THE n-DUAL STRUCTURE OF THE SPACE OF p-SUMMABLE SEQUENCE SPACES

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Abstract. In this paper, we shall investigate the n-dual structure of the sequence space \( l^p \) regarded as normed space and n-normed space, where the given norm is derived by n-norm and they have been studied in [5, 6, 7].

Keywords: Sequence space, normed space, multilinear n-functional, isometric linear bijection.

1. Introduction

Similar to the theory of the space of the bounded linear functionals defined on a normed space, bounded multilinear n-functionals have been defined on an n-normed space and these theories have been studied by White [1], Gunawan [9, 10].

The concept of 2-normed spaces was initially investigated by Gahler [8]. After that, it has been generalized to n-normed spaces and has been studied by many others (see [2, 3, 4, 5, 6, 7]).

Definition 1.1. Let \( X \) be a vector space over \( \mathbb{K}(= \mathbb{R} \text{ or } \mathbb{C}) \) of dimension \( d \geq n(n \geq 2) \). A non-negative real valued function \( \| \cdot , \ldots , \cdot \| \) defined on \( X^n \) satisfying the four conditions:

(N1) \( \| x^1, x^2, \ldots, x^n \| = 0 \) if and only if \( x^1, x^2, \ldots, x^n \) are linearly dependent;

(N2) \( \| x^1, x^2, \ldots, x^n \| \) is invariant under any permutation of \( x^1, x^2, \ldots, x^n \);

(N3) \( \| \alpha \cdot x^1, x^2, \ldots, x^n \| = \| \alpha \| \cdot \| x^1, x^2, \ldots, x^n \| \);

(N4) \( \| x^1 + y, x^2, \ldots, x^n \| \leq \| x^1, x^2, \ldots, x^n \| + \| y, x^2, \ldots, x^n \| \);

for all \( x^1, x^2, \ldots, x^n, y \in X \) and for all \( \alpha \in \mathbb{K} \), is called an \( n \)-norm on \( X \), and the pair \( (X, \| \cdot , \ldots , \cdot \|) \) is called an \( n \)-normed space.
Definition 1.2. Let \((X, \|\cdot\|)\) be an \(n\)-normed space and \(\{e^1, \ldots, e^n\}\) is a linearly independent set of \(n\) vectors, let us define:

1. \(\|x\|_o = \max \{\|x, e^1, \ldots, e^{n-1}\| : \{t_1, \ldots, t_{n-1}\} \subset \{1, \ldots, n\}\}\)
2. \(\|x\|_q = \left(\sum_{|t_1| + \ldots + |t_{n-1}| = 1} |x, e^1, \ldots, e^{n-1}|^{|t|} \right)^{1/q} ; \quad 1 \leq q < \infty.\)

In [3, 4], Gunawan proved that these two functions \((\|\cdot\|_o, \|\cdot\|_q)\) define norms (known as derived norms) on the vector space \(X\) and they are equivalent.

Definition 1.3. Let \((X, \|\cdot\|)\) is a normed space then a linear functional \(f : X \to K\) is said to be bounded if \(\exists\) a real number \(k > 0\) such that

\[|f(x)| \leq k\|x\|, \quad \text{for all } x \in X.\]

The linear functional \(f\) is said to be continuous at a point \(x_0 \in X\) if for every given \(\epsilon > 0, \exists\delta > 0\) such that

\[x \in X, \|x - x_0\| < \delta \implies |f(x) - f(x_0)| < \epsilon.\]

Lemma 1.1. Let \((X, \|\cdot\|)\) is a normed space then a linear functional \(f : X \to K\) is bounded if and only if \(f\) is continuous.

Analogous to the above definitions a bounded multilinear \(n\)-functional has been defined on \(n\)-normed space (for detail see [1, 9, 10]).

