FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 38, No 3 (2023), 519 - 533
ttps://doi.org/10.22190/FUMI221014034G
Original Scientific Paper

FURTHER BEREZIN NUMBER INEQUALITIES OF OPERATOR MATRICES

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Abstract. In this paper, we have some inequalities for the Berezin number of operator matrices using the convex functions. Also, we obtain some upper bounds for the Berezin number of operator matrices. These results improve some earlier related Berezin number inequalities.
Keywords: Berezin number, Berezin norm, reproducing kernel Hilbert space, operator matrices.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated operator norm $\|\cdot\|$ and let $\mathcal{B}(\mathcal{H})$ denote the C^*- algebra of all bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $A \geq 0$. The absolute value of A is denoted by $|A|$, that is $|A| = (A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint of A.

For $A \in \mathcal{B}(\mathcal{H})$, the numerical range of A is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

while the numerical radius is defined as

$$\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

Received October 14, 2022. accepted March 23, 2023.
Communicated by Dijana Mosić
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2010 *Mathematics Subject Classification.* 47A30; 15A60; 47A12

It is well known that $\omega(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$ which is equivalent to the usual operator norm $\|\cdot\|$ defined for $A \in \mathcal{B}(\mathcal{H})$ by

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\}.$$

Further, for all $A \in \mathcal{B}(\mathcal{H})$, we have [15]

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|.$$

More recently, results concerning numerical radius can be found in [8, 14, 21, 24].

We recall that for $A \in \mathcal{B}(\mathcal{H})$, the spectral radius A is defined as

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A.

It is well known that for any operator $A \in \mathcal{B}(\mathcal{H})$, we have

$$r(A) \leq \omega(A) \leq \|A\|.$$

Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensues that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see, e.g., [16, 26]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. Let A be a bounded linear operator on $\mathcal{H}$, the Berezin symbol of A, which was firstly introduced by Berezin [9] is the function $\tilde{A}$ on Ω defined by

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

The Berezin set and the Berezin number of the operator A are defined respectively by

$$\mathbf{Ber}(A) := \left\{ \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\},$$

and

$$\mathbf{ber}(A) := \sup\left\{ \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on Ω whose value lies in the numerical range of the operator A and hence for any $A \in \mathcal{B}(\mathcal{H})$,

$$\mathbf{Ber}(A) \subset W(A) \text{ and } \mathbf{ber}(A) \leq \omega(A).$$

Thus the Berezin number is strongly connected with the numerical radius and the operator norm.

The Berezin symbol of an operator provides important information about it. Especially, the Berezin symbol has been studied in details for Toeplitz and Hankel operators on Hardy and Bergman spaces. For example, it is well known that the Berezin symbol uniquely determines the operator, i.e., $T_1 = T_2$ if and only if $\widetilde{T_1} = \widetilde{T_2}$ (see, Zhu [32].). The concept of the Berezin symbol of an operator arose in connection with quantum mechanics and non-commutative geometry (see, for instance, [9]). On the other hand, the method of reproducing kernels is actively developed by Saitoh & Castro and their collaborators, in order to solve various problems of applied mathematics (see, [12, 13], and the references therein).

Since the collection of normalized reproducing kernel of $\mathcal{H}$ is a subset of the unit sphere of $\mathcal{H}$, the numerical radius and the Berezin number of an operator on $\mathcal{H}$ may not be equal. For example, Karaev [19] showed that for $T = S \otimes S \in \mathcal{B}\left(H^2\left(\mathbb{D}\right)\right)$, where S is the shift operator defined by $Sf\left(z\right) = zf\left(z\right)$ on the Hardy-Hilbert space $H^2\left(\mathbb{D}\right)$, $\widetilde{T}\left(\lambda\right) = \left|\lambda\right|^2\left(1 - \left|\lambda\right|^2\right)$ and $Ber\left(T\right) = \left\{\left|\lambda\right|^2\left(1 - \left|\lambda\right|^2\right) : \lambda \in \mathbb{D}\right\} = [0, 1/4] \subsetneq [0, 1] = W\left(T\right)$. So, $ber\left(T\right) = 1/4 < 1 = w\left(T\right)$.

Other immediate properties of the the Berezin number of an operator A can be stated as follows [19]:

(i) $\mathbf{ber}\left(A\right) \leq \|A\|$.

(ii) $\mathbf{ber}\left(\alpha A\right) = |\alpha|\,\mathbf{ber}\left(A\right)$ for all $\alpha \in \mathbb{C}$.

