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2-RULED HYPERSURFACES IN A WALKER 4-MANIFOLD

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Abstract. The hypersurface is one of the most important objects in a space. Many authors studied diffrent geometric aspects of hypersurfaces in a space. In this paper, we define three types of 2-ruled hypersurfaces in a Walker 4-manifold. We obtain the Gaussian and mean curvatures of the 2-ruled hypersurfaces of type-1, type-2 and type-3. We give some characterizations about its minimality. We also deal with the first Laplace-Beltrami operators of these types of 2-ruled hypersurfaces in the considered Walker 4-manifold.
Keywords: 2-ruled hypersurface, Walker manifolds.

1. Introduction

The study of hypersurface of a given ambiant space M is an interesting problem which enriches our knowledge and understanding of the geometry of the space itself. The theory of ruled surfaces in $\mathbb{R}^3$ is a classical subject in diffrential geometry. The study of ruled surfaces of a given ambiant space M is also a natural and interesting problem. A surface Σ in M is said to be ruled if every point of Σ is on (a open geodesic segment) in M that lies in Σ (see [21]). Locally a ruled surface is made by a one parameter family of geodesic segments [7]. Ruled surfaces are one parameter set

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of lines and they are one of the important topics of classifical differential geometry. A ruled surface is defined as

$$\varphi(s,t) = \alpha(s) + tX(s), s, t \in I \subset \mathbb{R},$$

where the curve $\alpha(s)$ is called base curve and $X(s)$ is called the ruling of the ruled surface. A lots of studies have been done about different characterizations of ruled surfaces in 3-dimensional Euclidean, Minkowskian, Galilean and pseudo-Galilean space (see [9, 10, 12, 13, 14, 18] and references therein). In [2], the authors define a quaternionic operator whose scalar part is a real parameter and vector part is a curve in three dimensional real vector space $\mathbb{R}^3$. They prove that quaternion product of this operator and a spherical curve represent a ruled surface in $\mathbb{R}^3$ if the vector part of the quaternionic operator is perpendicular to the position vector of the spherical curve. Also in [3], the authors show that the split quaternion product of a split quaternion operator and a curve, which lies on Lorentzian unit sphere or on hyperbolic unit sphere, parametrizes a ruled surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ if the vector part of the operator is perpendicular to the position vector of the spherical curve. Recently, in [19], the authors have constructed two special families of ruled surfaces in a three dimensional strict Walker manifold. They show that the local degeneracy (resp. non-degeneracy) to one of this family has a strong consequence on the geometry of the ambiant Walker manifold. Ruled hypersurfaces in higher dimensions have also been studied by many authors [4, 5]. In [6], the intrinsic classification of irreducible ruled hypersurfaces of $\mathbb{R}^4$ has been given. In [16], a new approach to investigating ruled real hypersurfaces in complex hyperbolic space $\mathbb{CH}^n$ is given. In the paper [15], the authors study ruled real hypersurfaces in the complex quadric.

A 2-ruled hypersurface in $\mathbb{R}^4$ is a one-parameter family of planes in $\mathbb{R}^4$. This is a generalization of ruled surfaces in $\mathbb{R}^3$. In [25], the author study singularities of 2-ruled hypersurfaces in Euclidean 4-space. After defining a non-degenerate 2-ruled hypersurface, he gives a necessary and sufficient condition for such a map germ to be right-left equivalent to the cross cap $\times$ interval. Also, the author in [25] discusses the behavior of a generic 2-ruled hypersurface map. In [1], the authors obtain the Gauss map (unit normal vector field) of a 2-ruled hypersurface in Euclidean 4-space with the aid of its general parametric equation. They also obtain Gaussian and mean curvatures of the 2-ruled hypersurface and they give some characterizations about its minimality. Finally, they deal with the first and second Laplace-Beltrami operators of 2-ruled hypersurfaces in $\mathbb{E}^4$. Recently, in [17] the authors have defined three types of 2-ruled hypersurfaces in the Minkowski 4-space $\mathbb{E}^4_1$. They obtain Gaussian and mean curvatures of the 2-ruled hypersurfaces of type-1 and type-2, and some characterizations about its minimality. They also deal with the first Laplace-Beltrami operators of these types of 2-ruled hypersurfaces in $\mathbb{E}^4_1$.

Motivated by the above two works, in this paper we study the 2-ruled hypersurfaces in a Walker 4-manifold. We define three types of 2-ruled hypersurfaces and we gives Gaussian and mean curvatures of the 2-ruled hypersurface and some characterizations about its minimality. Our paper is organized as follows: We introduce the topic in section 1., then we recall some basics notions on pseudo-Riemannian manifolds in section 2.. Finally, we study 2-ruled hypersurfaces on a Walker 4-manifold in section 3.

2. Preliminaries

In this section, we recall some basics notions on pseudo-Riemannian manifolds taken from the book [11]. We begin with some algebraic preliminaries on non-degenerate bilinear forms on an m-dimensional real vector space V.

Let $g : V \times V \to \mathbb{R}$ be a symmetric bilinear form. We say that g is non-degenerate if $g(u, v) = 0$ for each $v \in V$ implies $u = 0$, otherwise g is called degenerate. A non-degenerate symmetric bilinear form on V is called a pseudo-Euclidean metric on V. It may induce either a non-degenerate or a degenerate symmetric bilinear form on a subspace W of V; then W is said to be a non-degenerate or a degenerate subspace, respectively. We say that g is positive (negative) definite provided that $u \neq 0$ implies $g(u, u) > 0 (< 0)$. If g is non-degenerate, there exists an ordered basis $(e_1, e_2, \ldots, e_m)$ of V such that:

$$\begin{aligned} g(e_i, e_i) &= -1, \quad 1 \leq i \leq q, \\ g(e_i, e_i) &= 1, \quad q+1 \leq i \leq m, \\ g(e_i, e_j) &= 0, \quad i \neq j, \end{aligned}$$

where q is uniquely determined and $(q, m-q)$ is the signature of g. Obviously, in the case $q = 0$ or $q = m$, the first or the second condition has to be dropped. The integer q is called the index of g on V and it is the largest dimension of a subspace $W \subset V$ on which the induced metric is negative definite.

