{s\times p_2}$, $B_1 \in H^{q_1\times t}$, $B_2 \in H^{q_2\times t}$, $C_3 \in H^{s\times t}$, *and the matrix equation*

$$A_3X_1B_1 + A_4X_2B_2 = C_3. \tag{3.2}$$

is consistent. Then the extremal ranks of the solution to (3.2) are given by

$$\begin{aligned}
\max_{X_1\in J_3} r(X_1) &= \min\left\{ \begin{array}{c} p_1, q_1, p_1+q_1 + r\begin{bmatrix} C_3 & A_4 \end{bmatrix} - r\begin{bmatrix} A_3 & A_4 \end{bmatrix} - r(B_1), \\ p_1+q_1 + r\begin{bmatrix} B_2 \\ C_3 \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_3). \end{array} \right\}. \\
\min_{X_1\in J_3} r(X_1) &= r\begin{bmatrix} C_3 & A_4 \end{bmatrix} + r\begin{bmatrix} B_2 \\ C_3 \end{bmatrix} - r\begin{bmatrix} C_3 & A_4 \\ B_2 & 0 \end{bmatrix}. \\
\max_{X_2\in J_4} r(X_2) &= \min\left\{ \begin{array}{c} p_2, q_2, p_2+q_2 + r\begin{bmatrix} C_3 & A_3 \end{bmatrix} - r\begin{bmatrix} A_3 & A_4 \end{bmatrix} - r(B_2), \\ p_2+q_2 + r\begin{bmatrix} B_1 \\ C_3 \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_4). \end{array} \right\}. \\
\min_{X_2\in J_4} r(X_2) &= r\begin{bmatrix} C_3 & A_3 \end{bmatrix} + r\begin{bmatrix} B_1 \\ C_3 \end{bmatrix} - r\begin{bmatrix} C_3 & A_3 \\ B_1 & 0 \end{bmatrix}.
\end{aligned}$$

Lemma 3.2. *[16] Let* $A_1 \in \mathcal{F}^{m\times p}$, $B_1 \in \mathcal{F}^{q\times n}$, $A_2 \in \mathcal{F}^{m\times s}$, $B_2 \in \mathcal{F}^{t\times n}$ *and* $C \in \mathcal{F}^{m\times n}$ *be given over an arbitrary field* $\mathcal{F}$, *and suppose that the matrix equation*

$$A_1XB_1 + A_2YB_2 = C \tag{3.3}$$

is solvable. Then its general solutions for X *and* Y *can be expressed as:*

$$X = X_0 + S_1L_GUR_HT_1 + L_{A_1}V_1 + V_2R_{B_1}. \tag{3.4}$$

$$Y = Y_0 + S_2L_GUR_HT_2 + L_{A_2}W_1 + W_2R_{B_2}. \tag{3.5}$$

where $S_1 = [I_p,\, 0]$, $S_2 = [0, I_s]$, $T_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 \\ I_t \end{bmatrix}$, $G = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, $H = \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}$ *and* X_0, Y_0 *are a pair of particular solutions to Eq (3.3),* U, V_1, V_2, W_1 *and* W_2 *are arbitrary*

Theorem 3.1. *Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix and let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$ are given, suppose that the matrix equation (3.1) has a reflexive solution.$X \in \mathbb{C}_r^{n\times n}(P)$ Then*
(a) The maximal and minimal ranks of the two submatrices X_1 and X_2 in a reflexive solution $X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ to the matrix equation (3.1) are given by

$$\max_{X_1} r(X_1) = \min \left\{ \begin{array}{c} k, 2k + r\begin{bmatrix} C & A(I_n - P) \end{bmatrix} - r(A) - r((I_n + P)B), \\ 2k + r\begin{bmatrix} (I_n - P)B \\ C \end{bmatrix} - r(B) - r(A(I_n + P)). \end{array} \right\}. \tag{3.6}$$

$$\min_{X_1} r(X_1) = r\begin{bmatrix} C & A(I_n - P) \end{bmatrix} + r\begin{bmatrix} (I_n - P)B \\ C \end{bmatrix} - r\begin{bmatrix} C & A(I_n - P) \\ (I_n - P)B & 0 \end{bmatrix}. \tag{3.7}$$

$$\max_{X_2} r(X_2) = \min \left\{ \begin{array}{c} n - k, 2(n - r) + r\begin{bmatrix} C & A(I_n + P) \end{bmatrix} - r(A) - r((I_n - P)B), \\ 2(n - k) + r\begin{bmatrix} (I_n + P)B \\ C \end{bmatrix} - r(B) - r(A(I_n - P)). \end{array} \right\}. \tag{3.8}$$

$$\min_{X_2} r(X_2) = r\begin{bmatrix} C & A(I_n + P) \end{bmatrix} + r\begin{bmatrix} (I_n + P)B \\ C \end{bmatrix} - r\begin{bmatrix} C & A(I_n + P) \\ (I_n + P)B & 0 \end{bmatrix}. \tag{3.9}$$