(iii) $\mathbf{ber}\left(A + B\right) \leq \mathbf{ber}\left(A\right) + \mathbf{ber}\left(B\right)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

At this point, we should remark that, in general, the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z < 1|\}$), then $\mathbf{ber}\left(\cdot\right)$ defines a norm on $\mathcal{B}(\mathcal{H}\left(D\right))$ (see [18, 19]). We refer the reader to [4, 5, 10, 16, 30, 31] as a sample where the reader can find some progress on the study of Berezin symbol and Berezin number.

The Berezin norm of A is defined as

$$\|A\|_{ber} := \sup\left\{\left|\left\langle A\hat{k}_\lambda, \hat{k}_\mu\right\rangle\right| : \lambda, \mu \in \Omega\right\},$$

where $\hat{k}_\lambda, \hat{k}_\mu$ are normalized reproducing kernels for λ, μ, respectively. It is clear from definition, Berezin norm satisfies the following properties:

(i) $\mathbf{ber}\left(A\right) \leq \|A\|_{ber}$,

(ii) $\|A\|_{ber} \leq \|A\|$,

(iii) $\|A^*\|_{ber} = \|A\|_{ber}$.

The direct sum of two copies of $\mathcal{H}$ is denoted by $\mathcal{H}^2 := \mathcal{H} \oplus \mathcal{H}$. If $A, B, C, D \in \mathcal{B}\left(\mathcal{H}\right)$, then the operator matrix $T := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ can be considered as an operator in $\mathcal{B}\left(\mathcal{H} \oplus \mathcal{H}\right)$, which is defined by $Tx = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix}$ for every vector $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$.

Recently, inspired by the numerical radius inequalities for operator matrices in [1, 2, 6, 8, 17, 28], some inequalities for the Berezin number of operator matrices have been presented in [3, 7, 27]. For example, the authors in [16] proved that for $r \geq 1$

$$\mathbf{ber}^r\left(\left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right]\right) \leq \frac{1}{2}\max\left\{\mathbf{ber}\left(|A|^r + |A^*|^r\right), \mathbf{ber}\left(|D|^r + |D^*|^r\right)\right\}.$$

They also obtained that

$$\begin{aligned} \mathbf{ber}\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) &\leq \tfrac{1}{2}\max\left\{\mathbf{ber}\left(|C| + |B^*|\right), \mathbf{ber}\left(|B| + |C^*|\right)\right\} + \\ &+ \tfrac{1}{2}\max\left\{\mathbf{ber}\left(|A| + |A^*|\right), \mathbf{ber}\left(|D| + |D^*|\right)\right\} \end{aligned}$$

In [3], the authors proved that

$$\mathbf{ber}\left(\left[\begin{array}{cc} 0 & B \\ C & 0 \end{array}\right]\right)^r \leq \frac{2^r}{2}\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}\left(|B|\right) + g^{2r}\left(|C^*|\right)\right)\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}\left(|C|\right) + g^{2r}\left(|B^*|\right)\right)$$

and

$$\mathbf{ber}\left(\left[\begin{array}{cc} 0 & B \\ C & 0 \end{array}\right]\right)^r \leq \frac{2^r}{2}\mathbf{ber}^{\frac{1}{2}}\left(f^{2r}\left(|B|\right) + f^{2r}\left(|C^*|\right)\right)\mathbf{ber}^{\frac{1}{2}}\left(g^{2r}\left(|C|\right) + g^{2r}\left(|B^*|\right)\right)$$

for all $r \geq 1$ and f, g are nonnegative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, \infty))$.

Our aim in this paper is to present some inequalities for the Berezin number of operator matrices using the convex functions. Also, we obtain some upper bounds for the Berezin number of operator matrices. These results improve the earlier results in [3, 7, 16].

In order to prove our results, we need the following sequence of lemmas.

Lemma 1.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*
(i) $\mathbf{ber}\left(\left[\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right]\right) \leq \max\left\{\mathbf{ber}(A), \mathbf{ber}(B)\right\}$.
(ii) $\mathbf{ber}\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) \leq \dfrac{\|A\| + \|B\|}{2}$.
(iii) In particular, $\mathbf{ber}\left(\left[\begin{array}{cc} 0 & A \\ A & 0 \end{array}\right]\right) \leq \|A\|$ *([6]).*

Lemma 1.2. *Let $A \in B(\mathcal{H})$ and $U \in B(\mathcal{H})$ be a unitary operator. Then*

$$\mathbf{ber}\left(U^*AU\right) = \mathbf{ber}(A).$$

Proof. Follows immediately by using definition of the Berezin number of A. □

Lemma 1.3. *([1]) Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\omega\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) = \frac{1}{2}\|A + B\|.$$

Lemma 1.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

(i) If A is self-adjoint, then $\omega(A) = r(A)$.
(ii) $r(AB) = r(BA)$ ([22]).