A pseudo-Riemannian metric g on an m-dimensional manifold M is a symmetric tensor field of type $(0, 2)$ on M such that for any $p \in M$ the tensor g is a non-degenerate symmetric bilinear form on the tangent space T_pM of constant index. We call (M, g) a pseudo-Riemannian manifold. Frequently, we denote by M_q^m an m-dimensional pseudo-Riemannian manifold of index q. In the particular case $m > 2$ and $q = 1$, we call (M, g) a Lorentzian manifold. Obviously, if $q = 0, (M, g)$ is a Riemannian manifold.

Let N_s^n be a submanifold of a pseudo-Riemannian manifold M_q^m. If the pseudo-Riemannian metric tensor g_M of M_q^m induces a pseudo-Riemannian metric tensor, a Riemannian metric tensor or a degenerate metric tensor g_N on N_s^n, then N_s^n is called a pseudo-Riemannian submanifold, a Riemannian submanifold or a degenerate submanifold, respectively, of M_q^m. Let M_q^m be an m-dimensional pseudo-Riemannian manifold with pseudo- Riemannian metric tensor g_M of index q. Denoting by $\langle , \rangle$ the associated nondegenerate inner product on M_q^m, a tangent vector X to M_q^m is said to be spacelike if $\langle X, X \rangle > 0$ (or $X = 0$), timelike if $\langle X, X \rangle < 0$ or lightlike (null) if $\langle X, X \rangle = 0$ and $X \neq 0$. The set of null vectors of T_pM is called the null cone at $p \in M$.

Let $M_1^m(c)$ be an m-dimensional Lorentzian space form of constant curvature c, that is, $M_1^m(c)$ is the de Sitter space-time $\mathbb{S}_1^m(c)$, Minkowski space-time $\mathbb{R}_1^4(c)$ or the anti-de Sitter space-time $\mathbb{H}_1^m(c)$ according to $c > 0, c = 0$ or $c < 0$. For simplicity, we suppose that the constant curvature c of $M_1^m(c)$ is equal to $1, 0, -1$ according to whether $c > 0, c = 0, c < 0$.

Now, we describe some basic examples of pseudo-Riemannian manifolds. Let $\mathbb{R}_q^m$ be an m-dimensional pseudo-Euclidean space with metric tensor given by

$$g = -\sum_{i=1}^{q}(du_i)^2 + \sum_{i=q+1}^{m}(du_i)^2,$$

where $(u_1, \ldots, u_m)$ is a coordinate system of $\mathbb{R}_q^m$. So $(\mathbb{R}_q^m, g)$ is a flat pseudo-Riemannian manifold of index q. Putting:

$$\mathbb{S}_1^m(1) = \{u \in \mathbb{R}_1^{m+1}, \langle u, u\rangle = 1\},$$

one obtains an m-dimensional pseudo-Riemannian manifold of index q and of constant curvature $c = 1$. In the theory of general relativity, $\mathbb{S}_1^4(c)$ is called the de Sitter space-time. Putting:

$$\mathbb{H}_1^m(-1) = \{u \in \mathbb{R}_2^{m+1}, \langle u, u\rangle = -1\},$$

one obtains an m-dimensional pseudo-Riemannian manifold of index q and of constant curvature $c = -1$. $\mathbb{H}_1^m(-1)$ is called the anti-de Sitter space. We end this section by the following remark.

Remark 2.1. In contrast to the Riemannian case, there are topological obstructions to the existence of a Lorentz metric on a manifold M. Such a metric exists if either M is non-compact, or M is compact and has Euler number $\chi(M) = 0$.

3. 2-ruled hypersurfaces on a Walker 4-manifold

Hypersurfaces are one of the important objects in a space. Hypersurfaces in a manifold of constant curvature have been studied by many authors. Many ambiant spaces are not always of constant curvature. In this paper, we will studied 2-ruled hypersurfaces in a Walker 4-manifold.

A Walker 4-manifold noted M, is a pseudo-Riemannian manifold, which admits a field of parallel null 2-planes with signature $(++--)$. This class of manifold is locally isometric to (U, g_f) where U is an open of $\mathbb{R}^4$ and g_f is the metric given, respectively to the local coordinates basis by $\{\partial_i = \frac{\partial}{\partial_{u_1}}\}_{i=1,2,3,4}$ by

$$\begin{aligned} g_f(\partial_1, \partial_3) &= g_f(\partial_2, \partial_4) = 1, \\ g_f(\partial_i, \partial_j) &= g_{f_{ij}}(u_1, u_2, u_3, u_4) \quad \text{for} \quad i, j = 3, 4. \end{aligned}$$

The pseudo-Riemannian geometry of Walker metrics satisfying $g_{f_{34}} = 0$ has been studied by Chaichi et al. [8]. The purpose of this paper is to characterize some metrics propertiers of Walker satisfying : $g_{f_{33}} = g_{f_{44}} = 0$. More precesily, we will consider Walker metrics of the following form:

$$g_f = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & f \\ 0 & 1 & f & 0 \end{pmatrix}, \tag{3.1}$$

where $f = f(u_3, u_4)$ denotes a differentiable function defined U. We denote by $f_3 = \frac{\partial f(u_3,u_4)}{\partial u_3}$ and $f_4 = \frac{\partial f(u_3,u_4)}{\partial u_4}$ for any function $f(u_3, u_4)$. It follows after some straightforward calculations that the non zero christoffel symbols of a Walker metric (3.1) are:

$$\Gamma^2_{33} = f_3 \quad \text{and} \quad \Gamma^1_{44} = f_4.$$

We deduce that the Levita-Civita connection of a Walker metric is given by

$$\nabla_{\partial_3}\partial_3 = f_3\partial_2 \quad \text{and} \quad \nabla_{\partial_4}\partial_4 = f_4\partial_1.$$

Since we will deal with 2-ruled hypersurface in Walker 4-manifold, we now define the de Sitter 3-space, the anti-de Sitter space 3-space and the light cone at the origin, respectivily, by

$$\mathbb{S}^3_1 = \{x \in M, \|u\| = 1\}, \tag{3.2}$$

$$\mathbb{H}^3_+(-1) = \{u \in M, \|u\| = -1\}, \tag{3.3}$$

$$\mathcal{LC} = \{x \in M, \|u\| = 0\}, \tag{3.4}$$

where $\|u\| = \sqrt{g_f(u,u)}$.

