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THE GROWTH OF SOLUTIONS OF SOME LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS BEING LACUNARY SERIES OF (P,Q)-ORDER

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Abstract. In this paper, we study the growth of meromorphic solutions of certain linear differential equations with entire coefficients being Lacunary series. We extend some previous results due to L. M. Li and T. B. Cao [9] and S. Z. Wu and X. M. Zheng [13] and others.

Keywords: Entire functions, meromorphic functions, differential equations, Lacunary series, (p,q) −order.

1. Introduction and main results

We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's theory (see e.g. [5, 8, 15]). For $r \in [0,+\infty)$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. For all r sufficiently large, we define $\log_1 r = \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$, $\log_{-1} r = \exp_1 r$ and $\exp_{-1} r = \log_1 r$. Furthermore, we define the linear measure of a set $E \subset [0,+\infty)$ by $m(E) = \int_E dt$ and the logarithmic measure of a set $F \subset [1,+\infty)$ by $m_l(F) = \int_F \frac{dt}{t}$. Now, we shall introduce the definition of meromorphic functions of (p,q)-order, where p, q are positive integers satisfying $p \geq q \geq 1$, (see e.g. [9, 10]).

Definition 1.1. The (p,q)-order of a meromorphic function $f(z)$ is defined by

$$\sigma_{(p.q)}(f) = \varlimsup_{r\to\infty} \frac{\log_p T(r,f)}{\log_q r},$$

where $T(r,f)$ is the characteristic function of Nevanlinna of the function f. If f is an entire function, then

$$\sigma_{(p.q)}(f) = \varlimsup_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log_q r},$$

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where $M(r, f)$ is the maximum modulus of f in the circle $|z| = r$.

Definition 1.2. The (p, q)-exponent of convergence of the sequence of a-points of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}(f-a) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f-a})}{\log_q r}$$

and the (p, q)-exponent of convergence of the sequence of distinct a-points of a meromorphic function $f(z)$ is defined by

$$\bar{\lambda}_{(p.q)}(f-a) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f-a})}{\log_q r}.$$

If $a = 0$, the (p, q)-exponent of convergence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f})}{\log_q r}$$

and the (p, q)-exponent of convergence of distinct zeros of a meromorphic function $f(z)$ is defined by

$$\bar{\lambda}_{(p.q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log_q r}.$$

If $a = \infty$, the (p, q)-exponent of convergence of the sequence of poles of a meromorphic function $f(z)$ is defined by

$$\lambda_{(p.q)}\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, f)}{\log_q r}.$$

Through the past years, many authors investigated the growth of solutions of the higher order linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1.2}$$

when $A_j(z)$ $(j = 0, 1, \ldots, k-1)$, $F(z)$ are entire functions and obtained some valuable results (see e.g. [8, 9, 10, 11, 12, 13, 14, 16]). In 2013, J. Tu, H. Y. Xu, H. M. Liu and Y. Liu [12] investigated (1.2) and obtained the properties of solutions of (1.2) when some coefficient A_d $(0 \le d \le k-1)$ is dominant and being Lacunary series.

Theorem 1.1. [12] *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$, $F(z)$ be entire functions of finite iterated order and satisfying*

$$\max\{\sigma_p(A_j), j \neq d\} \leq \sigma_p(A_d) < \infty$$

and

$$\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_d)\} < \tau_p(A_d)\ (0 \leq d \leq k-1).$$

Suppose that $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is an entire function such that the sequence of exponents $\{\lambda_n\}$ satisfies

$$\frac{\lambda_n}{n} > (\log n)^{2+\eta},\ (\eta > 0, n \in \mathbb{N}), \tag{1.3}$$

then, one has

(i) If $\sigma_p(F) < \sigma_p(A_d)$ or $\sigma_p(F) = \sigma_p(A_d)$ and $\tau_p(F) < \tau_p(A_d)$, then every transcendental solution $f(z)$ of (1.2) satisfies $\sigma_{p+1}(f) = \sigma_p(A_d)$; furthermore if $F(z) \not\equiv 0$, then every transcendental solution $f(z)$ of (1.2) satisfies

$$\bar{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_d).$$

(ii) If $\sigma_p(F) > \sigma_p(A_d)$ and $\sigma_{p+1}(F) \leq \sigma_p(A_d)$, then all solutions of (1.2) satisfy $\sigma_p(f) \geq \sigma_p(F)$ and $\sigma_{p+1}(f) \leq \sigma_p(A_d)$.

(iii) If $\sigma_{p+1}(F) > \sigma_p(A_d)$, then all solutions of (1.2) satisfy $\sigma_{p+1}(f) = \sigma_{p+1}(F)$ and $\bar{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(F)$ holds for all solutions of (1.2) with at most one exceptional solution f_0 satisfying $\lambda_{p+1}(f_0) < \sigma_{p+1}(F)$.