Lemma 1.5. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}(A) = \sup_{\theta \in \mathbb{R}} \mathbf{ber}\left(\operatorname{Re}\left(e^{i\theta}A\right)\right),$$

where $\operatorname{Re}(A) = \dfrac{A + A^*}{2}$ *([5]).*

Lemma 1.6. *([10]) Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then*

$$\|A\|_{ber} = \mathbf{ber}(A).$$

Lemma 1.7. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$|\langle Ax, y\rangle|^2 \leq |\langle |A| x, x\rangle| \, |\langle |A^*| y, y\rangle|,$$

for all $x, y \in \mathcal{H}$ ([20]).

Lemma 1.8. *Let $A \in B(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t) g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$|\langle Ax, y\rangle|^2 \leq \left\langle f^2(|A|) x, x\right\rangle \left\langle g^2(|A^*|) y, y\right\rangle,$$

for all $x, y \in \mathcal{H}$ ([20]).

Lemma 1.9. *If f is a convex function on a real interval J containing the spectrum of the self-adjoint operator A, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A) x, x\rangle$$

for any unit vector $x \in \mathcal{H}$ ([23]).

Lemma 1.10. *([25]) Let f be a twice differentiable convex function such that $\alpha \leq f''$ and $\alpha \in \mathbb{R}$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2} - \frac{1}{8}\alpha (a-b)^2.$$

Lemma 1.11. *([11]) Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. Then*

$$|\langle x, z\rangle \langle z, y\rangle| \leq \frac{1}{2}\left(\|x\| \|y\| + |\langle x, y\rangle|\right).$$

Lemma 1.12. *([8]) Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. Then*

$$|\langle x, z\rangle \langle z, y\rangle|^2 \leq \frac{1}{4}\left(3\|x\|^2 \|y\|^2 + \|x\| \|y\| |\langle x, y\rangle|\right).$$

2. Main results

In this section we present our results. We begin with the following theorem.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let f be a twice differentiable nonnegative non-decreasing convex function on $[0,\infty)$ such that $\alpha \le f^{''}$ and $\alpha \in \mathbb{R}$. Then*

$$f\left(\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right)$$

$$\le \tfrac{1}{2}\max\left\{\left\|f(|A|)+f(|A^*|)\right\|_{ber}, \left\|(f(|B|))+f((|B^*|))\right\|_{ber}\right\} - \inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right),$$

where

$$\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right) = \frac{1}{8}\alpha\left\langle\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2.$$

Proof. Put $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. For any $(\lambda_1,\lambda_2) \in \Omega\times\Omega$, let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathcal{H}\oplus\mathcal{H}$. Then, in view of Lemma 1.7 and using the Lemma 1.10, Lemma 1.9, we have

$$\begin{aligned}
f\left(\left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right) &\le f\left(\left\langle |T|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\left\langle |T^*|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^{\frac{1}{2}}\right) \\
&\le f\left(\frac{\left\langle |T|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle |T^*|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle}{2}\right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&\le \frac{f\left(\left\langle |T|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right) + f\left(\left\langle |T^*|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)}{2} \\
&\quad -\frac{1}{8}\alpha\left(\left\langle |T|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle - \left\langle |T^*|\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)^2 \\
&\le \frac{1}{2}\left\langle (f(|T|)+f(|T^*|))\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad -\frac{1}{8}\alpha\left\langle (|T|-|T^*|)\,\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2 \\
&= \frac{1}{2}\left\langle\left(f\left(\begin{bmatrix} |A| & 0 \\ 0 & |B| \end{bmatrix}\right) + f\left(\begin{bmatrix} |A^*| & 0 \\ 0 & |B^*| \end{bmatrix}\right)\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad -\frac{1}{8}\alpha\left\langle\left(\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2 \\
&= \frac{1}{2}\left\langle\begin{bmatrix} f(|A|)+f(|A^*|) & 0 \\ 0 & f(|B|)+f(|B^*|) \end{bmatrix}\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
&\quad -\frac{1}{8}\alpha\left\langle\left(\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2.
\end{aligned}$$