If $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ are three vectors in M, then the vector product is defined by

$$\overrightarrow{u} \times_f \overrightarrow{v} \times_f \overrightarrow{w} = \begin{pmatrix} 0 & -f & 1 & 0 \\ -f & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \det \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}. \tag{3.5}$$

If

$$\begin{aligned} \varphi : I_1 \times I_2 \times I_3 &\to M \\ (u_1, u_2, u_3) &\mapsto \varphi(u_1, u_2, u_3), \end{aligned}$$

with

$$\varphi(u_1, u_2, u_3) = (\varphi_1, \varphi_2, \varphi_3, \varphi_4), \tag{3.6}$$

where $\varphi_i = \varphi_i(u_1, u_2, u_3), i = 1, 2, 3$, is a hypersurface in M. The Gauss map (i.e., the unit normal vector field), the matrix forms of the first and second fundamental forms are

$$G_f = \frac{\varphi_{u_1} \times_f \varphi_{u_2} \times_f \varphi_{u_2}}{\|\varphi_{u_1} \times_f \varphi_{u_2} \times \varphi_{u_3}\|}, \tag{3.7}$$

$$[g_{ij}] = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{3.8}$$

and

$$[h_{ij}] = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}, \tag{3.9}$$

respectively, where: $g_{ij} = g_f(\varphi_{u_i}, \varphi_{u_j}), h_{ij} = g_f(\varphi_{u_i u_j}, G_f)_{i,j \in \{1,2,3\}}$, with $\varphi_{uv} = \sum_{k=1}^{4} \Big\{ \frac{\partial^2 \varphi_k}{\partial v \partial u} + \sum_{ij} \Gamma_{ij}^k \frac{\partial \varphi_i}{\partial u} \frac{\partial \varphi_j}{\partial v} \Big\} \partial_k$. Also, the matrix of shape operator of the hypersurface φ (3.6) is

$$S_f = [s_{ij}] = [g^{ij}] \cdot [h_{ij}], \tag{3.10}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$. With aid of (3.8)-(3.10), the Gaussian curvature and mean curvature of a hypersurface in M are given by

$$K_f = \frac{\det[h_{ij}]}{\det[g_{ij}]}, \tag{3.11}$$

and

$$3H_f = \mathrm{trace}(S_f), \tag{3.12}$$

respectively.

3.1. 2-ruled hypersurfaces of type-1 in M

In this subsection, we give the definition of 2-ruled hypersurfaces of type-1 and state some results on Gaussian and mean curvatures. By a 2-ruled hypersurface of type-1 in M, we mean a map $\varphi : I_1 \times I_2 \times I_3 \to M$ of the form

$$\varphi(u_1, u_2, u_3) = \alpha(u_1) + u_2 \beta(u_1) + u_3 \gamma(u_1), \tag{3.13}$$

where $\alpha : I_1 \to M, \beta : I_2 \to \mathbb{S}_1^3$ and $\gamma : I_3 \to \mathbb{S}_1^3$ are smooth maps, $\mathbb{S}_1^3$ is the de Sitter 3-space of M and I_1, I_2, I_3 are open intervals. We call α a base curve and two curves β and γ director curves. The planes $(u_2, u_3) \to \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1)$ are called rulings [25].

Putting:

$$\left\{\begin{array}{lcl}\alpha(u_1) & = & \big(\alpha_1(u_1),\alpha_2(u_1),\alpha_3(u_1),\alpha_4(u_1)\big)\\ \beta(u_1) & = & \big(\beta_1(u_1),\beta_2(u_1),\beta_3(u_1),\beta_4(u_1)\big)\\ \gamma(u_1) & = & \big(\gamma_1(u_1),\gamma_2(u_1),\gamma_3(u_1),\gamma_4(u_1)\big),\end{array}\right. \tag{3.14}$$

then, the equation (3.13) becomes:

$$\varphi(u_1,u_2,u_3)=\begin{pmatrix}\alpha_1(u_1)+u_2\beta_1(u_1)+u_3\gamma_1(u_1)\\ \alpha_2(u_1)+u_2\beta_2(u_1)+u_3\gamma_2(u_1)\\ \alpha_3(u_1)+u_2\beta_3(u_1)+u_3\gamma_3(u_1)\\ \alpha_4(u_1)+u_2\beta_4(u_1)+u_3\gamma_4(u_1)\end{pmatrix}. \tag{3.15}$$

We see that $\langle\beta_i,\beta_i\rangle=\langle\gamma_i,\gamma_i\rangle=1$ and we state: $\alpha_i=\alpha_i(u_1)$, $\beta_i=\beta_i(u_1)$, $\gamma_i=\gamma_i(u_1)$, $\varphi_i=\varphi_i(u_1,u_2,u_3)$, $f'=\frac{\partial f(u_1)}{\partial u_1}$, $f''=\frac{\partial^2 f(u_1)}{\partial u_1\partial u_1}$, $i\in\{1,2,3,4\}$ and $f\in\{\alpha,\beta,\gamma\}$. We denote by

$$E_{ij} = \gamma_i(\alpha'_j+u_2\beta'_j+u_3\gamma'_j) \tag{3.16}$$

$$F_{ij} = \beta_i(\alpha'_j+u_2\beta'_j+u_3\gamma'_j). \tag{3.17}$$

Now, let us prove the following theorem which contains the Gauss map of the 2-ruled hypersurface of type-1 defined in (3.15).

Theorem 3.1. *The Gauss map of the* 2*-ruled hypersurface of type-*1 *of the form (3.15) is given by*

$$\begin{array}{rcl}G_f(u_1,u_2,u_3) & = & \dfrac{G_1(u_1,u_2,u_3)\partial_1+G_2(u_1,u_2,u_3)\partial_2}{A}\\ & & +\dfrac{G_3(u_1,u_2,u_3)\partial_3+G_4(u_1,u_2,u_3)\partial_4}{A},\end{array} \tag{3.18}$$

where

$$\begin{array}{rcl}G_1(u_1,u_2,u_3) & = & -f\Big(\beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})+\beta_4(E_{31}-E_{13})\Big)\\ & & +\beta_1(E_{24}-E_{42})+\beta_2(E_{41}-E_{14})+\beta_4(E_{12}-E_{21}),\\ G_2(u_1,u_2,u_3) & = & -f\Big(\beta_2(E_{34}-E_{43})+\beta_3(E_{42}-E_{24})+\beta_4(E_{23}-E_{32}\Big)\\ & & +\beta_1(E_{32}-E_{23})+\beta_2(E_{13}-E_{31})+\beta_3(E_{21}-E_{12}),\\ G_3(u_1,u_2,u_3) & = & \beta_2(E_{34}-E_{43})+\beta_3(E_{42}-E_{24})+\beta_4(E_{23}-E_{32}),\\ G_4(u_1,u_2,u_3) & = & \beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})\\ & & +\beta_4(E_{31}-E_{13}),\end{array} \tag{3.19}$$

and

$$A=\sqrt{2G_1G_3+2G_2G_4+2fG_3G_4}, \tag{3.20}$$

with $G_1=G_1(u_1,u_2,u_3)$, $G_2=G_2(u_1,u_2,u_3)$, $G_3=G_3(u_1,u_2,u_3)$ *and* $G_4=G_4(u_1,u_2,u_3)$.