Definition 1.4. Let \(X\) be a vector space then a scalar valued function \(f : X^n \to K\) is called a multilinear \(n\)-functional if it satisfies:

1. \(f(x^1 + y^1, \ldots, x^n + y^n) = \sum_{h \in [x^1, \ldots, x^n]} f(h^1, \ldots, h^n),\)
2. \(f(\alpha_1 x^1, \ldots, \alpha_n x^n) = \alpha_1 \ldots \alpha_n f(x^1, x^2, \ldots, x^n),\)

for every \(x^1, x^2, \ldots, x^n \in X\) and for every \(\alpha_i \in K\).

A multilinear \(n\)-functional \(f\) defined on a normed space \((X, \|\cdot\|)\) is said to be bounded (i.e. bounded with respect norm) if \(\exists K > 0\) such that

\[|f(x^1, x^2, \ldots, x^n)| \leq K\|x^1\| \ldots \|x^n\|, \quad \text{for every } x^1, x^2, \ldots, x^n \in X.\]

Similarly, a multilinear \(n\)-functional \(f\) defined on an \(n\)-normed space \((X, \|\cdot\|)\) is said to be bounded (i.e. bounded with respect \(n\)-norm) if \(\exists K > 0\) such that

\[|f(x^1, x^2, \ldots, x^n)| \leq K\|x^1, x^2, \ldots, x^n\|, \quad \text{for every } x^1, x^2, \ldots, x^n \in X.\]

As we know that, the set \(B(X, K)\) of all bounded linear functional \(f\) defined on the normed space \((X, \|\cdot\|)\) forms a normed space with norm defined by

\[\|f\| = \sup \{|f(x)| : x \in X, \|x\| = 1\}.\]
The normed space \( \mathbf{B}(X, K) \) is called \textit{dual space} of the normed space \( (X, \| \cdot \|) \) and is usually denoted by \( X^* \).

Similarly, the space of bounded multilinear \( n \)-functionals on \( (X, \| \cdot \|) \) on \( (X, \| \cdot \|, \ldots, \| \cdot \|) \) is called the \textit{n-dual space} of the normed space \( (X, \| \cdot \|) \) \( \text{[the n-dual space of the n-normed space \( (X, \| \cdot \|, \ldots, \| \cdot \|) \)} \) respectively see \([9, 10]\), with norms

\[
\| f \|_{n,1} := \sup_{\| x^1 \|, \ldots, \| x^n \| \neq 0} \frac{|f(x^1, x^2, \ldots, x^n)|}{\|x^1\| \cdots \|x^n\|};
\]

\[
\| f \|_{n,n} := \sup_{\| x^1, x^2, \ldots, x^n \| \neq 0} \frac{|f(x^1, x^2, \ldots, x^n)|}{\|x^1, x^2, \ldots, x^n\|} \quad \text{respectively}.
\]

Here, we shall consider the well-known sequence space \( l^p \), \( 1 \leq p < \infty \); where

\[
l^p = \left\{ x = (x_i)_{i=0}^{\infty} \mid \sum_{i=0}^{\infty} |x_i|^p < \infty \quad \text{and} \quad x_i \in K; i = 0, 1, 2, \ldots \right\}
\]

with norms

\[
\| x \|_p = \left( \sum_{i=0}^{\infty} |x_i|^p \right)^{1/p}
\]

and

\[
\| x \|_\infty = \sup_{0 \leq i < \infty} |x_i|.
\]

We know that \( (l^p, \| \cdot \|_p) \) is a Banach space whereas \( (l^p, \| \cdot \|_\infty) \) is not a Banach space.

In \([5]\), for our convenience and need, we have denoted the set of whole numbers as \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \) will also be written in the form of a sequence \( \mathbb{N} = (0, 1, 2, 3, \ldots) \) as well as in the form of \( n \)-consecutive terms notation as:

\[
\mathbb{N} = (nl, nl + 1, \ldots, nl + (n - 1))_{i=0}^{\infty}
\]

where “\( n \)” is fixed positive integer and refer to the integer “\( n \)” of \( n \)-normed space.