Remark 1.1. Suppose that $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is an entire function of infinite order such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition (1.3), then the series $\sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is called Lacunary series.

In the following result, Zhan and Zheng [16] investigated the growth of solutions of (1.2) when the coefficients are meromorphic functions and extended the results in Theorem 1.1 to the (p,q)-order case.

Theorem 1.2. [16] *Suppose that $A_0(z), \ldots, A_{k-1}(z)$, $F(z)$ are meromorphic functions satisfying that there exists some $d \in \{0, 1, \ldots, k-1\}$ such that*

$$\sigma_1 = \max\{\sigma_{(p,q)}(A_j), (j \neq d), \sigma_{(p,q)}(F)\} < \mu_{(p,q)}(A_d) \leq \sigma_{(p,q)}(A_d) < \infty.$$

Suppose that $A_d(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is also an entire function such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition (1.3). If $f(z)$ is a meromorphic solution to (1.2) satisfying $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(A_d)$, then the following results hold:

(*a*) *If* $f(z)$ *is a rational solution, then* $f(z)$ *must be a polynomial with* $\deg f \leq d-1$.

(*b*) *If* $f(z)$ *is a transcendental solution, then* $f(z)$ *satisfies*

$$\mu_{(p+1,q)}(f) = \mu_{(p,q)}(A_d) \leq \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_d).$$

Furthermore, if $F(z) \not\equiv 0$, *then we have*

$$\underline{\overline{\lambda}}_{(p+1,q)}(f) = \mu_{(p+1,q)}(f) = \mu_{(p,q)}(A_d) \leq \sigma_{(p,q)}(A_d) = \sigma_{(p+1,q)}(f) = \overline{\lambda}_{(p+1,q)}(f).$$

In [9], Li and Cao have considered the equation (1.2) with meromorphic coefficients of finite (p,q)-order and obtained the following results.

Theorem 1.3. [9] *Assume that* $A_0(z), A_1(z), \ldots, A_{k-1}(z), F(z) \not\equiv 0$ *are meromorphic functions in the plane satisfying*

$$\max\{\sigma_{(p,q)}(A_j), \lambda_{(p,q)}\left(\frac{1}{A_0}\right), \sigma_{(p+1,q)}(F) : j = 1, 2, \ldots, k-1\} < \sigma_{(p,q)}(A_0),$$

then all meromorphic solutions $f(z)$, *whose poles are of uniformly bounded multiplicities, of (1.2), satisfy*

$$\overline{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_0),$$

with at most one exceptional solution f_0 *satisfying* $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_0)$.

Theorem 1.4. [9] *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$, $F(z) \not\equiv 0$ *be meromorphic functions in the plane satisfying*

$$\max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(p+1,q)}(F).$$

Suppose that all solutions of (1.2) are meromorphic functions whose poles are of uniformly bounded multiplicities, then $\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F)$ *holds for all solutions of (1.2).*

Recently, Wu and Zheng [13] have considered the linear differential equations

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.4}$$

and

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z), \tag{1.5}$$

where $A_j(z)$ $(j = 0, 1, \ldots, k)$, $F(z)$ are entire functions, such that $A_0(z)A_k(z)F(z) \not\equiv 0$ and obtained the following result when the coefficient $A_k(z)$ is of maximal order and Fabry gap series.

Theorem 1.5. [13] *Suppose that* $k \geq 2$, $A_j(z)$ $(j = 0, 1, \ldots, k)$ *are entire functions satisfying* $A_k(z)A_0(z) \not\equiv 0$ *and* $\sigma(A_j) < \sigma(A_k) < \infty$, $j = 0, 1, \ldots, k-1$. *Suppose that* $A_k(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ *and the sequence of exponents* $\{\lambda_n\}$ *satisfies the Fabry gap condition*

$$\frac{\lambda_n}{n} \to \infty, \ (n \to \infty). \tag{1.6}$$

Then every rational solution $f(z)(\not\equiv 0)$ *of (1.4) is a polynomial with* $\deg f \leq k-1$ *and every transcendental meromorphic solution* $f(z)$, *whose poles are of uniformly bounded multiplicities, of (1.4) such that* $\lambda\left(\frac{1}{f}\right) < \mu(f)$, *satisfies*

$$\bar{\lambda}(f-\varphi) = \lambda(f-\varphi) = \sigma(f) = \infty,$$
$$\bar{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \sigma_2(f) = \sigma(A_k),$$

where φ *is a finite order meromorphic function and doesn't solve (1.4).*

Remark 1.2. Suppose that $A_k(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is an entire function of finite order such that the sequence of exponents $\{\lambda_n\}$ satisfies Fabry gap condition (1.6), then the series $\sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ is called Fabry gap series.