Proof. If we differentiate (3.15), we get:

$$\begin{cases} \varphi_{u_1}(u_1,u_2,u_3) &= \big(\alpha_1'+u_2\beta_1'+u_3\gamma_1', \alpha_2'+u_2\beta_2'+u_3\gamma_2', \\ & \quad \alpha_3'+u_2\beta_3'+u_3\gamma_3', \alpha_4'+u_2\beta_4'+u_3\gamma_4'\big) \\ \varphi_{u_2}(u_1,u_2,u_3) &= \big(\beta_1,\beta_2,\beta_3,\beta_4\big) \\ \varphi_{u_3}(u_1,u_2,u_3) &= \big(\gamma_1,\gamma_2,\gamma_3,\gamma_4\big). \end{cases}$$

By using, the vector product define in (3.5), we get:

$$\begin{aligned} \varphi_{u_1}\times_f\varphi_{u_2}\times_f\varphi_{u_3} &= \Big(-f\Big(\beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})+\beta_4(E_{31}-E_{13})\Big) \\ &+\Big(\beta_1(E_{24}-E_{42})+\beta_2(E_{41}-E_{14})+\beta_4(E_{12}-E_{21})\Big)\Big)\partial_1 \\ &+\Big(-f\Big(\beta_2(E_{34}-E_{43})+\beta_3(E_{42}-E_{24})+\beta_4(E_{23}-E_{32})\Big) \\ &+\Big(\beta_1(E_{32}-E_{23})+\beta_2(E_{13}-E_{31})+\beta_3(E_{21}-E_{12})\Big)\Big)\partial_2 \\ &+\Big(\beta_2(E_{43}-E_{34})+\beta_3(E_{24}-E_{42})+\beta_4(E_{32}-E_{23})\Big)\partial_3 \\ &+\Big(\beta_1(E_{43}-E_{34})+\beta_3(E_{14}-E_{41})+\beta_4(E_{31}-E_{13})\Big)\partial_4. \end{aligned}$$

Now using the unit normal vector formula in (3.7), we get the result. □

From (3.8), we obtain the matrix of the first fundamental form:

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & 1 & e \\ c & e & 1 \end{bmatrix}, \tag{3.21}$$

where

$$\begin{aligned} a &= 2f(\alpha_3'+u_2\beta_3'+u_3\gamma_3')(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\ &\quad +2\sum_{i=1}^{2}(\alpha_i'+u_2\beta_i'+u_3\gamma_i')(\alpha_{i+2}'+u_2\beta_{i+2}'+u_3\gamma_{i+2}'), \\ b &= f(F_{34}-F_{43})+\sum_{i=1}^{2}(F_{i(i+2)}+F_{(i+2)i}), \\ c &= f(E_{34}-E_{43})+\sum_{i=1}^{2}(E_{i(i+2)}+E_{(i+2)i}), \\ e &= f(\beta_3\gamma_4+\beta_4\gamma_3)+\sum_{i=1}^{2}(\beta_i\gamma_{i+2}+\beta_{i+2}\gamma_i), \end{aligned} \tag{3.22}$$

and we obtain the inverse matrix $[g^{ij}]$ of $[g_{ij}]$ as:

$$[g^{ij}] = \frac{1}{\det[g_{ij}]}\begin{bmatrix} 1-e^2 & ce-b & be-c \\ ce-b & a-c^2 & bc-ae \\ be-c & bc-ae & a-b^2 \end{bmatrix}, \tag{3.23}$$

where

$$\det[g_{ij}] = -b^2 + 2cbe - c^2 - ae^2 + a = B. \tag{3.24}$$

Furthermore, from (3.9), the matrix form of the second fundamental from of the 2-ruled hypersurface (3.15) is obtained by

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{3.25}$$

where

$$\begin{aligned}
h_{11} &= \frac{f_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4')}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{\sum_{i=1}^2 G_{i+2}(\alpha_{i''} + u_2\beta_{i''} + u_3\gamma_{i''})}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\
h_{12} = h_{21} &= \frac{f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' - u_3\gamma_4')}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{\sum_{i=1}^2 G_{i+2}\beta_i'}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
h_{13} = h_{31} &= \frac{f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4')}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{\sum_{i=1}^2 G_{i+2}\gamma_i'}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
h_{22} &= \frac{f_3\beta_3^2G_4 + f_4\beta_4^2G_3}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\
h_{33} &= \frac{f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\
h_{23} = h_{32} &= \frac{f_3\beta_3\gamma_3G_4}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}} \\
&\quad + \frac{f_4\beta_4\gamma_4G_3}{\sqrt{2fG_3(u_1,u_2,u_3)G_4(u_1,u_2,u_3) + \sum_{i=1}^2 G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}. \qquad (3.26)
\end{aligned}$$

We can see easily that the $\det[h_{ij}] = h_{11}h_{22}h_{33} + 2h_{12}h_{13}h_{23} - h_{12}^2h_{33} - h_{13}^2h_{22} - h_{23}^2h_{11} \neq 0$.

Then we can give the following theorem by using (3.11)

Theorem 3.2. *The 2-ruled hypersurface of type-1 defined in (3.15) is no flat.*

Corollary 3.1. *The 2-ruled hypersurface of type-1 defined in (3.15) is flat if f is nonzero constant.*

Proof. From (3.9), the matrix of second fundamental form of the 2-ruled hypersurface (3.15) is obtained by

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & 0 & 0 \\ h_{31} & 0 & 0 \end{bmatrix}, \tag{3.27}$$

where

$$\begin{aligned} h_{11} &= \frac{\sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')}{\sqrt{\sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\ h_{12} &= h_{21} = \frac{\sum_{i=1}^{2} G_{i+2}\beta_i'}{\sqrt{\sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\ h_{13} &= h_{31}\frac{\sum_{i=2}^{2} G_{i+2}\gamma_i'}{\sqrt{\sum_{i=1}^{2} G_i(u_1,u_2,u_3)G_{i+2}(u_1,u_2,u_3)}}, \\ h_{22} &= h_{23} = h_{33} = 0. \end{aligned} \tag{3.28}$$

So we have $\det(h_{ij}) = 0$, hence $G_f = 0$. □

Now, we will prove the following theorem about the mean curvature.