Taking \( \mathbb{N} = (\overline{m}_{nk}, \overline{m}_{nk+1}, \ldots, \overline{m}_{nk+(n-1)})_{i=0}^{\infty} \) as a rearrangement of the sequence \( \mathbb{N} \). In \([5]\), we have seen that \( (l^p, \| \cdot \|_p, \ldots, \| \cdot \|_p) \), \( 1 \leq p < \infty \) is an \( n \)-normed space, but not complete where

\[
\| x^1, x^2, \ldots, x^n \|_p = \sup \left\{ \left\| \overline{x}^1, \overline{x}^2, \ldots, \overline{x}^l \right\| : \overline{x}^1, \overline{x}^2, \ldots, \overline{x}^l \text{ are parallel rearrangements of } x^1, x^2, \ldots, x^n \text{ respectively} \right\}
\]

(1.3)

and
where their derived norms are non-equivalent. It is easy to check that the sequence\( f \) is bounded with respect to the linearly independent set\( \{e^1, \ldots, e^n\} \) are equivalent to\( \|\cdot\|_{\infty} \), where \( e^i = (\delta^i_{t=0}) \). For details see [5, 6, 7].

Here we shall consider the normed space \( (\mathbb{V}, \|\cdot\|_{\infty}) \) and \( n \)-normed space \( (\mathbb{V}, \|\cdot, \ldots, \cdot\|_{p}) \).

It is well known that, \( \mathbb{V} \subset C_0 \) and \( \|\cdot\|_{\infty} \leq \|\cdot\|_{p} \), by applying usual methods for finding dual spaces of sequence spaces we have the following Lemma:

**Lemma 1.2.** The dual space of \( (\mathbb{V}, \|\cdot\|_{\infty}) \) \( 1 \leq p < \infty \) is identified by \( (l^1, \|\cdot\|_{1}) \). Moreover, the mapping \( f \rightarrow (f(e^i))_{i=0}^{\infty} \) is a linear isometric bijection.

**Proof.** It is easy to check that the sequence \( (e^i)_{i=0}^{\infty} \) constitutes a Schauder basis for the space \( (\mathbb{V}, \|\cdot\|_{\infty}) \) also where \( e^i = (\delta^i_{t=0}) \); \( i = 0, 1, 2, \ldots \). Therefore, every \( x = (x_i)_{i=0}^{\infty} \in \mathbb{V} \) can be uniquely expressed as

\[
x = \sum_{i=0}^{\infty} x_i e^i.
\]

i.e. \( \|s_n - x\|_{\infty} \to 0 \) as \( n \to \infty \), where \( s_n = \sum_{i=0}^{n} x_i e^i \). Let \( f \) be bounded linear functional on \( (\mathbb{V}, \|\cdot\|_{\infty}) \) [It should be noted that \( f \) is bounded with respect to \( \|\cdot\|_{\infty} \).]
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Since \( f \) is continuous and \( s_n \to x \) it follows that \( f(s_n) \to f(x) \). Hence, \( f(x) \) can be uniquely expressed as

\[
f(x) = \sum_{i=0}^{\infty} x_i f(e^i).
\]

Now, we shall show that \( (f(e^i))_{i=0}^{\infty} \in (l^1, \| \cdot \|_1) \). Let \( n \in \mathbb{N} \) be arbitrary, define \( y^n = (y_i)_{i=0}^{\infty} \in l^p \) as follows

\[
y_i = \begin{cases} \frac{f(e^i)}{f(e^i)} & ; f(e^i) \neq 0 \text{ and } 0 \leq i \leq n \\ 0 & ; \text{otherwise} \end{cases}
\]

Obviously \( y^n \in l^p \) and

\[
\|y^n\|_\infty = \sup_{0 \leq i < \infty} |y_i| \leq 1
\]

(But \( \|y^n\|_p \leq (n + 1)^{1/p} \).) Now,

\[
f(y^n) = \sum_{i=0}^{\infty} y_i f(e^i) \leq \sum_{i=0}^{n} |f(e^i)| \leq \|f\| \cdot \|y^n\|_\infty \leq \|f\|.
\]