Thus, natural questions arises : What can we say about the growth of solutions of equations of the form (1.4) and (1.5) when the coefficient $A_k(z)$ is of maximal (p,q)-order and being Lacunary series and can we have similar results as in Theorems 1.3, 1.4 and 1.5 using the concept of (p,q)-order. In this paper, we proceed this way and we obtain the following results.

Theorem 1.6. *Suppose that* $k \geq 2$, $A_j(z)$ $(j = 0, 1, \ldots, k)$ *are entire functions satisfying* $A_k(z)A_0(z) \not\equiv 0$ *and*

$$\max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(p,q)}(A_k) < \infty.$$

Suppose that $A_k(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ *and the sequence of exponents* $\{\lambda_n\}$ *satisfies the gap condition (1.3). Then every rational solution* $f(z)(\not\equiv 0)$ *of (1.4) is a polynomial with* $\deg f \leq k-1$ *and every transcendental meromorphic solution* $f(z)$, *of (1.4) such that* $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, *satisfies*

$$\bar{\lambda}_{(p+1,q)}(f-\varphi) = \lambda_{(p+1,q)}(f-\varphi) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_k),$$

where $\varphi(z)$ *is a meromorphic function satisfying* $\sigma_{(p,q)}(\varphi) < \infty$ *and doesn't solve (1.4).*

Theorem 1.7. *Suppose that* $k \geq 2$, $A_j(z)$ $(j = 0, 1, \ldots, k)$ *and* $F(z)$ *are entire functions satisfying* $A_k(z)A_0(z)F(z) \not\equiv 0$ *and*

$$\max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F) : j = 0, 1, \ldots, k-1\} < \sigma_{(p,q)}(A_k) < \infty.$$

Suppose that $A_k(z)=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ *and the sequence of exponents* $\{\lambda_n\}$ *satisfies (1.3). Then every rational solution* $f(z)$ *of (1.5) is a polynomial with* $\deg f\le k-1$ *and every transcendental meromorphic solution* $f(z)$ *of (1.5) such that* $\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu_{(p,q)}(f)$*, satisfies*

$$\bar{\lambda}_{(p+1,q)}(f-\varphi)=\lambda_{(p+1,q)}(f-\varphi)=\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_k),$$

where $\varphi(z)$ *is a meromorphic function satisfying* $\sigma_{(p,q)}(\varphi)<\infty$ *and does not solve (1.5).*

Theorem 1.8. *Suppose that* $k\ge 2$, $A_j(z)$ $(j=0,1,\dots,k)$ *are entire functions satisfying hypotheses of Theorem 1.6 and* $F(z)\not\equiv 0$ *is an entire function.*

(i) If $\sigma_{(p+1,q)}(F)<\sigma_{(p,q)}(A_k)$*, then every transcendental meromorphic solution* $f(z)$ *of (1.5), satisfies*

$$\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_k),$$

with at most one exceptional solution f_0 *satisfying* $\sigma_{(p+1,q)}(f_0)<\sigma_{(p,q)}(A_k)$.

(ii) If $\sigma_{(p+1,q)}(F)>\sigma_{(p,q)}(A_k)$*, then every transcendental meromorphic solution* $f(z)$ *of (1.5) satisfies* $\sigma_{(p+1,q)}(f)=\sigma_{(p+1,q)}(F)$.

2. Preliminary lemmas

Lemma 2.1. [2] *Let* f *be a transcendental meromorphic function in the plane, and let* $\alpha>1$ *be a given constant. Then there exist a set* $E_1\subset(1,+\infty)$ *that has a finite logarithmic measure, and a constant* $B>0$ *depending only on* α *and* (m,n) $(m,n\in\{0,1,\dots,k\})$ $m<n$ *such that for all* z *with* $|z|=r\notin[0,1]\cup E_1$*, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right|\le B\left(\frac{T(\alpha r,f)}{r}(\log^{\alpha}r)\log T(\alpha r,f)\right)^{n-m}.$$

By using similar proof of Lemma 2.5 in [4], we can easily extend Lemma 3.3 in [16] to the case $\sigma_{(p,q)}(g)=\sigma_{(p,q)}(f)=+\infty$.