Theorem 3.3. *The 2-ruled hypersurface of type-1 defined in (3.15) is minimal, if*

$$\begin{aligned} 0 = {} & (1-e^2)\Big[f_3G_4(\alpha_3' + u_2\beta_2' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\ & + 2(ce-b)\Big[f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\ & + 2(be-c)\Big[f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\ & + 2(bc-ae)\Big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\Big] + (a-c^2)\Big[f_3\beta_3^2G_4 + f_4\beta_4^2G_3\Big] \\ & + (a-b^2)\Big[f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3\Big]. \end{aligned} \tag{3.29}$$

Proof. By (3.10), the matrix of the shape operator is

$$S = \begin{bmatrix} 1-e^2 & ce-b & be-c \\ ce-b & a-c^2 & bc-ae \\ be-c & bc-ae & a-b^2 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix},$$

where $h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}$ are the same in (3.26). Then we get the coefficients of S by

$$\begin{aligned} S_{11} &= (1-e^2)h_{11} + (ce-b)h_{12} + (be-c)h_{13} \\ S_{22} &= (ce-b)h_{12} + (a-c^2)h_{22} + (bc-ae)h_{23} \\ S_{33} &= (be-c)h_{13} + (bc-ae)h_{23} + (a-b^2)h_{33}. \end{aligned}$$

And using (3.26) and (3.12), we see that the 2-ruled hypersurface is minimal if

$$S_{11} + S_{22} + S_{33} = 0.$$

Then, the proof is complete. □

Corollary 3.2. *If the curves β and γ are orthogonal, then the 2-ruled hypersurface of type-1 defined in (3.15) is minimal if*

$$\begin{aligned} 0 = & \Big[f_3 G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4 G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\ & -2b\Big[f_3\beta_3 G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_4') + f_4\beta_4 G_3(\alpha_4' + u_2\beta_4' + \gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\ & -2c\Big[f_3\gamma_3 G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4 G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\ & +2bc\Big[f_3\beta_3\gamma_3 G_4 + f_4\beta_4\gamma_4 G_3\Big] + (a-c^2)\Big[f_3\beta_3^2 G_4 + f_4\gamma_4^2 G_3\Big] \\ & +(a-b^2)\Big[f_3\gamma_3^2 G_4 + f_4\gamma_4^2 G_3\Big]. \end{aligned} \tag{3.30}$$

The Laplace-Beltrami operator of a smooth function $\varphi = \varphi(u_1, u_2, u_3)$ of class C^3 with respect to the first fundamental form of a hypersurface is defined as follows:

$$\Delta\varphi = \frac{1}{\sqrt{\det[g_{ij}]}} \sum_{i,j}^{3} \frac{\partial}{\partial u_i}\left(\sqrt{\det[g_{ij}]}\, g^{ij} \frac{\partial \varphi}{\partial u_j}\right). \tag{3.31}$$

Using (3.31), we get the Laplace-Beltrami operator of the 2-ruled hypersurface of type-1 (3.15) by

$$\Delta\varphi = (\Delta\varphi_1, \Delta\varphi_2, \Delta\varphi_3, \Delta\varphi_4),$$

where

$$\begin{aligned}\Delta\varphi_i &= \frac{1}{\sqrt{B}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{(1-e^2)\varphi_{iu_1} + (ce-b)\varphi_{iu_2} + (be-c)\varphi_{iu_3}}{\sqrt{\det[g_{ij}]}}\Big)\\ &\quad + \frac{\partial}{\partial u_2}\Big(\frac{(ce-b)\varphi_{iu_1} + (a-c^2)\varphi_{iu_2} + (bc-ae)\varphi_{iu_3}}{\sqrt{\det[g_{ij}]}}\Big)\\ &\quad + \frac{\partial}{\partial u_3}\Big(\frac{(be-c)\varphi_{iu_1} + (bc-ae)\varphi_{iu_2} + (a-b^2)\varphi_{iu_3}}{\sqrt{\det[g_{ij}]}}\Big)\Big]. \qquad (3.32)\end{aligned}$$

That is

$$\begin{aligned}\Delta\varphi_i &= \frac{1}{\sqrt{B}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{(1-e^2)(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (ce-b)\beta_i + (be-c)\gamma_i}{\sqrt{\det[g_{ij}]}}\Big)\\ &\quad + \frac{\partial}{\partial u_2}\Big(\frac{(ce-b)(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (a-c^2)\beta_i + (bc-ae)\gamma_i}{\sqrt{\det[g_{ij}]}}\Big)\\ &\quad + \frac{\partial}{\partial u_3}\Big(\frac{(be-c)(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (bc-ae)\beta_i + (a-b^2)\gamma_i}{\sqrt{\det[g_{ij}]}}\Big)\Big]. \qquad (3.33)\end{aligned}$$

If we suppose that β and γ are orthogonal, then the Laplace-Beltrami operator of the 2-ruled hypersuface of type-1 (3.15) is given by

$$\begin{aligned}\Delta\varphi_i &= \frac{1}{\sqrt{a-b^2-c^2}}\Big[\frac{\partial}{\partial u_1}\Big(\frac{(\alpha_i' + u_2\beta_i' + u_3\gamma_i') - b\beta_i - c\gamma_i}{\sqrt{a-b^2-c^2}}\Big)\\ &\quad + \frac{\partial}{\partial u_2}\Big(\frac{-b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + (a-c^2)\beta_i + bc\gamma_i}{\sqrt{a-b^2-c^2}}\Big)\\ &\quad + \frac{\partial}{\partial u_3}\Big(\frac{-c(\alpha_i' + u_2\beta_i' + u_3\gamma_i') + bc\beta_i + (a-b^2)\gamma_i}{\sqrt{a-b^2-c^2}}\Big)\Big]. \qquad (3.34)\end{aligned}$$

