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GENERALIZED η-RICCI SOLITONS ON TRANS-SASAKIAN MANIFOLDS ASSOCIATED TO THE SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In this paper, we study generalized η-Ricci solitons with respect to the Schouten-van Kampen connection on trans-Sasakian manifolds. We give an example of generalized η-Ricci solitons on a trans-Sasakian manifold with respect to the Schouten-van Kampen connection to prove our results.
Keywords: manifolds, vector field, generalized Ricci solutions.

1. Introduction

The trans-Sasakian manifold was introduced by Oubina [37] as a class of almost contact metric manifolds. Later, Blair and Oubina [10] obtaned some properties of this manifolds. A trans-Sasakian manifold is usually denoted by $(M, \varphi, \xi, \eta, g, \sigma, \theta)$, where both σ and θ are smooth functions on M and (φ, ξ, η, g) is an almost contact metric structure. In this case, it is said to be of type (σ, θ). A trans-Sasakian manifold of type $(0, 0)$, $(0, \theta)$ and $(\sigma, 0)$ are cosymplectic, θ-Kenmotsu [1, 30, 36, 48] and σ-Sasakian [31], respectively. In [18, 19, 20, 21, 22, 23, 28, 34, 35, 49], the authors studied compact trans-Sasakian manifolds with some restrictions on the smooth functions σ, θ and the vector field ξ appearing in their definition for getting conditions under which a trans-Sasakian manifold is homothetic to a Sasakian manifold. In addition, in [43, 44, 49], interesting results on the geometry of trans-Sasakian manifolds are obtained.

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Hamilton [25] introduced the concept of Ricci flow on a Riemannian manifold as follows:

$$\frac{\partial}{\partial t} g = -2S$$

where S is the Ricci tensor of a manifold. A self-similar solution to the Ricci flow is called a Ricci soliton which is a generalization of Einstein metric. A Ricci soliton [25] is a triplet (g, V, λ) on a pseudo-Riemannian manifold M such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\mathcal{L}_V$ is the Lie derivative in direction of the potential vector field V, S is the Ricci tensor, and λ is a real constant. Ricci solitons are important in physics and are often referred as quasi-Einstein [12, 13]. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively. If the vector field V is the gradient of a potential function ψ, that is, $V = \nabla\psi$, then g is called a gradient Ricci soliton. In 2016, Nurowski and Randall [33] introduced the concept of generalized Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat - 2\alpha S - 2\lambda g = 0, \tag{1.2}$$

where $V^\flat$ is the canonical 1-form associated to V. Also, as a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [16] which it is a 4-tuple (g, V, λ, ρ), where V is a vector field on M, λ and ρ are constants, and g is a pseudo-Riemannian metric satisfying the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. Many authors studied the η-Ricci solitons [5, 6, 7, 26, 29, 38, 42]. In particular, if $\rho = 0$, then the η-Ricci soliton equation becomes the Ricci soliton equation. Motivated by the above works M. D. Siddiqi [40] introduced the notion of generalized η-Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0. \tag{1.4}$$

Motivated by [2, 3, 11, 32] and the above works, we study generalized η-Ricci solitons on trans-Sasakian manifolds assoicated to the Schouten-van Kampen connection. We give an example of generalized η-Ricci soliton on a trans-Sasakian manifold with respect to the Schouten-van Kampen connection.

The paper is organized as follows. In Section 2, we recall some necessary and fundamental concepts and formulas on trans-Sasakian manifolds which be used throughout the paper. In Section 3, we give the main results and their proofs. In Section 4, we give an example of trans-Sasakian admitting the generalized η-Ricci solitons with respect to the Schouten-van Kampen connection.