Taking the supremum over all $\hat{k}_{(\lambda_1,\lambda_2)} \in \mathcal{H} \oplus \mathcal{H}$ with $\left\|\hat{k}_{(\lambda_1,\lambda_2)}\right\| = 1$ and using the Lemma 1.1(i), Lemma 1.6, we obtain

$$\begin{aligned} f(\mathbf{ber}(T)) &\leq \frac{1}{2}\mathbf{ber}\left(\begin{bmatrix} f(|A|)+f(|A^*|) & 0 \\ 0 & f(|B|)+f(|B^*|) \end{bmatrix}\right) \\ &\quad -\frac{1}{8}\alpha\left\langle\left(\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2 \\ &\leq \frac{1}{2}\max\left\{\mathbf{ber}(f(|A|)+f(|A^*|)), \mathbf{ber}(f(|B|)+f(|B^*|))\right\} \\ &\quad -\frac{1}{8}\alpha \inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}\left(\left\langle\left(\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2\right) \\ &= \frac{1}{2}\max\left\{\|f(|A|)+f(|A^*|)\|_{ber}, \|(f(|B|))+f((|B^*|))\|_{ber}\right\} \\ &\quad - \inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right), \end{aligned}$$

where

$$\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right) = \frac{1}{8}\alpha\left\langle\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2.$$

□

By considering $A = B$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ and let f be a twice differentiable nonnegative non-decreasing convex function on $[0,\infty)$ such that $\alpha \leq f^{''}$ and $\alpha \in \mathbb{R}$. Then*

$$f\left(\mathbf{ber}\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\right)\right) \leq \frac{1}{2}\|f(|A|)+f(|A^*|)\|_{ber} - \inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right),$$

where

$$\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right) = \frac{1}{8}\alpha\left\langle(|A|-|A^*|)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2.$$

Remark 2.1. It is clear that $\inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right)$ is positive. Indeed, $\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}$ is self-adjoint. Thus,

$$\left\langle\left(\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \in \mathbb{R}.$$

So,

$$\left\langle\left(\begin{bmatrix} |A|-|A^*| & 0 \\ 0 & |B|-|B^*| \end{bmatrix}\right)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \geq 0.$$

Using similar argument as used in Theorem 2.1, we have the following result.

Theorem 2.2. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and let f be a twice differentiable nonnegative non-decreasing convex function on $[0,\infty)$ such that $\alpha \leq f^{''}$ and $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned} & f\left(\mathbf{ber}\left(\left[\begin{array}{cc} 0 & B \\ C & 0 \end{array}\right]\right)\right) \\ \leq\ & \frac{1}{2}\max\left\{\left\|f(|B|)+f(|C^*|)\right\|_{ber}, \left\|(f(|C|))+f((|B^*|))\right\|_{ber}\right\} - \inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right), \end{aligned}$$

where

$$\gamma\left(\hat{k}_{(\lambda_1,\lambda_2)}\right) = \frac{1}{8}\alpha\left\langle\left[\begin{array}{cc} |C|-|B^*| & 0 \\ 0 & |B|-|C^*| \end{array}\right]\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2.$$

The following corollary follows from Theorem 2.1 and Theorem 2.2.

Corollary 2.2. *Let $B, C \in \mathcal{B}(\mathcal{H})$ and let f be a twice differentiable nonnegative non-decreasing convex function on $[0,\infty)$ such that $\alpha \leq f^{''}$ and $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned} f\left(\frac{1}{2}\mathbf{ber}\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right)\right) \leq\ & \frac{1}{4}\max\left\{\left\|f(|A|)+f(|A^*|)\right\|_{ber}, \left\|(f(|D|))+f((|D^*|))\right\|_{ber}\right\} \\ & +\frac{1}{4}\max\left\{\left\|f(|B|)+f(|C^*|)\right\|_{ber}, \left\|(f(|C|))+f((|B^*|))\right\|_{ber}\right\} \\ & -\inf_{(\lambda_1,\lambda_2)\in\Omega\times\Omega}(\gamma_1+\gamma_2)\left(\hat{k}_{(\lambda_1,\lambda_2)}\right), \end{aligned}$$

where

$$\gamma_1\left(\hat{k}_{(\lambda_1,\lambda_2)}\right) = \frac{1}{16}\alpha\left\langle\left[\begin{array}{cc} |A|-|A^*| & 0 \\ 0 & |D|-|D^*| \end{array}\right]\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2,$$

and

$$\gamma_2\left(\hat{k}_{(\lambda_1,\lambda_2)}\right) = \frac{1}{16}\alpha\left\langle\left[\begin{array}{cc} |C|-|B^*| & 0 \\ 0 & |B|-|C^*| \end{array}\right]\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2.$$