Lemma 2.2. *Let* $f(z)=\frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire functions satisfying* $\mu_{(p,q)}(g)=\mu_{(p,q)}(f)=\mu\le\sigma_{(p,q)}(g)=\sigma_{(p,q)}(f)\le\infty$ *and* $\lambda_{(p,q)}(d)=\sigma_{(p,q)}(d)=\lambda_{(p,q)}\left(\frac{1}{f}\right)<\mu$*. Then there exists a set* $E_2\subset(1,+\infty)$ *of finite logarithmic measure such that for all* $|z|=r\notin[0,1]\cup E_2$ *and* $|g(z)|=M(r,g)$ *we have*

$$\left|\frac{f(z)}{f^{(k)}(z)}\right|\le r^{2k},\ (k\in\mathbb{N}).$$

Lemma 2.3. [7] *Let* $f(z) = \sum_{n=0}^{\infty} c_{\lambda_n} z^{\lambda_n}$ *be an entire function and the sequence of exponents* $\{\lambda_n\}$ *satisfies the gap condition (1.3). Then for any given* $\varepsilon > 0$,

$$\log L(r, f) > (1 - \varepsilon) \log M(r, f)$$

holds outside a set E_3 *of finite logarithmic measure, where*

$$M(r, f) = \sup_{|z|=r} \left| f(z) \right|, \quad L(r, f) = \inf_{|z|=r} \left| f(z) \right|.$$

Lemma 2.4. [10] *Let* $f(z)$ *be an entire function of* (p, q)*-order satisfying* $0 < \sigma_{(p,q)}(f) = \sigma < \infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_4 \subset (1, +\infty)$ *having infinite logarithmic measure such that for all* $r \in E_4$, *we have*

$$\sigma = \lim_{r\to\infty, r\in E_4} \frac{\log_p T(r, f)}{\log_q r} = \lim_{r\to\infty, r\in E_4} \frac{\log_{p+1} M(r, f)}{\log_q r}$$

and

$$M(r, f) > \exp_{p+1}\{(\sigma - \varepsilon) \log_q r\}.$$

Lemma 2.5. [1, 3] *Let* $g : [0, +\infty) \to \mathbb{R}$ *and* $h : [0, +\infty) \to \mathbb{R}$ *be monotone nondecreasing functions. If* (i) $g(r) \le h(r)$ *outside of an exceptional set of finite linear measure, or* (ii) $g(r) \le h(r)$, $r \notin E_5 \cup [0, 1]$ *where* $E_5 \subset (1, +\infty)$ *is a set of finite logarithmic measure, then for any* $\alpha > 1$, *there exists an* $r_0 = r_0(\alpha) > 0$ *such that* $g(r) \le h(\alpha r)$ *for all* $r > r_0$.

Lemma 2.6. [9] *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ *and* $F(z) \not\equiv 0$ *be meromorphic functions. If* $f(z)$ *is a meromorphic solution to (1.2) satisfying*

$$\max\{\sigma_{(p+1,q)}(F), \sigma_{(p+1,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(p+1,q)}(f),$$

then we have

$$\bar{\lambda}_{(p+1,q)}(f) = \lambda_{(p+1,q)}(f) = \sigma_{(p+1,q)}(f).$$

By using similar proof of Lemma 3.5 in [11], we can easily extend Lemma 3.6 in [16] to the case $\sigma_{(p,q)}(g) = \sigma_{(p,q)}(f) = +\infty$.

Lemma 2.7. *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire functions satisfying*

$$\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \le \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f) \le \infty$$

and

$$\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu.$$

Then there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_6$ and $|g(z)| = M(r, g)$ we have

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)), \quad (n \in \mathbb{N}),$$

where $\nu_g(r)$ is the central index of $g(z)$.

Lemma 2.8. [16] *Let $f(z)$ be a meromorphic function satisfying $\sigma_{(p,q)}(f) = \sigma < \infty$. Then there exist entire functions $\pi_1(z)$, $\pi_2(z)$ and $D(z)$ such that*

$$f(z) = \frac{\pi_1(z)e^{D(z)}}{\pi_2(z)}$$

and

$$\sigma_{(p,q)}(f) = \max\{\sigma_{(p,q)}(\pi_1), \sigma_{(p,q)}(\pi_2), \sigma_{(p,q)}(e^{D(z)})\}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$|f(z)| \leq \exp_{p+1}\{(\sigma + \varepsilon)\log_q r\}, \quad r \notin E_7,$$

where E_7 is a set of r of finite linear measure.

Lemma 2.9. [6] *Let $f(z)$ be an entire function of (p, q)-order, and let $\nu_f(r)$ be a central index of $f(z)$. Then*

$$\sigma_{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log_q r}.$$

Lemma 2.10. [9] *If $f(z)$ is a meromorphic function, then $\sigma_{(p,q)}(f') = \sigma_{(p,q)}(f)$.*

3. Proof of Theorem 1.6

Proof. Assume that $f(z) \not\equiv 0$ is a rational solution of (1.4). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $\lambda \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq k$, then $f^{(k)}(z) \not\equiv 0$. Since

$$\max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(p,q)}(A_k) < \infty,$$

then

$$\begin{aligned}\sigma_{(p,q)}(0) &= \sigma_{(p,q)}(A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f) \\ &= \sigma_{(p,q)}(A_k) > 0,\end{aligned}$$

which is a contradiction. Thus, $f(z)$ is a polynomial with $\deg f \leq k - 1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.4) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$. By Lemma 2.1, there exists a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left(T(2r, f)\right)^{k+1},\ j = 0, 1, \ldots, k. \tag{3.1}$$