Theorem 3.4. *The components of the Laplace-Beltrami operator of the* 2*-ruled hypersurface of type-*1 *defined in (3.15) are*

$$\begin{aligned}\Delta\varphi_i &= \frac{1}{W^{\frac{3}{2}}\sqrt{W}}\Big[(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'' - (b\beta_i)_{u_1} - (c\gamma_i)_{u_1})W\\ &\quad - V_1(\alpha_i' + u_2\beta_i' + u_2\gamma_i' - b\beta_i - c\gamma_i)\\ &\quad + (-b\beta_i' + ((a-c^2)\beta_i)_{u_2} + (bc\gamma_i)_{u_2})W - V_2(-b(\alpha_i' + u_2\beta_i' + u_3\gamma_i')\\ &\quad + (a-c^2)\beta_i + bc\gamma_i)\\ &\quad + (-c\gamma_i' + (bc\beta_i)_{u_3} + ((a-b^2)\gamma_i)_{u_3})W - V_3(-c(\alpha_i' + u_2\beta_i' + u_3\gamma_i')\\ &\quad + bc\beta_i + (a-b^2)\gamma_i)\Big], \qquad (3.35)\end{aligned}$$

where $i = 1, 2, 3, 4$*;* β *and* γ *are orthogonal;* $W = a - b^2 - c^2$, $V_1 = a_{u_1} - 2bb_{u_1} - 2cc_{u_1}$, $V_2 = a_{u_2} - 2bb_{u_2} - 2cc_{u_2}$, $V_3 = a_{u_3} - 2bb_{u_3} - 2cc_{u_3}$.

3.2. 2-Ruled hypersurfaces of type-2 in M

A 2-ruled hypersurface of type-2 in M means (the image of) a map $\varphi : I_1 \times I_2 \times I_3 \to M$ of the form

$$\varphi(u_1, u_2, u_3) = \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1), \tag{3.36}$$

where $\alpha : I_1 \to M$, $\beta : I_2 \to \mathbb{H}^3_+(-1)$, $\gamma : I_3 \to \mathbb{H}^3_+(-1)$ are smooth maps, $\mathbb{H}^3_+(-1)$ is the anti-de Sitter space 3-space of M and I_1, I_2, I_3 are open intervals. We call α a base curve, β and γ director curves. The planes $(u_2, u_3) \mapsto \alpha(u_1)+u_2\beta(u_1)+u_3\gamma(u_1)$ are called rulings. So, if we take

$$\begin{cases} \alpha(u_1) &= \big(\alpha_1(u_1), \alpha_2(u_1), \alpha_3(u_1), \alpha_4(u_1)\big) \\ \beta(u_1) &= \big(\beta_1(u_1), \beta_2(u_1), \beta_3(u_1), \beta_4(u_1)\big) \\ \gamma(u_1) &= \big(\gamma_1(u_1), \gamma_2(u_1), \gamma_3(u_1), \gamma_4(u_1)\big) \end{cases} \tag{3.37}$$

in (3.36), then we can write the 2-ruled hypersurface of type-2 as

$$\varphi(u_1, u_2, u_3) = \begin{pmatrix} \alpha_1(u_1) + u_2\beta_1(u_1) + u_3\gamma_1(u_1) \\ \alpha_2(u_1) + u_2\beta_2(u_1) + u_3\gamma_2(u_1) \\ \alpha_3(u_1) + u_2\beta_3(u_1) + u_2\gamma_3(u_1) \\ \alpha_4(u_1) + u_2\beta_4(u_1) + u_3\gamma_4(u_1) \end{pmatrix}. \tag{3.38}$$

We see that $\langle \beta_i, \beta_i \rangle = \langle \gamma_i, \gamma_i \rangle = -1$ and we state $\alpha_i = \alpha_i(u_1)$, $\beta_i = \beta_i(u_1)$, $\gamma_i = \gamma_i(u_1)$, $\varphi_i = \varphi_i(u_1, u_2, u_3)$, $f' = \frac{\partial f(u_1)}{\partial u_1}$, $f'' = \frac{\partial^2 f(u_1)}{\partial u_1 \partial u_1}$, $i \in \{1, 2, 3, 4\}$ and $f \in \{\alpha, \beta, \gamma\}$.

From (3.8), we obtain the matrix of the first fundamental form

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & -1 & e \\ c & e & -1 \end{bmatrix}. \tag{3.39}$$

And we obtain the inverse matrix $[g^{ij}]$ of $[g_{ij}]$ as

$$[g^{ij}] = \frac{1}{\det[g_{ij}]} \begin{bmatrix} 1 - e^2 & ce + b & be + c \\ ce + b & -a - c^2 & bc - ae \\ be + c & bc - ae & -a - b^2 \end{bmatrix}. \tag{3.40}$$

where a, b, c and e are the same in (3.22) and

$$\det[g_{ij}] = b^2 + 2cbe + c^2 - ae^2 + a = C. \tag{3.41}$$

Furthermore, from (3.9), the matrix form of the second fundamental from of the 2-ruled hypersurface (3.38) is the same given in (3.25) and (3.26). And we have the following theorem since the $\det[h_{ij}] \neq 0$.

Theorem 3.5. *The 2-ruled hypersurface of type-2 defined in (3.38) is not flat.*

Corollary 3.3. *The* 2*-ruled hypersurface of type-*2 *defined in (3.38) is flat if* f *is nonzero constant.*

For the mean curvature we have:

Theorem 3.6. *The* 2*-ruled hypersurface of type-*2 *defined in (3.38) is minimal in* M*, if*

$$\begin{aligned}
0 \;=\; & (1-e^2)\Big[f_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4G_3(\alpha_4'+u_2\gamma_4'+u_3\gamma_4') \\
& +\sum_{i=1}^{2}G_{i+2}(\alpha_i''+u_2\beta_i''+u_3\gamma_i'')\Big] \\
& +2(ce+b)\Big[f_3\beta_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4\beta_4G_3(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\
& +\sum_{i=1}^{2}G_{i+2}\beta_i'\Big] \\
& +2(be+c)\Big[f_3\gamma_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4\gamma_4G_3(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\
& +\sum_{i=1}^{2}G_{i+2}\gamma_i'\Big] \\
& +2(bc-ae)\Big[f_3\beta_3\gamma_3G_4+f_4\beta_4\gamma_4G_3\Big]+(-a-c^2)\Big[f_3\beta_3^2G_4+f_4\beta_4^2G_3\Big] \\
& +(-a-b^2)\Big[f_3\gamma_3^2G_4+f_4\gamma_4^2G_3\Big]. \qquad (3.42)
\end{aligned}$$