Proof. By using the triangle inequality for $\mathbf{ber}(\cdot)$ and by the convexity of f, we observe that

$$\begin{aligned} f\left(\frac{1}{2}\mathbf{ber}\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right)\right) &= f\left(\frac{1}{2}\mathbf{ber}\left(\left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right]+\left[\begin{array}{cc} 0 & B \\ C & 0 \end{array}\right]\right)\right) \\ &\leq f\left(\frac{1}{2}\mathbf{ber}\left(\left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right]\right)+\frac{1}{2}\mathbf{ber}\left(\left[\begin{array}{cc} 0 & B \\ C & 0 \end{array}\right]\right)\right) \\ &\leq \frac{1}{2}f\left(\mathbf{ber}\left(\left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right]\right)\right)+\frac{1}{2}f\left(\mathbf{ber}\left(\left[\begin{array}{cc} 0 & B \\ C & 0 \end{array}\right]\right)\right). \end{aligned}$$

Now, using Theorem 2.1 and Theorem 2.2, we get the desired inequality. □

In the following we obtain an upper bound for the Berezin number of operator matrix $\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ which is similiar to the relation between the numerical radius and the operator norm of the same operator matrix, as in [29, Theorem 3.2].

Theorem 2.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2}\omega(A) + \frac{1}{4}\|I + AA^* + BB^*\|.$$

Proof. Let $T = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$. Using the Lemma 1.4 (i), we have

$$\begin{aligned} 2\mathbf{ber}\left(\operatorname{Re}\left(e^{i\theta}T\right)\right) &= \mathbf{ber}\left(e^{i\theta}T + e^{-i\theta}T^*\right) \\ &= \mathbf{ber}\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\ &\leq \omega\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{i\theta}A + e^{-i\theta}A^* & e^{i\theta}B \\ e^{-i\theta}B^* & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} A^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 \\ A & B \end{bmatrix}\right). \end{aligned}$$

Using the commutative property of the spectral radius and Lemma 1.3, we get

$$\begin{aligned} 2\mathbf{ber}\left(\operatorname{Re}\left(e^{i\theta}T\right)\right) &\leq r\left(\begin{bmatrix} A^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 \\ A & B \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{-i\theta}I & 0 \\ A & B \end{bmatrix}\begin{bmatrix} A^* & e^{i\theta}I \\ B^* & 0 \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{-i\theta}A^* & I \\ AA^* + BB^* & e^{i\theta}A \end{bmatrix}\right) \\ &= r\left(\begin{bmatrix} e^{-i\theta}A^* & 0 \\ 0 & e^{i\theta}A \end{bmatrix} + \begin{bmatrix} 0 & I \\ AA^* + BB^* & 0 \end{bmatrix}\right) \\ &\leq r\left(\begin{bmatrix} e^{-i\theta}A^* & 0 \\ 0 & e^{i\theta}A \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & I \\ AA^* + BB^* & 0 \end{bmatrix}\right) \\ &\leq \omega\left(\begin{bmatrix} e^{-i\theta}A^* & 0 \\ 0 & e^{i\theta}A \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & I \\ AA^* + BB^* & 0 \end{bmatrix}\right) \\ &\quad (\text{since } r(X) \leq \omega(X), \text{ for any } X \in \mathcal{B}(\mathcal{H})) \\ &\leq \omega(A) + \frac{1}{2}\|I + AA^* + BB^*\|. \end{aligned}$$

Therefore, we have

$$\mathbf{ber}\left(\operatorname{Re}\left(e^{i\theta}T\right)\right) \leq \frac{1}{2}\omega(A) + \frac{1}{4}\|I + AA^* + BB^*\|.$$

Taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality, and noting Lemma 1.5, we get desired result. $\square$

As applications of Theorem 2.3, we give the following result.

Corollary 2.3. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{1}{2}\left(\omega(A)+\omega(D)\right)+\frac{1}{4}\left(\|I+AA^*+BB^*\|+\|I+CC^*+DD^*\|\right).$$

Proof. Let $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ be a unitary operator on $\mathcal{H} \oplus \mathcal{H}$. By the triangle inequality for $\mathbf{ber}(\cdot)$ and Lemma 1.2, we observe that

$$\begin{aligned}
\mathbf{ber}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &= \mathbf{ber}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right) \\
&\leq \mathbf{ber}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right) \\
&= \mathbf{ber}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + \mathbf{ber}\left(U^*\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}U\right) \\
&= \mathbf{ber}\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) + \mathbf{ber}\left(\begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix}\right) \\
&\leq \frac{1}{2}\omega(A) + \frac{1}{2}\omega(D) + \frac{1}{4}\|I+AA^*+BB^*\| + \frac{1}{4}\|I+CC^*+DD^*\|.
\end{aligned}$$