Thus, for all \( n \in \mathbb{N} \) we have \( \sum_{i=0}^{n} |f(e^i)| \leq \|f\| \) therefore

\[
\sum_{i=0}^{\infty} |f(e^i)| \leq \|f\|
\]

and hence

\[
(f(e^i))_{i=0}^{\infty} \in (l^1, \| \cdot \|_1).
\]

Now, define a mapping \( T : (l^p, \| \cdot \|_\infty)^* \to (l^1, \| \cdot \|_1) \) as follows

\[
T(f) = (f(e^i))_{i=0}^{\infty}
\]

where \( (l^p, \| \cdot \|_\infty)^* \) is dual space of \( (l^p, \| \cdot \|_\infty) \). Clearly \( T \) is well-defined and linear and from above it follows that \( T(f) = 0 \Rightarrow f = 0 \) therefore \( T \) is one-one.

To prove \( T \) is onto, let \( \lambda = (\lambda_i)_{i=0}^{\infty} \in (l^1, \| \cdot \|_1) \) and \( x = (x_i)_{i=0}^{\infty} \in l^p \) be arbitrary, clearly \( x \) is bounded in \( K \), therefore

\[
\sum_{i=0}^{\infty} |x_i \lambda_i| < \infty \quad \Rightarrow \quad \sum_{i=0}^{\infty} x_i \lambda_i < \infty.
\]

Define \( f : l^p \to K \) as

\[
f(x) = \sum_{i=0}^{\infty} x_i \lambda_i < \infty.
\]
Obviously, $f$ is linear and for every $x \in l^p$:

$$|f(x)| \leq \sum_{i=0}^{\infty} |x_i| |\lambda_i| \leq \left( \sum_{i=0}^{\infty} |\lambda_i| \right) \|x\|_\infty < \infty$$

i.e. $f$ is bounded and $T(f) = \left( f(e^i)^{\infty}_{i=0} = (\lambda_i)^{\infty}_{i=0} \right.$ and

$$\|f\| \leq \sum_{i=0}^{\infty} |f(e^i)|.$$ 

Moreover, above inequalities give

$$\|f\| = \sum_{i=0}^{\infty} |f(e^i)|.$$ 

Thus $T$ is a linear isometric bijection. □

**Remark**: Moreover, above Lemma 1.2 says that if $1 \leq p \leq q < \infty$ then

$$(l^p, \|\cdot\|) = (l^p, \|\cdot\|)^*$$

and

$$(c_0, \|\cdot\|) = (c, \|\cdot\|)^* = (l^p, \|\cdot\|)^* = (l^p, \|\cdot\|)^*.$$ 

**But it need not be true for** $$(l^p, \|\cdot\|)^*$$ **and** $$(l^p, \|\cdot\|)^*.$$ 

From [9, 10], we have the following results:

**Lemma 1.3.** Every bounded multilinear $n$-functional $f$ defined on the $n$-normed space $(X, \|\cdot\|)$ satisfies

1. $f(x^1, x^2, \ldots, x^n) = 0;$ whenever $x^1, x^2, \ldots, x^n$ are linearly dependent
2. $f(x^1, x^2, \ldots, x^n) = \text{sgn}(\sigma) f(x^{\sigma(1)}, x^{\sigma(2)}, \ldots, x^{\sigma(n)})$ for every $x^1, x^2, \ldots, x^n \in X$

for every permutation $\sigma$ of $(1, 2, \ldots, n)$ where $\text{sgn}(\sigma) = 1$ if $\sigma$ is an even permutation and $\text{sgn}(\sigma) = -1$ if $\sigma$ is an odd permutation.