Since $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$, then by Hadamard's factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying

$$\mu_{(p,q)}(g) = \mu_{(p,q)}(f) = \mu \leq \sigma_{(p,q)}(g) = \sigma_{(p,q)}(f),$$

$$\lambda_{(p,q)}(d) = \sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu.$$

Then by Lemma 2.2, there exists a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E_2$ and $\left|g(z)\right| = M(r, g)$, we have

$$\left|\frac{f(z)}{f^{(k)}(z)}\right| \leq r^{2k}. \tag{3.2}$$

Set

$$\alpha = \max\{\sigma_{(p,q)}(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_{(p,q)}(A_k) = \sigma < \infty.$$

Then, for any given ε $(0 < 2\varepsilon < \sigma - \alpha)$, we have

$$\left|A_j(z)\right| \leq \exp_{p+1}\{(\alpha + \varepsilon)\log_q r\}, \quad j = 0, 1, \ldots, k-1. \tag{3.3}$$

By Lemma 2.3 and Lemma 2.4, there exists a set $E_8 \subset (1, +\infty)$ of infinite logarithmic measure such that for all $|z| = r \in E_8$, we have

$$|A_k(z)| \geq L(r, A_k) > (M(r, A_k))^{1-\varepsilon} > (\exp_{p+1}\{(\sigma - \frac{\varepsilon}{2})\log_q r\})^{1-\varepsilon}$$

$$\geq \exp_{p+1}\{(\sigma - \varepsilon)\log_q r\}. \tag{3.4}$$

It follows from (1.4)

$$|A_k(z)| \leq \left|\frac{f}{f^{(k)}}\right|\left[|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right| + |A_0(z)|\right]. \tag{3.5}$$

Hence, by substituting (3.1)-(3.4) into (3.5), for all $|z| = r \in E_8 \backslash ([0, 1] \cup E_1 \cup E_2)$, we obtain

$$\exp_{p+1}\{(\sigma - \varepsilon)\log_q r\} \leq r^{2k}\exp_{p+1}\{(\alpha + \varepsilon)\log_q r\}kB\left(T(2r, f)\right)^{k+1}. \tag{3.6}$$

By Lemma 2.5 and (3.6), we have $\sigma - \varepsilon \leq \sigma_{(p+1,q)}(f)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\sigma_{(p+1,q)}(f) \geq \sigma_{(p,q)}(A_k)$. On the other hand, by (1.4), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq \left|\frac{A_{k-1}(z)}{A_k(z)}\right|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + \left|\frac{A_1(z)}{A_k(z)}\right|\left|\frac{f'}{f}\right| + \left|\frac{A_0(z)}{A_k(z)}\right|. \tag{3.7}$$

By Lemma 2.7, there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_6$ and $\left|g(z)\right| = M(r, g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)), \quad (j = 0, \ldots, k). \tag{3.8}$$

Since

$$\max\left\{\sigma_{(p,q)}(\frac{A_{k-1}}{A_k}), \ldots, \sigma_{(p,q)}(\frac{A_0}{A_k})\right\} = \sigma_{(p,q)}(A_k) = \sigma < \infty,$$

then by Lemma 2.8, there exists a set $E_7 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin E_7$ and for sufficiently large r, we have

$$\left|\frac{A_j(z)}{A_k(z)}\right| \leq \exp_{p+1}\{(\sigma + \varepsilon)\log_q r\}, \quad (j = 0, \ldots, k-1), \tag{3.9}$$

Then, it follows from (3.7), (3.8) and (3.9), for sufficiently large $r \notin [0,1] \cup E_6 \cup E_7$

$$\left(\frac{\nu_g(r)}{r}\right)|1 + o(1)| \leq k\,|1 + o(1)|\exp_{p+1}\{(\sigma + \varepsilon)\log_q r\}. \tag{3.10}$$

By (3.10), Lemma 2.5 and Lemma 2.9, one can verify

$$\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(g) \leq \sigma_{(p,q)}(A_k) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\sigma_{(p+1,q)}(f) \leq \sigma_{(p,q)}(A_k)$. Thus, we have

$$\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_k).$$

Next, we prove that

$$\bar{\lambda}_{(p+1,q)}(f - \varphi) = \lambda_{(p+1,q)}(f - \varphi) = \sigma_{(p+1,q)}(f).$$