This completes the proof. $\square$

Theorem 2.4. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\mathbf{ber}^4\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq\; & 8\max\left\{\mathbf{ber}^4(A), \mathbf{ber}^4(D)\right\} \\
& +3\max\left\{\left\||C|^4+|B^*|^4\right\|_{ber}, \left\||B|^4+|C^*|^4\right\|_{ber}\right\} \\
& +\max\left\{\mathbf{ber}(BC), \mathbf{ber}(CB)\right\}\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber}, \left\||B|^2+|C^*|^2\right\|_{ber}\right\}.
\end{aligned}$$

Proof. Put $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $P = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Then, $T = P + M$. Let $\hat{k}_{(\lambda_1,\lambda_2)} = \begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathcal{H} \oplus \mathcal{H}$. Then, using the Lemma 1.12, Lemma 1.11, Lemma 1.1 (i) and Lemma 1.6, respectively, we have

$$
\begin{aligned}
& \left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 \\
= \; & \left|\left\langle (P+M)\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 \\
= \; & \left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle M\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 \\
\leq \; & \left(\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| + \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|\right)^4 \\
= \; & \left(\frac{2\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| + 2\left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|}{2}\right)^4 \\
\leq \; & 8\left(\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 + \left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4\right) \text{ (by the convexity of } f(t)=t^4) \\
\leq \; & 8\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 + 2\left(3\left\|M\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\left\|M^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|^2\right) \\
& +2\left\|M\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left\|M^*\hat{k}_{(\lambda_1,\lambda_2)}\right\|\left|\left\langle M\hat{k}_{(\lambda_1,\lambda_2)},M^*\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq \; & 8\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 + 6\left\langle |M|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left\langle |M^*|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
& +2\sqrt{\left\langle |M|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left\langle |M^*|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle}\left|\left\langle M^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right| \\
\leq \; & 8\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 + 3\left(\left\langle |M|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2 + \left\langle |M^*|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle^2\right) \\
& +\left(\left\langle |M|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle + \left\langle |M^*|^2\,\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right)\left\langle M^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
& \text{(by the arithmetic-geometric mean inequality)} \\
\leq \; & 8\left|\left\langle P\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4 + 3\left\langle\left(|M|^4+|M^*|^4\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
& +\left\langle\left(|M|^2+|M^*|^2\right)\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\left\langle M^2\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle \\
\leq \; & 8\mathbf{ber}^4(P) + 3\mathbf{ber}\left(|M|^4+|M^*|^4\right) + \mathbf{ber}\left(|M|^2+|M^*|^2\right)\mathbf{ber}\left(M^2\right) \\
= \; & 8\mathbf{ber}^4\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + 3\mathbf{ber}\left(\begin{bmatrix} |C|^4+|B^*|^4 & 0 \\ 0 & |B|^4+|C^*|^4 \end{bmatrix}\right) \\
& +\mathbf{ber}\left(\begin{bmatrix} |C|^2+|B^*|^2 & 0 \\ 0 & |B|^2+|C^*|^2 \end{bmatrix}\right)\mathbf{ber}\left(\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right) \\
\leq \; & 8\max\left\{\mathbf{ber}^4(A),\mathbf{ber}^4(D)\right\} + 3\max\left\{\mathbf{ber}\left(|C|^4+|B^*|^4\right),\mathbf{ber}\left(|B|^4+|C^*|^4\right)\right\} \\
& +\max\left\{\mathbf{ber}\left(|C|^2+|B^*|^2\right),\mathbf{ber}\left(|B|^2+|C^*|^2\right)\right\}\max\left\{\mathbf{ber}(BC),\mathbf{ber}(CB)\right\} \\
= \; & 8\max\left\{\mathbf{ber}^4(A),\mathbf{ber}^4(D)\right\} + 3\max\left\{\left\||C|^4+|B^*|^4\right\|_{ber},\left\||B|^4+|C^*|^4\right\|_{ber}\right\} \\
& +\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber},\left\||B|^2+|C^*|^2\right\|_{ber}\right\}\max\left\{\mathbf{ber}(BC),\mathbf{ber}(CB)\right\}
\end{aligned}
$$

Thus

$$\begin{aligned}&\left|\left\langle\begin{bmatrix} A & B \\ C & D \end{bmatrix}\hat{k}_{(\lambda_1,\lambda_2)},\hat{k}_{(\lambda_1,\lambda_2)}\right\rangle\right|^4\\ \leq\ & 8\max\left\{\mathbf{ber}^4(A),\mathbf{ber}^4(D)\right\}+3\max\left\{\left\||C|^4+|B^*|^4\right\|_{ber},\left\||B|^4+|C^*|^4\right\|_{ber}\right\}\\ &+\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber},\left\||B|^2+|C^*|^2\right\|_{ber}\right\}\max\left\{\mathbf{ber}(BC),\mathbf{ber}(CB)\right\}.\end{aligned}$$

Taking the supremum over all $\hat{k}_{(\lambda_1,\lambda_2)}\in\mathcal{H}\oplus\mathcal{H}$ with $\left\|\hat{k}_{(\lambda_1,\lambda_2)}\right\|=1$ in the above inequality, we get the desired result. □

As an immediate consequence of Theorem 2.4, we have the following.