**Lemma 1.4.** The norm of every bounded multilinear $n$-functional $f$ defined on an $n$-normed space $(X, \|\cdot\|)$ is given by:

$$\|f\|_{n,n} := \sup_{\|x^1, x^2, \ldots, x^n\| \neq 0} \frac{|f(x^1, x^2, \ldots, x^n)|}{\|x^1, x^2, \ldots, x^n\|}$$

or equivalently

$$\|f\|_{n,n} := \sup_{\|x^1, x^2, \ldots, x^n\| = 1} |f(x^1, x^2, \ldots, x^n)|.$$
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or equivalently

$$
\|f\|_{n,n} := \sup_{\|x^1, x^2, \ldots, x^n\| \leq 1} |f(x^1, x^2, \ldots, x^n)|
$$

or equivalently

$$
\|f\|_{n,n} := \inf \{ K : |f(x^1, x^2, \ldots, x^n)| \leq K \|x^1, x^2, \ldots, x^n\|, \text{ for every } x^1, x^2, \ldots, x^n \in X \}.
$$

A similar result can be obtained for bounded multilinear $n$-functional defined on a normed space.

2. Results

In [5, 7], we have already investigated the equivalence relations between different norms and $n$-norms defined on $l^p$ as

1. $\|x^1, x^2, \ldots, x^n\|_p \leq n! \|x^1\|_p \|x^2\|_p \cdots \|x^n\|_p$

2. $\|x^1, x^2, \ldots, x^n\|_p \leq (n!)^{1/p} \|x^1, x^2, \ldots, x^n\|_p$.

for every $x^1, x^2, \ldots, x^n \in l^p$; where the $n$-norm $\|., ., \|_p$ is defined by Gunawan [4] as

$$
\|x^1, x^2, \ldots, x^n\|_p = \left[ \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |\det(x^t_{j_1})|^p \right]^{1/p} \quad t = 1, 2, \ldots, n.
$$

The above relations give the following propositions:

**Proposition 2.1.** A bounded multilinear $n$-functional defined on $(l^p, \|., ., \|_p)$ is a bounded multilinear $n$-functional on $(l^p, \|., ., \|_p)$.

**Proposition 2.2.** A bounded multilinear $n$-functional defined on $(l^p, \|., ., \|_p)$ is a bounded multilinear $n$-functional on $(l^p, \|., ., \|_p)$.

In [10], Gunawan investigated the $n$-dual structure of the Banach space $(l^p, \|., ., \|_p)$ and $n$-Banach space $(l^p, \|., ., \|_p)$, respectively. In [7] we have investigated that the two $n$-normed spaces $(l^p, \|., ., \|_p)$ and $(l^p, \|., ., \|_p)$ are non-equivalent. Inspired by Gunawan [10] here, we shall investigate the $n$-dual spaces of the normed space $(l^p, \|., ., \|_{\infty})$ and $n$-normed space $(l^p, \|., ., \|_p)$, respectively.
We shall begin our investigations by finding the 2-dual structure of \( l^p \) with respect to norm \( \| \|_\infty \) and 2-norm \( \| \|_p \), respectively.

First of all, let us define the normed space \( (l^1_{NNN}, \| \|_2) \) of double indexed sequences as follows:

\[
\Theta := (\theta_{ij})_{i,j=0}^\infty \in l^1_{NNN} \quad \Theta_{ij} \in \mathbb{K} \quad \text{if and only if} \quad (2.1) \quad \| \Theta \|_2^2 = \sup_{\| x \|_\infty = 1} \left( \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} x_{ij} \theta_{ij} \right) < \infty, \quad \text{where} \quad x = (x_{ij})_{i,j=0}^\infty.
\]

and the normed space \( (l^A_{NNN}, \| \|_2^A) \) as follows:

\[
\Theta := (\theta_{ij})_{i,j=0}^\infty \in l^A_{NNN} \quad \Theta_{ij} \in \mathbb{K} \quad \text{if and only if} \quad (2.2) \quad \| \Theta \|_2^A = \sup_{\| x \|_p = 0} \left( \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} x_{ij} \theta_{ij}}{\| x \|_p} \right) < \infty \quad \text{and} \quad \theta_{ij} = -\theta_{ji}
\]

where \( x = (x_{ij})_{i,j=0}^\infty \) and \( y = (y_{ij})_{i,j=0}^\infty \).