Proof. By (3.10), the matrix of the shape operator is

$$S=\begin{bmatrix} 1-e^2 & ce+b & be+c \\ ce+b & -a-c^2 & bc-ae \\ be+c & bc-ae & -a-b^2 \end{bmatrix}\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix},$$

where $h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}$ are the same in (3.26). Then, we get the coefficients of S by

$$\begin{aligned}
S_{11} &= (1-e^2)h_{11}+(ce+b)h_{12}+(be+c)h_{13}, \\
S_{22} &= (ce+b)h_{12}+(-a-c^2)h_{22}+(bc-ae)h_{23}, \\
S_{33} &= (be+c)h_{13}+(bc-ae)h_{23}+(-a-b^2)h_{33}.
\end{aligned}$$

And using (3.26) and (3.12), we see that the 2-ruled hypersurface of type-2 defined in (3.38) is minimal if

$$S_{11}+S_{22}+S_{33}=0.$$

Then, the proof is complete. □

Corollary 3.4. *If the curves β and γ are orthogonal, then the 2-ruled hypersurface of type-2 defined in (3.38) is minimal if*

$$\begin{aligned} 0 \;=\; & \Big[f_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'')\Big] \\ & +2b\Big[f_3\beta_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\beta_4G_3(\alpha_4' + u_2\beta_4' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\beta_i'\Big] \\ & +2c\Big[f_3\gamma_3G_4(\alpha_3' + u_2\beta_3' + u_3\gamma_3') + f_4\gamma_4G_3(\alpha_4' + u_2\beta_2' + u_3\gamma_4') \\ & + \sum_{i=1}^{2} G_{i+2}\gamma_i'\Big] \\ & +2bc\Big[f_3\beta_3\gamma_3G_4 + f_4\beta_4\gamma_4G_3\Big] + (-a-c^2)\Big[f_3\beta_3^2G_4 + f_4\gamma_4^2G_3\Big] \\ & +(-a-b^2)\Big[f_3\gamma_3^2G_4 + f_4\gamma_4^2G_3\Big]. \end{aligned} \tag{3.43}$$

To end this subsection, we will give the operator of Laplace-Beltrami in the following theorem:

Theorem 3.7. *The components of the Laplace-Beltrami operator of the 2-ruled hypersurface of type-2 defined in (3.38) are*

$$\begin{aligned} \Delta\varphi_i \;=\; & \frac{1}{R^{\frac{3}{2}}\sqrt{T}}\Big[(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'') + (b\beta_i)_{u_1} + (c\gamma_i)_{u_1})T \\ & -R_1(\alpha_i' + u_2\beta_i' + u_3\gamma_i' + b\beta_i + c\gamma_i)\Big] \\ & +(b\beta_i' + ((-a-c^2)\beta_i)_{u_2} + (bc\gamma_i)_{u_2})T - R_2(b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\ & +(-a-c^2)\beta_i + bc\gamma_i) \\ & +(c\gamma_i' + (bc\beta_i)_{u_3} + ((-a-b^2)\gamma_i)_{u_3})T - R_3(c(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\ & +bc\beta_i + (-a-b^2)\gamma_i)\Big], \end{aligned} \tag{3.44}$$

where $i = 1, 2, 3, 4$; β and γ are orthogonal; $T = a+b^2+c^2$, $R_1 = a_{u_1}+2bb_{u_2}+2cc_{u_1}$, $R_2 = a_{u_2} + 2bb_{u_2} + 2cc_{u_2}$, $R_3 = a_{u_3} + 2bb_{u_3} + 2cc_{u_3}$.

3.3. 2-Ruled hypersurfaces of type-3 in M

A 2-ruled hypersurface of type-3 in M means (the image of) a map $\varphi : I_1 \times I_2 \times I_3 \to M$ of the form

$$\varphi(u_1, u_2, u_3) = \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1), \tag{3.45}$$

where $\alpha : I_1 \to M$, $\beta : I_2 \to \mathcal{LC}$, $\gamma : I_3 \to \mathcal{LC}$ are smooth maps, $\mathcal{LC}$ is the light cone of M and I_1, I_2, I_3 are open intervals. We call α a base curve, β and γ director curves. The planes $(u_2, u_3) \mapsto \alpha(u_1) + u_2\beta(u_1) + u_3\gamma(u_1)$ are called rulings. So, if we take

$$\left\{\begin{array}{lcl} \alpha(u_1) & = & \big(\alpha_1(u_1), \alpha_2(u_1), \alpha_3(u_1), \alpha_4(u_1)\big), \\ \beta(u_1) & = & \big(\beta_1(u_1), \beta_2(u_1), \beta_3(u_1), \beta_4(u_1)\big), \\ \gamma(u_1) & = & \big(\gamma_1(u_1), \gamma_2(u_1), \gamma_3(u_1), \gamma_4(u_1)\big), \end{array}\right. \tag{3.46}$$

in (3.45), then we can write the 2-ruled hypersurface of type-3 as

$$\varphi(u_1, u_2, u_3) = \begin{pmatrix} \alpha_1(u_1) + u_2\beta_1(u_1) + u_3\gamma_1(u_1) \\ \alpha_2(u_1) + u_2\beta_2(u_1) + u_3\gamma_2(u_1) \\ \alpha_3(u_1) + u_2\beta_3(u_1) + u_2\gamma_3(u_1) \\ \alpha_4(u_1) + u_2\beta_4(u_1) + u_3\gamma_4(u_1) \end{pmatrix}. \tag{3.47}$$

We see that $\langle \beta_i, \beta_i \rangle = \langle \gamma_i, \gamma_i \rangle = -1$ and we state $\alpha_i = \alpha_i(u_1)$, $\beta_i = \beta_i(u_1)$, $\gamma_i = \gamma_i(u_1)$, $\varphi_i = \varphi_i(u_1, u_2, u_3)$, $f' = \frac{\partial f(u_1)}{\partial u_1}$, $f'' = \frac{\partial^2 f(u_1)}{\partial u_1 \partial u_1}$, $i \in \{1, 2, 3, 4\}$ and $f \in \{\alpha, \beta, \gamma\}$.