Corollary 2.4. *Let $A,B\in B(\mathcal{H})$. Then*

$$\mathbf{ber}^4\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right)\leq 8\mathbf{ber}^4(A)+3\left\||B|^4+|B^*|^4\right\|_{ber}+3\left\||B|^2+|B^*|^2\right\|_{ber}\mathbf{ber}\left(B^2\right).$$

Proof. The result follows from Theorem 2.4, by putting $A=D$ and $B=C$. □

If $A=D=0$ inTheorem 2.4, then we get the following inequality.

Corollary 2.5. *Let $B,C\in\mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\mathbf{ber}^4\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\ \leq\ & 3\max\left\{\left\||C|^4+|B^*|^4\right\|_{ber},\left\||B|^4+|C^*|^4\right\|_{ber}\right\}\\ &+\max\left\{\mathbf{ber}(BC),\mathbf{ber}(CB)\right\}\max\left\{\left\||C|^2+|B^*|^2\right\|_{ber},\left\||B|^2+|C^*|^2\right\|_{ber}\right\}.\end{aligned}$$

At the end of this paper, we get an upper bound for the Berezin norm of a 3×3 operator matrices $\begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix}$ defined on $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$.

Theorem 2.5. *Let $T=\begin{bmatrix} 0 & A & 0 \\ B & 0 & C \\ 0 & D & 0 \end{bmatrix}$, where $A,B,C,D\in B(\mathcal{H})$. Then*

$$\mathbf{ber}(T)\leq\frac{1}{2}\left(\sqrt{\||A^*|+|B|\|_{ber}\,\||A|+|B^*|\|_{ber}}+\sqrt{\||C^*|+|D|\|_{ber}\,\||C|+|D^*|\|_{ber}}\right).$$

Proof. For any $(\lambda_1,\lambda_2,\lambda_3)\in\Omega\times\Omega\times\Omega$, let $\hat{k}_{(\lambda_1,\lambda_2,\lambda_2)}=\begin{bmatrix} k_{\lambda_1} \\ k_{\lambda_2} \\ k_{\lambda_2} \end{bmatrix}$ be the normalized reproducing kernel in $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$ (i.e., $\|k_{\lambda_1}\|^2+\|k_{\lambda_2}\|^2+\|k_{\lambda_3}\|^2=1$). Using the