**Theorem 2.1.** The 2-dual space of \((l^p, \| \|_\infty) ; 1 \leq p < \infty\) is identified by \( (l^1_{NNN}, \| \|_2^1) \). Moreover, the mapping \( f \rightarrow \Theta := (f(e^i, e^j))_{i,j=0}^\infty \) is an isometric linear bijection.

**Proof.** Let \( f \) be a bounded bilinear 2-functional on \((l^p, \| \|_\infty) \), then for every \( x = (x_{ij})_{i,j=0}^\infty \) and \( y = (y_{ij})_{i,j=0}^\infty \), \( f(x, y) \) can be expressed as

\[
(2.3) \quad f(x, y) = \sum_{j=0}^{\infty} y_j \sum_{i=0}^{\infty} x_i f(e^i, e^j)
\]

We shall first show that \((f(e^i, e^j))_{i,j=0}^\infty \in l^1_{NNN} \). Since for any arbitrary \( x \) in \( l^p \) with \( \| x \|_\infty = 1 \) the function \( f_x \) defined on \( l^p \) as \( f_x(y) = f(x, y) \) is bounded linear functional on \((l^p, \| \|_\infty) \), that is

\[
(2.4) \quad \| f_x(y) \| = \| f(x, y) \| \leq \| f \|_{l^2,1} \| x \|_\infty \| y \|_\infty = \| f \|_{l^2,1} \| y \|_\infty
\]

Therefore by lemma 1.2, \( f_x \) can be identified as \( f_x \equiv (f_x(e^i))_{i=0}^\infty \) with norm \( \| f_x \| = \sum_{j=0}^{\infty} \| f(x, e^j) \| = \sum_{j=0}^{\infty} |f(x, e^j)| = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} x_i f(e^i, e^j)|, \) as well as \( \| f_x \| = \sup \{ \| f_x(y) \| : \| y \|_\infty = 1 \} \), therefore from \((2.4)\), we have

\[
\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} x_i f(e^i, e^j) \leq \| f \|_{l^2,1}, \quad \text{for every arbitrary} \quad \| x \|_\infty = 1.
\]

Which shows that \( \Theta := (f(e^i, e^j))_{i,j=0}^\infty \in l^1_{NNN} \) with

\[
(2.5) \quad \| \Theta \|_2^1 = \| f(e^i, e^j) \|_2^1 \leq \| f \|_{l^2,1}.
\]
Theorem 2.2. The 2-dual space of the 2-normed space $l^2_{\text{NNX}}$ such that $T(f) = (f(e', e'))_{i,j=0}^{\infty}$ then obviously, $T$ is well defined and linear. From (2.3), it is clear that $T$ is zero function, whenever $T(f) = O$, thus $T$ is one-one.

Next, let $\Theta := (\theta_{ij})_{i,j=0}^{\infty} \in l^1_{\text{NNX}}$ is arbitrary, for $x = (x_i)_{i=0}^{\infty}$ and $y = (y_i)_{i=0}^{\infty}$ define $f : \ell^p \times \ell^p \rightarrow K$ as follows:

$$f(x, y) = \sum_{j=0}^{\infty} y_j \sum_{i=0}^{\infty} x_i \theta_{ij};$$

obviously $f(e', e') = \theta_{ij}$. For $x, y \in \ell^p$ with $||x||_{\infty} = 1$ and $||y||_{\infty} = 1$, we have

$$|f(x, y)| \leq \sum_{j=0}^{\infty} |y_j| \sum_{i=0}^{\infty} x_i \theta_{ij} \leq \|y\|_{\infty} \sum_{j=0}^{\infty} |y_j| \sum_{i=0}^{\infty} x_i \theta_{ij} \leq ||\Theta||_2^1.$$