b) The general reflexive solution to (3.1) can be expressed as

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$$

where

$$\begin{aligned} X_1 &= X_0 + S_1 L_{AU} Z R_{U^*B} T_1 + L_{\left(\frac{1}{2}A(I_n+P)U\right)} Z_1 + Z_2 R_{\left(\frac{1}{2}U^*(I_n+P)B\right)}, \\ X_2 &= Y_0 + S_2 L_{AU} Z R_{U^*B} T_2 + L_{\left(\frac{1}{2}A(I_n-P)U\right)} Z_3 + Z_4 R_{\left(\frac{1}{2}U^*(I_n-P)B\right)}. \end{aligned}$$

where $S_1 = [I_k,\ 0]$, $S_2 = [0, I_{n-k}]$, $T_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix}$, and $\begin{bmatrix} X_0 & 0 \\ 0 & Y_0 \end{bmatrix}$ is a particular reflexive solution to equation (3.1),
Z, Z_1, Z_2, Z_3 and Z_4 are arbitrary matrices with appropriate sizes.

Proof. a) From lemma 1.2 the reflexive solution to $AXB = C$ can be written as

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} U^*$$

for arbitrary unitary matrix $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, with $U_1 \in \mathbb{C}^{n\times k}$, $U_2 \in \mathbb{C}^{n\times(n-k)}$. We denote

$$AU = \begin{bmatrix} A_1 & A_2 \end{bmatrix}, U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $A_1 \in \mathbb{C}^{m\times k}$, $A_2 \in \mathbb{C}^{m\times(n-k)}$, $B_1 \in \mathbb{C}^{k\times l}$, $B_2 \in \mathbb{C}^{(n-k)\times l}$. So,

$$\begin{aligned} AXB = C &\iff AU\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}U^*B = C \\ &\iff \begin{bmatrix} A_1 & A_2 \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = C \\ &\iff A_1X_1B_1 + A_2X_2B_2 = C. \end{aligned} \tag{3.10}$$

Then, the two equations (3.1) and (3.10) are equivalent, now we adopt the following notations:

$$S_1 = \{X_1 \in \mathbb{C}^{k\times k} \mid A_1X_1B_1 + A_2X_2B_2 = C\},$$
$$S_2 = \{X_2 \in \mathbb{C}^{(n-k)\times(n-k)} \mid A_1X_1B_1 + A_2X_2B_2 = C\}.$$

From Lemma 3.1 we have

$$\max_{X_1\in S_1} r(X_1) = \min\left\{ \begin{array}{c} r, 2k + r\begin{bmatrix} C & A_2 \end{bmatrix} - r\begin{bmatrix} A_1 & A_2 \end{bmatrix} - r(B_1), \\ 2k + r\begin{bmatrix} B_2 \\ C \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_1). \end{array} \right\} \tag{3.11}$$

$$\min_{X_1\in S_1} r(X_1) = r\begin{bmatrix} C & A_2 \end{bmatrix} + r\begin{bmatrix} B_2 \\ C \end{bmatrix} - r\begin{bmatrix} C & A_2 \\ B_2 & 0 \end{bmatrix} \tag{3.12}$$

$$\max_{X_2\in S_2} r(X_2) = \min\left\{ \begin{array}{c} n-k, 2(n-r) + r\begin{bmatrix} C & A_1 \end{bmatrix} - r\begin{bmatrix} A_1 & A_2 \end{bmatrix} - r(B_2), \\ 2(n-k) + r\begin{bmatrix} B_1 \\ C \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(A_2). \end{array} \right\} \tag{3.13}$$

$$\min_{X_2\in S_2} r(X_2) = r\begin{bmatrix} C & A_1 \end{bmatrix} + r\begin{bmatrix} B_1 \\ C \end{bmatrix} - r\begin{bmatrix} C & A_1 \\ B_1 & 0 \end{bmatrix}. \tag{3.14}$$

From Lemmas 1.1 and 1.2 we can simplify:

$$\begin{aligned} r\begin{bmatrix} C & A_2 \end{bmatrix} &= r\begin{bmatrix} C & 0 & A_2 \end{bmatrix} \\ &= r\begin{bmatrix} C & \tfrac{1}{2}A(I_n - P)U \end{bmatrix} \\ &= r\begin{bmatrix} C & A(I_n - P) \end{bmatrix}, \\ r\begin{bmatrix} C & A_1 \end{bmatrix} &= r\begin{bmatrix} C & A_1 & 0 \end{bmatrix} \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&= r\begin{bmatrix} C & \frac{1}{2}A\left(I_n+P\right)U \end{bmatrix} \\
&= r\begin{bmatrix} C & A\left(I_n+P\right) \end{bmatrix}, \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} B_2 \\ C \end{bmatrix} &= r\begin{bmatrix} 0 \\ B_2 \\ C \end{bmatrix} \\
&= r\begin{bmatrix} \frac{1}{2}U^*\left(I_n-P\right)B \\ C \end{bmatrix} \\
&= r\begin{bmatrix} \left(I_n-P\right)B \\ C \end{bmatrix}, \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} B_1 \\ C \end{bmatrix} &= r\begin{bmatrix} B_1 \\ 0 \\ C \end{bmatrix} \\
&= r\begin{bmatrix} \frac{1}{2}U^*\left(I_n+P\right)B \\ C \end{bmatrix} \\
&= r\begin{bmatrix} \left(I_n+P\right)B \\ C \end{bmatrix}. \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} C & A_2 \\ B_2 & 0 \end{bmatrix} &= r\begin{bmatrix} C & 0 & A_2 \\ 0 & 0 & 0 \\ B_2 & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} C & \frac{1}{2}A\left(I_n-P\right)U \\ \frac{1}{2}U^*\left(I_n-P\right)B & 0 \end{bmatrix} \\
&= r\begin{bmatrix} C & A\left(I_n-P\right) \\ \left(I_n-P\right)B & 0 \end{bmatrix} \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
r\begin{bmatrix} C & A_1 \\ B_1 & 0 \end{bmatrix} &= \begin{bmatrix} C & A_1 & 0 \\ B_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} C & \frac{1}{2}A\left(I_n+P\right)U \\ \frac{1}{2}U^*\left(I_n+P\right)B & 0 \end{bmatrix} \\
&= r\begin{bmatrix} C & A\left(I_n+P\right) \\ \left(I_n+P\right)B & 0 \end{bmatrix} \qquad (3.20)
\end{aligned}
$$

Substituting (3.15)-(3.20) into (3.11)-(3.14) yields results of Theorem 3.1.
b) Obvious from formulas (3.4)-(3.5) of Lemma (3.2) and necessary changes from (3.15)-(3.20). □

Corollary 3.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $C \in \mathbb{C}^{m\times l}$ are given, we suppose that the matrix equation (3.1) has a reflexive solution. Then* *a) Equation (3.1) has a reflexive solution of the form* $X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} C & A(I_n+P) \\ (I_n+P)B & 0 \end{bmatrix} = r \begin{bmatrix} C & A(I_n+P) \end{bmatrix} + r \begin{bmatrix} (I_n+P)B \\ C \end{bmatrix}.$$

b) All reflexive solutions of equation (3.1) have the form $X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$ *if and only if*

$$\begin{aligned} r \begin{bmatrix} C & A(I_n+P) \end{bmatrix} &= r(A) + r((I_n-P)B) - 2(n-k), \\ r \begin{bmatrix} (I_n+P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n-P)) - 2(n-k). \end{aligned}$$

c) Equation (3.1) has a reflexive solution of the form $X = U \begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$r \begin{bmatrix} C & A(I_n-P) \\ (I_n-P)B & 0 \end{bmatrix} = r \begin{bmatrix} C & A(I_n-P) \end{bmatrix} + r \begin{bmatrix} (I_n-P)B \\ C \end{bmatrix}.$$

d) All reflexive solutions of equation (3.1) have the form $X = U \begin{bmatrix} 0 & 0 \\ 0 & X_2 \end{bmatrix} U^*$ *if and only if*

$$\begin{aligned} r \begin{bmatrix} C & A(I_n-P) \end{bmatrix} &= r(A) + r((I_n+P)B) - 2k, \\ r \begin{bmatrix} (I_n-P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n+P)) - 2k. \end{aligned}$$

e) Equation (3.1) has a null reflexive solution if and only if

$$\begin{aligned} r \begin{bmatrix} C & A(I_n+P) \\ (I_n+P)B & 0 \end{bmatrix} &= r \begin{bmatrix} C & A(I_n+P) \end{bmatrix} + r \begin{bmatrix} (I_n+P)B \\ C \end{bmatrix}, \\ \text{and } r \begin{bmatrix} C & A(I_n-P) \\ (I_n-P)B & 0 \end{bmatrix} &= r \begin{bmatrix} C & A(I_n-P) \end{bmatrix} + r \begin{bmatrix} (I_n-P)B \\ C \end{bmatrix}. \end{aligned}$$

f) All reflexive solutions of equation (3.1) are nulls if and only if

$$\begin{aligned} r \begin{bmatrix} C & A(I_n+P) \end{bmatrix} &= r(A) + r((I_n-P)B) - 2(n-k), \\ r \begin{bmatrix} (I_n+P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n-P)) - 2(n-k). \end{aligned}$$

and

$$\begin{aligned} r \begin{bmatrix} C & A(I_n-P) \end{bmatrix} &= r(A) + r((I_n+P)B) - 2k, \\ r \begin{bmatrix} (I_n-P)B \\ C \end{bmatrix} &= r(B) + r(A(I_n+P)) - 2k. \end{aligned}$$

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