Lemma 1.7 and Lemma 1.6, we have

$$
\begin{aligned}
& \left|\left\langle T\hat{k}_{(\lambda_1,\lambda_2,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2,\lambda_2)}\right\rangle\right| \\
\leq\ & |\langle Ak_{\lambda_2}, k_{\lambda_1}\rangle| + |\langle Bk_{\lambda_1}, k_{\lambda_2}\rangle| + |\langle Ck_{\lambda_3}, k_{\lambda_2}\rangle| + |\langle Dk_{\lambda_2}, k_{\lambda_3}\rangle| \\
\leq\ & \langle |A|\, k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle |A^*|\, k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} + \langle |B|\, k_{\lambda_1}, k_{\lambda_1}\rangle^{\frac{1}{2}} \langle |B^*|\, k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \\
& + \langle |C|\, k_{\lambda_3}, k_{\lambda_3}\rangle^{\frac{1}{2}} \langle |C^*|\, k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} + \langle |D|\, k_{\lambda_2}, k_{\lambda_2}\rangle^{\frac{1}{2}} \langle |D^*|\, k_{\lambda_3}, k_{\lambda_3}\rangle^{\frac{1}{2}} \\
\leq\ & \sqrt{\langle |A^*|\, k_{\lambda_1}, k_{\lambda_1}\rangle + \langle |B|\, k_{\lambda_1}, k_{\lambda_1}\rangle}\sqrt{\langle |A|\, k_{\lambda_2}, k_{\lambda_2}\rangle \langle |B^*|\, k_{\lambda_2}, k_{\lambda_2}\rangle} \\
& + \sqrt{\langle |C^*|\, k_{\lambda_2}, k_{\lambda_2}\rangle + \langle |D|\, k_{\lambda_2}, k_{\lambda_2}\rangle}\sqrt{\langle |C|\, k_{\lambda_3}, k_{\lambda_3}\rangle \langle |D^*|\, k_{\lambda_3}, k_{\lambda_3}\rangle} \\
& \left(\text{using the inequality } ab + cd \leq \sqrt{a^2+c^2}\sqrt{b^2+d^2},\ a,b,c,d \in \mathbb{R}\right) \\
=\ & \sqrt{\langle (|A^*| + |B|)\, k_{\lambda_1}, k_{\lambda_1}\rangle}\sqrt{\langle (|A| + |B^*|)\, k_{\lambda_2}, k_{\lambda_2}\rangle} \\
& + \sqrt{\langle (|C^*| + |D|)\, k_{\lambda_2}, k_{\lambda_2}\rangle}\sqrt{\langle (|C| + |D^*|)\, k_{\lambda_3}, k_{\lambda_3}\rangle} \\
=\ & \sqrt{\|k_{\lambda_1}\|^2 \left\langle (|A^*| + |B|)\, \frac{k_{\lambda_1}}{\|k_{\lambda_1}\|}, \frac{k_{\lambda_1}}{\|k_{\lambda_1}\|}\right\rangle}\sqrt{\|k_{\lambda_2}\|^2 \left\langle (|A| + |B^*|)\, \frac{k_{\lambda_2}}{\|k_{\lambda_2}\|}, \frac{k_{\lambda_2}}{\|k_{\lambda_2}\|}\right\rangle} \\
& + \sqrt{\|k_{\lambda_2}\|^2 \left\langle (|C^*| + |D|)\, \frac{k_{\lambda_2}}{\|k_{\lambda_2}\|}, \frac{k_{\lambda_2}}{\|k_{\lambda_2}\|}\right\rangle}\sqrt{\|k_{\lambda_3}\|^2 \left\langle (|C| + |D^*|)\, \frac{k_{\lambda_3}}{\|k_{\lambda_3}\|}, \frac{k_{\lambda_3}}{\|k_{\lambda_3}\|}\right\rangle} \\
=\ & \sqrt{\left\langle (|A^*| + |B|)\, \hat{k}_{\lambda_1}, \hat{k}_{\lambda_1}\right\rangle}\sqrt{\left\langle (|A| + |B^*|)\, \hat{k}_{\lambda_2}, \hat{k}_{\lambda_2}\right\rangle}\, \|k_{\lambda_1}\|\, \|k_{\lambda_2}\| \\
& + \sqrt{\left\langle (|C^*| + |D|)\, \hat{k}_{\lambda_2}, \hat{k}_{\lambda_2}\right\rangle}\sqrt{\left\langle (|C| + |D^*|)\, \hat{k}_{\lambda_3}, \hat{k}_{\lambda_3}\right\rangle}\, \|k_{\lambda_2}\|\, \|k_{\lambda_3}\| \\
\leq\ & \sqrt{\left\langle (|A^*| + |B|)\, \hat{k}_{\lambda_1}, \hat{k}_{\lambda_1}\right\rangle}\sqrt{\left\langle (|A| + |B^*|)\, \hat{k}_{\lambda_2}, \hat{k}_{\lambda_2}\right\rangle}\frac{\|k_{\lambda_1}\|^2 + \|k_{\lambda_2}\|^2}{2} \\
& + \sqrt{\left\langle (|C^*| + |D|)\, \hat{k}_{\lambda_2}, \hat{k}_{\lambda_2}\right\rangle}\sqrt{\left\langle (|C| + |D^*|)\, \hat{k}_{\lambda_3}, \hat{k}_{\lambda_3}\right\rangle}\frac{\|k_{\lambda_2}\|^2 + \|k_{\lambda_3}\|^2}{2} \\
& \text{(by the arithmetic-geometric mean inequality)} \\
\leq\ & \frac{1}{2}\mathbf{ber}^{\frac{1}{2}}\left(|A^*| + |B|\right)\mathbf{ber}^{\frac{1}{2}}\left(|A| + |B^*|\right) \\
& + \frac{1}{2}\mathbf{ber}^{\frac{1}{2}}\left(|C^*| + |D|\right)\mathbf{ber}^{\frac{1}{2}}\left(|C| + |D^*|\right) \\
=\ & \frac{1}{2}\, \||A^*| + |B|\|_{ber}^{\frac{1}{2}}\, \||A| + |B^*|\|_{ber}^{\frac{1}{2}} \\
& + \frac{1}{2}\, \||C^*| + |D|\|_{ber}^{\frac{1}{2}}\, \||C| + |D^*|\|_{ber}^{\frac{1}{2}}.
\end{aligned}
$$

□

Acknowledgement. The authors are grateful to the referees for their constructive remarks and suggestions.

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