Therefore for every $||x||_{\infty} \neq 0$ and $||y||_{\infty} \neq 0$

$$\frac{|f(x, y)|}{||x||_{\infty} ||y||_{\infty}} \leq ||\Theta||_2^1$$

or equivalently,

$$|f(x, y)| \leq ||\Theta||_2^1 ||x||_{\infty} ||y||_{\infty}$$

which exhibits that $f$ is bounded bilinear 2-functional on $(\ell^p, ||.||_{\infty})$ and $T(f) = \Theta$ with

$$||f||_{2,1} \leq ||\Theta||_2^1 = ||f(e', e')||_2^1.$$

From (2.5) and (2.7) it is clear that, $T$ is isometric linear bijection. □

**Theorem 2.2.** The 2-dual space of the 2-normed space $(\ell^p, ||.||_{\infty})$ is identified as $(l^A_{\text{NNX}}, ||.||_2^A)$. Moreover, the mapping $f \rightarrow \Theta := (f(e', e'))_{i,j=0}^{\infty}$ is an isometric linear bijection.

**Proof.** Since $||x, y||_p \leq 2||x||_p ||y||_p$ see[5], therefore $f(x, y)$ can be expressed as

$$f(x, y) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_j x_i f(e', e').$$

Since $f$ is bounded therefore

$$|f(x, y)| = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_j x_i f(e', e') \leq ||f||_{2,1} ||x, y||_p.$$

Defining $\Theta := (f(e', e'))_{i,j=0}^{\infty}$ above equation exhibits that $\Theta := (f(e', e'))_{i,j=0}^{\infty} \in l^A_{\text{NNX}}$ and

$$||\Theta||_2^A = ||f(e', e')||_2^A \leq ||f||_{2,2}.$$

Now for any arbitrary $\Theta := (\theta_{i,j})_{i,j=0}^{\infty} \in l_{N\times N}^A$ define bilinear functional

$$f(x,y) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} y_{xj} \theta_{i,j};$$

it is easy to show that $f$ is bounded bilinear functional on $\left(\ell^p, \|\|_p\right)$ with $f(e^i, e^j) = \delta_{ij}$ and $\|f\|_2 \leq \|\Theta\|_2 = \|f(e^i, e^j)\|_2^2$.

Now proceeding as in theorem 2.1, we have the result. □

To achieve the $n$-dual spaces of $(\ell^p, \|\|_p)$; $1 \leq p < \infty$ and $(\ell^p, \|\|_p)$, let us generalize the definitions of $\left(l_{N\times N}^1, \|\|_1\right)$ and $\left(l_{N\times N}^A, \|\|_2\right)$ to following normed space of $n$-indexed sequence spaces as follows:

The normed space $\left(l_{N\times n}^1, \|\|_1\right)$ of $n$-indexed sequences $\Theta := (\theta_{i_1,\ldots,i_n})_{i_1=0}^{\infty}$ with $\theta_{i_1,\ldots,i_n} \in \mathbb{K}$ as follows:

$$\Theta := (\theta_{i_1,\ldots,i_n}) \in l_{N\times n}^1; \quad \text{if and only if} \quad \|\Theta\|_n = \sup_{\|x\|_n} \left(\sum_{i_0=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} x_{i_0},\ldots,x_{i_{n-1}} \theta_{i_1,\ldots,i_n}\right) < \infty,$$

(2.10)

and the normed space $\left(l_{N\times n}^A, \|\|_2\right)$ of $n$-indexed sequences $\Theta := (\theta_{i_1,\ldots,i_n})_{i_1=0}^{\infty}$ with $\theta_{i_1,\ldots,i_n} \in \mathbb{K}$ as follows:

$$\Theta := (\theta_{i_1,\ldots,i_n}) \in l_{N\times n}^1; \quad \text{if and only if} \quad \theta_{i_1,\ldots,i_n} = sgn(\sigma) \theta_{\sigma(i_1),\ldots,\sigma(i_n)} \quad \text{and} \quad \|\Theta\|_A = \sup_{\|x\|_{n} \neq 0} \left(\sum_{i_0=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} x_{i_0},\ldots,x_{i_{n-1}} \theta_{i_1,\ldots,i_n}\right) < \infty,$$

(2.11)

and for every permutation $\sigma$ of $(i_1, i_2, \ldots, i_n)$ where $sgn(\sigma) = 1$ if $\sigma$ is an even permutation and $sgn(\sigma) = -1$ if $\sigma$ is an odd permutation.