Set $g(z) = f(z) - \varphi(z)$. Then

$$\sigma_{(p+1,q)}(g) = \sigma_{(p+1,q)}(f).$$

By substituting $f(z) = g(z) + \varphi(z)$ into (1.4), we get

$$\begin{aligned} & g^{(k)} + \frac{A_{k-1}(z)}{A_k(z)}g^{(k-1)} + \cdots + \frac{A_1(z)}{A_k(z)}g' + \frac{A_0(z)}{A_k(z)}g \\ = \quad & -\left[\varphi^{(k)} + \frac{A_{k-1}(z)}{A_k(z)}\varphi^{(k-1)} + \cdots + \frac{A_1(z)}{A_k(z)}\varphi' + \frac{A_0(z)}{A_k(z)}\varphi\right]. \end{aligned}$$

Since $\varphi(z)$ doesn't solve (1.4), then we have

$$\varphi^{(k)} + \frac{A_{k-1}(z)}{A_k(z)}\varphi^{(k-1)} + \cdots + \frac{A_1(z)}{A_k(z)}\varphi' + \frac{A_0(z)}{A_k(z)}\varphi \not\equiv 0.$$

Then by Lemma 2.6 and $\sigma_{(p,q)}(\varphi) < \infty$, we have

$$\bar{\lambda}_{(p+1,q)}(g) = \lambda_{(p+1,q)}(g) = \sigma_{(p+1,q)}(g),$$

that is

$$\bar{\lambda}_{(p+1,q)}(f-\varphi) = \lambda_{(p+1,q)}(f-\varphi) = \sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_k).$$

□

4. Proof of Theorem 1.7

Proof. Assume that $f(z)$ is a rational solution of (1.5). If either $f(z)$ is a rational function, which has a pole at z_0 of degree $\lambda \geq 1$, or $f(z)$ is a polynomial with $\deg f \geq k$, then $f^{(k)}(z) \not\equiv 0$. Since

$$\max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F) : j = 0, 1, \dots, k-1\} < \sigma_{(p,q)}(A_k) < \infty,$$

then

$$\begin{aligned}\sigma_{(p,q)}(A_k) &= \sigma_{(p,q)}(A_k(z) f^{(k)}) \\ &= \sigma_{(p,q)}(F(z) - (A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f)) \\ &\leq \max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F) : j = 0, 1, \dots, k-1\} < \sigma_{(p,q)}(A_k),\end{aligned}$$

which is a contradiction. Thus, $f(z)$ is a polynomial with $\deg f \leq k-1$.

Now, we assume that $f(z)$ is a transcendental meromorphic solution of (1.5) such that $\lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f)$. Set

$$\beta = \max\{\sigma_{(p,q)}(A_j), \sigma_{(p,q)}(F) : j = 0, 1, \dots, k-1\} < \sigma_{(p,q)}(A_k) = \sigma < \infty.$$

Then, for any given ε $(0 < 2\varepsilon < \sigma - \beta)$, we have

(4.1) $$\left|A_j(z)\right| \leq \exp_{p+1}\{(\beta + \varepsilon)\log_q r\} \ (j = 0, 1, \dots, k-1),$$

(4.2) $$|F(z)| \leq \exp_{p+1}\{(\beta + \varepsilon)\log_q r\}.$$

Since

$$\sigma_{(p,q)}(d) = \lambda_{(p,q)}\left(\frac{1}{f}\right) < \mu_{(p,q)}(f) = \mu_{(p,q)}(g),$$

then for any given ε $(0 < 2\varepsilon < \mu_{(p,q)}(f) - \lambda_{(p,q)}\left(\frac{1}{f}\right))$ and for sufficiently large r we have

$$\frac{1}{\left|f(z)\right|} = \left|\frac{d(z)}{g(z)}\right| \leq \frac{\exp_{p+1}\{(\lambda_{(p,q)}\left(\frac{1}{f}\right) + \varepsilon)\log_q r\}}{\exp_{p+1}\{(\mu_{(p,q)}(f) - \varepsilon)\log_q r\}} \leq 1. \tag{4.3}$$

It follows from (1.5)

$$|A_k(z)| \leq \left|\frac{f}{f^{(k)}}\right| \left[|A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)| + |F(z)| \left|\frac{1}{f}\right|\right]. \tag{4.4}$$

Hence, by substituting (3.1), (3.2), (3.4), (4.1)-(4.3) into (4.4), for sufficiently large z such that

$$|z| = r \in E_8 \backslash ([0,1] \cup E_1 \cup E_2),$$

we obtain

$$\exp_{p+1}\{(\sigma - \varepsilon)\log_q r\} \leq r^{2k} \exp_{p+1}\{(\beta + \varepsilon)\log_q r\}(k+1)B\left(T(2r, f)\right)^{k+1}. \tag{4.5}$$