2. Preliminaries

Let M be a $(2n + 1)$-dimensional manifold, φ be a $(1, 1)$-tensor field, ξ be a vector field, η be a 1-form, and g be a compatible Riemannian metric on M such that

$$\varphi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y. Then manifold (M, g) is called an almost contact metric manifold [8, 9] with an almost contact structure (φ, ξ, η, g). In this case, we have $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $g(X, \varphi Y) = -g(\varphi X, Y)$, and $\eta(X) = g(X, \xi)$. The fundamental 2-form Φ of M is given by

$$\Phi(X,Y) = g(X, \varphi Y),$$

for all vector fields X, Y. An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called trans-Sasakian manifold [37] if $(M \times \mathbb{R}, J, G)$ belong to the class W_4 [24], where J is the almost complex structure on $M \times \mathbb{R}$ given by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

for all vector field X on M, smooth function f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [10]

$$(\nabla_X \varphi)Y = \sigma\left(g(X,Y)\xi - \eta(Y)X\right) + \theta\left(g(\varphi X, Y)\xi - \eta(Y)\varphi X\right), \tag{2.3}$$

for all vector fields X, Y, for some smooth functions σ, θ on M. In this case, we say that the trans-Sasakian structure is of type (σ, θ). By virtue of (2.3), we have

$$\nabla_X \xi = -\sigma\varphi X + \theta(X - \eta(X)\xi), \tag{2.4}$$

$$(\nabla_X \eta)Y = -\sigma g(\varphi X, Y) + \theta g(\varphi X, \varphi Y), \tag{2.5}$$

for all vector fields X, Y. Using (2.3) and (2.4), we have

$$2\sigma\theta + \xi(\sigma) = 0, \tag{2.6}$$

$$\varphi(\nabla\sigma) = 2n\nabla\theta. \tag{2.7}$$

Further, we have the following relations [17]

$$\begin{aligned}
R(X,Y)\xi &= (\sigma^2 - \theta^2)(\eta(Y)X - \eta(X)Y) + 2\sigma\theta(\eta(Y)\varphi X - \eta(X)\varphi Y) \\
&\quad +(Y(\sigma))\varphi X - (X(\sigma))\varphi Y + (Y(\theta))\varphi^2 X - (X(\theta))\varphi^2 Y,
\end{aligned}\tag{2.8}$$

$$R(X,\xi)\xi = (\sigma^2 - \theta^2 - \xi(\theta))\{X - \eta(X)\xi\}, \tag{2.9}$$

$$\begin{aligned}
R(\xi,X)Y &= (\sigma^2 - \theta^2)\{g(X,Y)\xi - \eta(Y)X\} + (Y(\theta))\{X - \eta(X)\xi\} \\
&\quad +2\sigma\theta\{g(\varphi Y, X)\xi + \eta(Y)\varphi X) + (Y(\sigma))\varphi X \\
&\quad +g(\varphi Y, X)\nabla\sigma - g(\varphi X, \varphi Y)\nabla\theta,
\end{aligned}\tag{2.10}$$

for all vector fields X, Y, where R is the Riemannian curvature tensor. From (2.8) and definition of the Ricci tensor S of a trans-Sasakian manifold M we also get

$$S(X,\xi) = \left(2n(\sigma^2 - \theta^2) - \xi(\theta)\right)\eta(X) - (2n-1)X(\theta) - (\varphi X)\sigma, \tag{2.11}$$

for all vector field X.

Let M be an almost contact metric manifold and TM be the tangent bundle of M. We have two naturally defined distribution on tangent bundle TM as follows

$$H = \ker\eta, \qquad \hat{H} = \operatorname{span}\{\xi\}, \tag{2.12}$$

thus we get $TM = H \oplus \hat{H}$. Therefore, by this composition we can define the Schouten-van Kampen connection $\bar{\nabla}$ [4, 41] on M with respect to Levi-Civita connection ∇ as follows

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + ((\nabla_X \eta)(Y))\xi \tag{2.13}$$

for all vector fields X, Y. From [41] we have

$$\bar{\nabla}\xi = 0, \qquad \bar{\nabla} g = 0, \qquad \bar{\nabla}\eta = 0, \tag{2.14}$$

and the torsion $\bar{T}$ of $\bar{\nabla}$ is given by

$$\bar{T}(X, Y) = \eta(X)\nabla_X \xi - \eta(X)\nabla_Y \xi + 2d\eta(X, Y)\xi, \tag{2.15}$$

for all vector fields X, Y. Let $\bar{R}$ and $\bar{S}$ be the curvature tensors and the Ricci tensors of the connection $\bar{\nabla}$, respectively. From [27] on a trans-Sasakian we have