**Theorem 2.3.** The $n$-dual space of $(\ell^p, \|\|_p)$ is identified by $\left(l_{N\times n}^1, \|\|_1\right)$. Moreover, the mapping $f \to \Theta := (f(e^1,\ldots,e^n))_{i_1,\ldots,i_n=0}^{\infty}$ is an isometric linear bijection.

**Proof.** The proof is similar to case $n=2$. For any $x_1, x_2, \ldots, x_n \in \ell^p$; $x^t = (x^t_{i_j})_{i_j=0}^{\infty}$; $1 \leq t \leq n$ the bounded multilinear $n$-functional $f(x^1, x^2, \ldots, x^n)$ can be expressed as

$$f(x^1, x^2, \ldots, x^n) = \sum_{i_0=0}^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} x_{i_0} x_{i_1} \cdots x_{i_{n-1}} f(e^1,\ldots,e^n).$$
The n-dual space of the n-normed space

First of all we shall show that \( (f(e^i, e^j, \ldots, e^n))_{\mathbf{N}^n} \in l_1^{n} \). To do this we shall use mathematical induction on \( n \). For \( n=2 \), we have already showed it. Let us assume that it is true for \( n-1 \), we have to prove it for \( n \). Let \( f \) be bounded multilinear \( n \)-functional and \( x^1 \in l^p \) with \( \|x^1\|_\infty = 1 \), if we define \( f_{x^1} : l^p \times l^p \times \cdots \rightarrow \mathbb{K} \) as
\[
f_{x^1}(x^2, \ldots, x^n) = f(x^1, x^2, \ldots, x^n),
\]
then \( f_{x^1} \) is bounded multilinear \((n-1)\)-functional on \( l^p \) and
\[
|f_{x^1}(x^2, \ldots, x^n)| = |f(x^1, x^2, \ldots, x^n)| \leq \|f\|_{l_1} \|x^2\|_{\infty} \cdots \|x^n\|_{\infty};
\]
which implies that
\[
\|f_{x^1}\|_{l_{n-1}} \leq \|f\|_{l_1}
\]
therefore it can be identified by \((f_{x^1}(e^i, \ldots, e^n)) \in l_1^{n-1} \) and
\[
\|f_{x^1}\|_{l_{n-1}} = \sup_{\|x^2\|_{\infty} \cdots \|x^n\|_{\infty}} \left( \sum_{i_k=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \sum_{i_1=0}^{\infty} x^2_{i_1} \cdots x^n_{i_{n-1}} f_{x^1}(e^{i_1}, \ldots, e^{i_{n-1}}, e^i) \right).
\]
That is,
\[
\sup_{\|x^2\|_{\infty} \cdots \|x^n\|_{\infty}} \left( \sum_{i_k=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \sum_{i_1=0}^{\infty} x^1_{i_1} x^2_{i_2} \cdots x^n_{i_{n-1}} f(e^{i_1}, \ldots, e^{i_{n-1}}, e^i) \right) \leq \|f\|_{l_1}
\]
for every arbitrary \( \|x^1\|_{\infty} = 1 \) therefore
\[
(2.12) \quad \|f(e^i, e^j, \ldots, e^n)\|_n \leq \|f\|_{l_1}.
\]
Thus \((f(e^i, e^j, \ldots, e^n))_{\mathbf{N}^n} \in l_1^{n} \). Rest part is similar to the case \( n=2 \). □

**Theorem 2.4.** The n-dual space of the n-normed space \( (\ell^p \prod_{i=1}^{n} \mathbb{K}) \) is identified by \((l_1^{\mathbf{N}^n}, \|\cdot\|_n)\). Moreover, the mapping \( f \rightarrow \Theta := (f(e^1, \ldots, e^n))_{\mathbf{N}^n} \) is an isometric linear bijection. 

**Proof.** The proof is similar to the proof of theorem 2.2. □

**REFERENCES**


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