From (3.8), we obtain the matrix of the first fundamental form

$$[g_{ij}] = \begin{bmatrix} a & b & c \\ b & 0 & e \\ c & e & 0 \end{bmatrix}. \tag{3.48}$$

And we obtain the inverse matrix $[g^{ij}]$ of $[g_{ij}]$ as

$$[g^{ij}] = \frac{1}{\det[g_{ij}]} \begin{bmatrix} -e^2 & ce & be \\ ce & -c^2 & bc - ae \\ be & bc - ae & -b^2 \end{bmatrix}, \tag{3.49}$$

where a, b, c and e are the same in (3.22) and

$$\det[g_{ij}] = 2bce - ae^2 = D. \tag{3.50}$$

Furthermore, from (3.9), the matrix form of the second fundamental from of the 2-ruled hypersurface (3.47) is the same given in (3.25) and (3.26). And we have the following theorem since the $\det[h_{ij}] \neq 0$.

Theorem 3.8. *The 2-ruled hypersurface of type-3 defined in (3.47) is no flat.*

Corollary 3.5. *The 2-ruled hypersurface of type-3 is flat if f is non zero constant.*

For the mean curvature, we have:

Theorem 3.9. *The* 2*-ruled hypersurface of type-*3 *defined in (3.47) is minimal in* M*, if*

$$\begin{aligned} 0 \;=\; & -e^2\Big[f_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4G_3(\alpha_4'+u_2\gamma_4'+u_3\gamma_4') \\ & +\sum_{i=1}^{2}G_{i+2}(\alpha_i''+u_2\beta_i''+u_3\gamma_i'')\Big] \\ & +2ce\Big[f_3\beta_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4\beta_4G_3(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\ & +\sum_{i=1}^{2}G_{i+2}\beta_i'\Big] \\ & +2be\Big[f_3\gamma_3G_4(\alpha_3'+u_2\beta_3'+u_3\gamma_3')+f_4\gamma_4G_3(\alpha_4'+u_2\beta_4'+u_3\gamma_4') \\ & +\sum_{i=1}^{2}G_{i+2}\gamma_i'\Big] \\ & +2(bc-ae)\Big[f_3\beta_3\gamma_3G_4+f_4\beta_4\gamma_4G_3\Big]-c^2\Big[f_3\beta_3^2G_4+f_4\beta_4^2G_3\Big] \\ & -b^2\Big[h_3\gamma_3^2G_4+h_4\gamma_4^2G_3\Big]. \end{aligned} \tag{3.51}$$

Proof. By (3.10) the matrix of the shape operator is

$$S=\begin{bmatrix} -e^2 & ce & be \\ ce & -c^2 & bc-ae \\ be & bc-ae & -b^2 \end{bmatrix}\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix},$$

where $h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}$ are the same in (3.26). Then we get the coefficients of S by

$$\begin{aligned} S_{11} &= -e^2h_{11}+ceh_{12}+beh_{13}, \\ S_{22} &= ceh_{12}-c^2h_{22}+(bc-ae)h_{23}, \\ S_{33} &= be+h_{13}+(bc-ae)h_{23}-b^2h_{33}. \end{aligned}$$

And using (3.26) and (3.12), we see that the 2-ruled hypersurface of type-3 defined in (3.47) is minimal if

$$S_{11}+S_{22}+S_{33}=0.$$

Then, the proof is complete. □

Corollary 3.6. *If the curves* β *and* γ *are orthogonal, then the* 3*-ruled hypersurface of type-*3 *defined in (3.47) is minimal if*

$$\begin{aligned} 0 \;=\; & 2bc\big[f_3\beta_3\gamma_3G_4+f_4\beta_4\gamma_4G_3\big] \\ & -c^2\big[f_3\beta_3^2G_4+h_4\gamma_4^2G_3\big] \\ & -b^2\big[f_3\gamma_3^2G_4+f_4\gamma_4^2G_3\big]. \end{aligned} \tag{3.52}$$

To end this subsection, we will give the operator of Laplace-Beltrami in the following theorem:

Theorem 3.10. *The components of the Laplace-Beltrami operator of the* 2*-ruled hypersurface of type-*3 *defined in (3.47) are:*

$$\begin{aligned}\Delta\varphi_i \quad = \quad & \frac{1}{L^{\frac{3}{2}}\sqrt{L}}\Big[(\alpha_i'' + u_2\beta_i'' + u_3\gamma_i'') + (b\beta_i)_{u_1} + (c\gamma_i)_{u_1})L \\ & -J_1(\alpha_i' + u_2\beta_i' + u_3\gamma_i' + b\beta_i + c\gamma_i) \\ & +(b\beta_i' + ((-a-c^2)\beta_i)_{u_2} + (bc\gamma_i)_{u_2})L - J_2(b(\alpha_i' + u_2\beta_i' + u_3\gamma_i') \\ & +(-a-c^2)\beta_i + bc\gamma_i) \\ & +(c\gamma_i' + (bc\beta_i)_{u_3} + ((-a-b^2)\gamma_i)_{u_3})L - J_3(c(\alpha_i' + y\beta_i' + z\gamma_i') \\ & +bc\beta_i + (-a-b^2)\gamma_i)\Big], \end{aligned} \tag{3.53}$$

where $i = 1, 2, 3, 4;\ ;\ L = 2bce - ae^22,\ J_1 = 2b_{u_1}ce + 2bc_{u_1}e + 2bce_{u_1} - 2aa_{u_1}$, $J_2 = 2b_{u_2}ce + 2bc_{u_2}e + 2bce_{u_2} - 2aa_{u_2},\ J_3 = 2b_{u_3}ce + 2bc_{u_3}e + 2bce_{u_3} - 2aa_{u_3}$.

Note that the hypersurfaces constructed in this paper are not flats. Unlike Euclidean and Minkowskian spaces, where the ruled hypersurfaces are flats.

4. Conclusion

We end this work by giving some applications of ruled surfaces and 2-ruled hypersurfaces as generalisations of the first one. Ruled surfaces have been applied in different areas such as CAD, electric discharge machining [1, 24]. The authors [23] present an elementary introduction to the theory of Betrand pairs of curves and ruled surfaces. Bertrand pairs of ruled surfaces are introduced as offsets in the context of line geometry. Also, the ruled surfaces has an important application area on kinematics [1, 22]. Additionally, ruled surfaces have an important application area in architecture [1]. For lightweight structures in the field of architecture and civil engineering, concrete shells with negative Gaussian curvature are frequently used. One class of such surfaces are the skew ruled surfaces [20].

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