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma\{\eta(Y)\varphi X - g(\varphi X, Y)\xi\} - \theta\{\eta(Y)X - g(X, Y)\xi\} \tag{2.16}$$

and

$$\begin{aligned} \bar{S}(X, Y) &= S(X, Y) - (2n-2)\sigma\theta g(\varphi X, Y) + \{\xi(\theta) + 2n\theta^2\}g(X, Y) \\ &\quad -2\sigma^2\eta(X)\eta(Y) + \{(\varphi X)\sigma + (2n-1)(X\theta)\}\eta(Y), \end{aligned} \tag{2.17}$$

for all vector fields X, Y, where S denotes the Ricci tensor of the connection ∇. Hence,

$$\bar{S}(X, \xi) = 0, \tag{2.18}$$

$$\bar{S}(\xi, X) = (2n-1)\{(\xi\theta)\eta(X) - X\theta\} - (\phi X)\sigma. \tag{2.19}$$

From (2.16), we get

$$\begin{aligned} \overline{\mathcal{L}}_V g(X, Y) &= g(\bar{\nabla}_X V, Y) + g(X, \bar{\nabla}_Y V) \\ &= \mathcal{L}_V g(X, Y) - \sigma g(\varphi X, V)\eta(Y) - \sigma g(\varphi Y, V)\eta(X) \\ &\quad -2\theta\eta(V)g(X, Y) + \theta g(X, V)\eta(Y) + \theta g(Y, V)\eta(X), \end{aligned}$$

for all vector fields X, Y, V, where $\overline{\mathcal{L}}_V g$ denotes the Lie derivative of g along the vector field V with respect to $\bar{\nabla}$. Using (2.17), the Ricci operator $\bar{Q}$ of the connection $\bar{\nabla}$ is determined by

$$\bar{Q}X = QX - (2n-2)\sigma\theta\varphi X + \{\xi(\theta) + 2n\theta^2\}X - 2\sigma^2\eta(X)\xi + \{(\varphi X)\sigma + (2n-1)(X\theta)\}\xi, \tag{2.20}$$

for any vector field X. Let r and $\bar{r}$ be the scalar curvature of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\bar{\nabla}$. The equation (2.17) yields

$$\bar{r} = r + 2n\{2(\xi\theta) - \sigma^2 + (2n+1)\theta^2\}. \tag{2.21}$$

The generalized η-Ricci soliton associated to the Schouten-van Kampen connection is defined by

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta\otimes\eta + \lambda g = 0, \tag{2.22}$$

where $\bar{S}$ denotes the Ricci tensor of the connection $\bar{\nabla}$,

$$(\overline{\mathcal{L}}_V g)(Y,Z) := g(\bar{\nabla}_Y V, Z) + g(Y, \bar{\nabla}_Z V),$$

$V^\flat$ is the canonical 1-form associated to V that is $V^\flat(X) = g(V,X)$ for all vector field X, λ is a smooth function on M, and α, β, μ, ρ are real constants such that $(\alpha, \beta, \mu) \neq (0,0,0)$.

The generalized η-Ricci soliton equation reduces to

(1) the η-Ricci soliton equation when $\alpha = 1$ and $\mu = 0$,

(2) the Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$,

(3) the generalized Ricci soliton equation when $\rho = 0$.