By Lemma 2.5 and (4.5), we have $\sigma - \varepsilon \leq \sigma_{(p+1,q)}(f)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\sigma_{(p+1,q)}(f) \geq \sigma_{(p,q)}(A_k)$. On the other hand, by (1.5), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq \left|\frac{A_{k-1}(z)}{A_k(z)}\right| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + \left|\frac{A_1(z)}{A_k(z)}\right| \left|\frac{f'}{f}\right| + \left|\frac{A_0(z)}{A_k(z)}\right| + \left|\frac{F(z)}{A_k(z)}\right| \left|\frac{1}{f}\right|. \tag{4.6}$$

By Lemma 2.7, there exists a set $E_6 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0,1] \cup E_6$ and $\left|g(z)\right| = M(r, g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)), \quad j = 0, \ldots, k. \tag{4.7}$$

Since

$$\max\left\{\sigma_{(p,q)}(\frac{A_{k-1}}{A_k}), \ldots, \sigma_{(p,q)}(\frac{A_0}{A_k}), \sigma_{(p,q)}(\frac{F}{A_k})\right\} = \sigma_{(p,q)}(A_k) = \sigma < \infty,$$

then by Lemma 2.8, there exists a set $E_7 \subset (1, +\infty)$ of finite logarithmic measure such that for all $|z| = r \notin E_7$ and for sufficiently large r, we have

$$\left|\frac{A_j(z)}{A_k(z)}\right| \leq \exp_{p+1}\{(\sigma + \varepsilon)\log_q r\} \ (j = 0, \ldots, k-1), \tag{4.8}$$

$$\left|\frac{F(z)}{A_k(z)}\right| \leq \exp_{p+1}\{(\sigma + \varepsilon)\log_q r\}. \tag{4.9}$$

Then, it follows from (4.3), (4.6)-(4.9), for sufficiently large $r \notin [0,1] \cup E_6 \cup E_7$

$$\left(\frac{\nu_g(r)}{r}\right) |1 + o(1)| \leq (k+1) |1 + o(1)| \exp_{p+1}\{(\sigma + \varepsilon)\log_q r\}. \tag{4.10}$$

By (4.10), Lemma 2.5 and Lemma 2.9, we obtain

$$\sigma_{(p+1,q)}(f)=\sigma_{(p+1,q)}(g)\leq\sigma_{(p,q)}(A_k)+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we get $\sigma_{(p+1,q)}(f)\leq\sigma_{(p,q)}(A_k)$. Thus, we have

$$\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_k).$$

Next, we prove that

$$\bar{\lambda}_{(p+1,q)}(f-\varphi)=\lambda_{(p+1,q)}(f-\varphi)=\sigma_{(p+1,q)}(f).$$

Set $g(z)=f(z)-\varphi(z)$. Then $\sigma_{(p+1,q)}(g)=\sigma_{(p+1,q)}(f)$. By substituting $f(z)=g(z)+\varphi(z)$ into

$$f^{(k)}+\frac{A_{k-1}(z)}{A_k(z)}f^{(k-1)}+\cdots+\frac{A_1(z)}{A_k(z)}f'+\frac{A_0(z)}{A_k(z)}f=\frac{F(z)}{A_k(z)},$$

we get

$$\begin{aligned}&g^{(k)}+\frac{A_{k-1}(z)}{A_k(z)}g^{(k-1)}+\cdots+\frac{A_1(z)}{A_k(z)}g'+\frac{A_0(z)}{A_k(z)}g\\&\quad=\frac{F(z)}{A_k(z)}-\left[\varphi^{(k)}+\frac{A_{k-1}(z)}{A_k(z)}\varphi^{(k-1)}+\cdots+\frac{A_1(z)}{A_k(z)}\varphi'+\frac{A_0(z)}{A_k(z)}\varphi\right].\end{aligned}$$

Since $\varphi(z)$ doesn't solve (1.5), then we have

$$\frac{F(z)}{A_k(z)}-\varphi^{(k)}-\frac{A_{k-1}(z)}{A_k(z)}\varphi^{(k-1)}-\cdots-\frac{A_1(z)}{A_k(z)}\varphi'-\frac{A_0(z)}{A_k(z)}\varphi\not\equiv0.$$

Then by Lemma 2.6 and $\sigma_{(p,q)}(\varphi)<\infty$, we have

$$\bar{\lambda}_{(p+1,q)}(g)=\lambda_{(p+1,q)}(g)=\sigma_{(p+1,q)}(g),$$

that is

$$\bar{\lambda}_{(p+1,q)}(f-\varphi)=\lambda_{(p+1,q)}(f-\varphi)=\sigma_{(p+1,q)}(f)=\sigma_{(p,q)}(A_k).$$