3. Main results and their proofs

A trans-Sasakian manifold is said to η-Einstein if its Ricci tensor S is of the form

$$S = ag + b\eta\otimes\eta,$$

where a and b are smooth functions on manifold. Let M be a trans-Sasakian manifold. Now, we consider M satisfies the generalized η-Ricci soliton (2.22) associated to the Schouten-van Kampen connection and the potential vector field V is a pointwise collinear vector field with the structure vector field ξ, that is, $V = \gamma\xi$ for some function γ on M. Using (2.4) we get

$$\begin{aligned} \overline{\mathcal{L}}_{\gamma\xi} g(X,Y) &= \mathcal{L}_{\gamma\xi} g(X,Y) - 2\gamma\theta\,(g(X,Y) - \eta(X)\eta(Y)) \\ &= X(\gamma)\eta(Y) + Y(\gamma)\eta(X), \end{aligned} \tag{3.1}$$

for all vector fields X, Y. Also, we have

$$\xi^\flat \otimes \xi^\flat(X,Y) = \eta(X)\eta(Y), \tag{3.2}$$

for all vector fields X, Y. Applying $V = \gamma\xi$, (2.17), (3.1), and (3.2) in the equation (2.22) we infer

$$\alpha\bar{S}(X,Y) + \frac{\beta}{2}X(\gamma)\eta(Y) + \frac{\beta}{2}Y(\gamma)\eta(X) + (\mu\gamma^2 + \rho)\eta(X)\eta(Y) + \lambda g(X,Y) = 0, \tag{3.3}$$

for all vector fields X, Y. We plug $Y = \xi$ in the above equation and using (2.18) to yield

$$\frac{\beta}{2}X(\gamma) + \frac{\beta}{2}\xi(\gamma)\eta(X) + (\mu\gamma^2 + \rho + \lambda)\eta(X) = 0. \tag{3.4}$$

Taking $X = \xi$ in (3.4) gives

$$\beta\xi(\gamma) = -(\mu\gamma^2 + \rho + \lambda). \tag{3.5}$$

Inserting (3.5) in (3.4), we conclude

$$\beta X(\gamma) = -(\mu\gamma^2 + \rho + \lambda)\eta(X), \tag{3.6}$$

which yields

$$\beta d\gamma = -(\mu\gamma^2 + \rho + \lambda)\eta. \tag{3.7}$$

Applying (3.7) in (3.3) we obtain

$$\alpha\bar{S}(X, Y) = \lambda(-g(X, Y) + \eta(X)\eta(Y)), \tag{3.8}$$

which implies $\alpha\bar{r} = -2n\lambda$. We plug $Y = \xi$ in the equation (3.8) and using (2.19) to obtain

$$(2n-1)\{(\xi\theta)\eta(X) - X\theta\} - (\phi X)\sigma = 0, \tag{3.9}$$

for any vector field X. Using (2.7) we have $\xi\theta = 0$ then $(2n-1)X\theta = -(\phi X)\sigma$. This conclude that $\nabla\theta = 0$. Thus θ is a constant and $\varphi(\nabla\sigma) = 0$. Therefore, this leads to the following:

Theorem 3.1. *Let $(M, g, \varphi, \xi, \eta)$ be a trans-Sasakian and it admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $\alpha \neq 0$ and $V = \gamma\xi$ for some smooth function γ on M, then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection. Also, θ is a constant and $\varphi(\nabla\sigma) = 0$*

From (3.8) we also have the following:

Corollary 3.1. *Let $(M, g, \varphi, \xi, \eta)$ be a trans-Sasakian 3-dimensional manifold. If M admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $V = \gamma\xi$ for some smooth function γ on M, then $\alpha\bar{r} = -2n\lambda$.*

Now, let M be an η-Einstein quasi-Sasakian 3-dimensional manifold with respect to the Schouten-van Kampen connection and $V = \xi$. Then we get $\bar{S} = ag + b\eta \otimes \eta$ for some functions a and b on M. We have $\overline{\mathcal{L}}_\xi g = 0$, then

$$\begin{aligned} &\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_\xi g + \mu\xi^\flat \otimes \xi^\flat + \rho\eta \otimes \eta + \lambda g \\ &= a\alpha g + b\alpha\eta \otimes \eta + \mu\eta \otimes \eta + \rho\eta \otimes \eta + \lambda g \\ &= (a\alpha + \lambda)g + (b\alpha + \mu + \rho)\eta \otimes \eta. \end{aligned}$$

From the above equation M admits a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection if $\lambda = -a\alpha$ and $\rho = -b\alpha - \mu$.