□

5. Proof of Theorem 1.8

Proof. (i) We assume that $f(z)$ is a transcendental meromorphic solution of (1.5) and $\{f_1,f_2,\ldots,f_k\}$ is a meromorphic solution base of the corresponding homogeneous equation (1.4) of (1.5). By Theorem 1.6, we get that

$$\sigma_{(p+1,q)}(f_j)=\sigma_{(p,q)}(A_k),\quad (j=1,2,\ldots,k).$$

By the elementary theory of differential equations, all solutions of (1.5) can be represented in the form

$$f(z) = f_0(z) + B_1 f_1(z) + B_2 f_2(z) + \cdots + B_k f_k(z), \tag{5.1}$$

where $B_1, \ldots, B_k \in \mathbb{C}$ and the function f_0 has the form

$$f_0(z) = C_1(z) f_1(z) + C_2(z) f_2(z) + \cdots + C_k(z) f_k(z), \tag{5.2}$$

where $C_1(z), \ldots, C_k(z)$ are suitable meromorphic functions satisfying

$$C_j' = F.G_j(f_1, \ldots, f_k). \left[W(f_1, \ldots, f_k)\right]^{-1}, \ j = 1, 2, \ldots, k, \tag{5.3}$$

where $G_j(f_1, \ldots, f_k)$ are differential polynomials in $f_1, \ldots, f_k$ and their derivatives with constant coefficients, and $W(f_1, \ldots, f_k)$ is the Wronskian of $f_1, \ldots, f_k$. Since the Wronskian $W(f_1, \ldots, f_k)$ is a differential polynomial in $f_1, \ldots, f_k$, it is easy to obtain

$$\sigma_{(p+1,q)}(W) \leq \max\{\sigma_{(p+1,q)}(f_j) : j = 1, 2, \ldots, k\} = \sigma_{(p,q)}(A_k). \tag{5.4}$$

Also, we have that $G_j(f_1, \ldots, f_k)$ are differential polynomials in $f_1, \ldots, f_k$ and their derivatives with constant coefficients, then

$$\sigma_{(p+1,q)}(G_j) \leq \max\{\sigma_{(p+1,q)}(f_j) : j = 1, 2, \ldots, k\} = \sigma_{(p,q)}(A_k), (j = 1, 2, \ldots, k). \tag{5.5}$$

By Lemma 2.10, (5.4) and (5.5) for $j = 1, \ldots, k$, we have from (5.3)

$$\sigma_{(p+1,q)}(C_j) = \sigma_{(p+1,q)}(C_j') \leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_k)\} = \sigma_{(p,q)}(A_k). \tag{5.6}$$

Hence, from (5.1), (5.2) and (5.6), we obtain

$$\sigma_{(p+1,q)}(f) \leq \max\{\sigma_{(p+1,q)}(C_j), \sigma_{(p+1,q)}(f_j) : j = 1, 2, \ldots, k\} = \sigma_{(p,q)}(A_k).$$

Now, we assert that all meromorphic solutions f of equation (1.5) satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_k)$, with at most one exceptional solution f_0 with $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_k)$. In fact, if there exists another meromorphic solution f_1 of (1.5) satisfying

$$\sigma_{(p+1,q)}(f_1) < \sigma_{(p,q)}(A_k),$$

then $f_0 - f_1$ is a nonzero meromorphic solution of (1.4) and satisfies $\sigma_{(p+1,q)}(f_0 - f_1) < \sigma_{(p,q)}(A_k)$. But by Theorem 1.6 we have every nonzero meromorphic solution of (1.4) satisfies $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_k)$. This is a contradiction. Therefore, we have that all meromorphic solutions f of equation (1.5) satisfy $\sigma_{(p+1,q)}(f) = \sigma_{(p,q)}(A_k)$, with at most one exceptional solution f_0 with $\sigma_{(p+1,q)}(f_0) < \sigma_{(p,q)}(A_k)$.

(ii) From (1.5), by a simple consideration of order, we get $\sigma_{(p+1,q)}(f) \geq \sigma_{(p+1,q)}(F)$. By Lemma 2.10 and (5.3)-(5.5), for $j = 1, \ldots, k$, we have

$$\sigma_{(p+1,q)}(C_j) = \sigma_{(p+1,q)}(C_j') \leq \max\{\sigma_{(p+1,q)}(F), \sigma_{(p,q)}(A_k)\} = \sigma_{(p+1,q)}(F). \tag{5.7}$$

By (5.1), (5.2) and (5.7), we have

$$\sigma_{(p+1,q)}(f) \leq \max\{\sigma_{(p+1,q)}(C_j), \sigma_{(p+1,q)}(f_j) : j = 1, 2, \ldots, k\} \leq \sigma_{(p+1,q)}(F).$$

Therefore, we have $\sigma_{(p+1,q)}(f) = \sigma_{(p+1,q)}(F)$. □

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