Hence, we can state the following theorem:

Theorem 3.2. *Suppose that M is a η-Einstein trans-Sasakian manifold with respect to the Schouten-van Kampen connection, that is, $\bar{S} = ag + b\eta \otimes \eta$ for some constants a and b on M. Then manifold M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -b\alpha - \mu, -a\alpha)$ with respect to the Schouten-van Kampen connection.*

Definition 3.1. A vector field V is said to a conformal Killing vector field with respect to the Schouten-van Kampen connection if

$$(\overline{\mathcal{L}}_V g)(X, Y) = 2hg(X, Y), \tag{3.10}$$

for all vector fields X, Y, where h is some function on M. The conformal Killing vector field V is called

- proper when h is not constant,
- homothetic vector field when h is a constant,
- Killing vector field when $h = 0$.

Let vector field V is a conformal Killing vector field and satisfies in (3.10). By (3.10), (2.17), and (2.22) we have

$$\alpha\bar{S}(X, Y) + \beta hg(X, Y) + \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X, Y) = 0. \tag{3.11}$$

for all vector fields X, Y. By inserting $Y = \xi$ in the above equation we get

$$g(\beta h\xi + \mu\eta(V)V + \rho\xi + \lambda\xi, X) = 0. \tag{3.12}$$

Since X is arbitraray vector field we have the following theorem.

Theorem 3.3. *If the metric g of a trans-Sasakian manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection where V is conformally Killing vector field, that is $\mathcal{L}_V g = 2hg$ then*

$$(\beta h + \rho + \lambda)\xi + \mu\eta(V)V = 0. \tag{3.13}$$

Definition 3.2. A nonvanishing vector field V on pseudo-Riemannian manifold (M, g) is called torse-forming [46] if

$$\nabla_X V = fX + \omega(X)V, \tag{3.14}$$

for any vector field X, where ∇ is the Levi-Civita connection of g, f is a smooth function and ω is a 1-form. The vector field V is called

- concircular [15, 45] whenever in the equation (3.14) the 1-form ω vanishes identically,
- concurrent [39, 47] if in equation (3.14) the 1-form ω vanishes identically and $f = 1$,

- parallel vector field if in equation (3.14) $f = \omega = 0$,
- torqued vector field [14] if in equation (3.14) $\omega(V) = 0$.

Let $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ be a generalized η-Ricci soliton on a trans-Sasakian manifold with respect to the Schouten-van Kampen connection where V is a torse-forming vector filed with respect to the Schouten-van Kampen connection that is, $\bar{\nabla}_X V = fX + \omega(X)V$. Then

$$\alpha\bar{S}(X,Y) + (\overline{\mathcal{L}}_V g)(X,Y) + \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0, \tag{3.15}$$

for all vector fields X, Y. On the other hand,

$$(\overline{\mathcal{L}}_V g)(X,Y) = 2fg(X,Y) + \omega(X)g(V,Y) + \omega(Y)g(V,X), \tag{3.16}$$

for all vector fields X, Y. Applying (3.16) into (3.15) we arrive at

$$\begin{aligned} &\alpha\bar{S}(X,Y) + [\beta f + \lambda]\, g(X,Y) + \rho\eta(X)\eta(Y) \\ &+\frac{\beta}{2}\,[\omega(X)g(V,Y) + \omega(Y)g(V,X)] + \mu g(V,X)g(V,Y) = 0. \end{aligned} \tag{3.17}$$

We take contraction of the above equation over X and Y to obtain

$$\alpha\bar{r} + (2n+1)\,[\beta f + \lambda] + \rho + \beta\omega(V) + \mu|V|^2 = 0. \tag{3.18}$$

Therefore we have the following theorem.

Theorem 3.4. *If the metric g of a trans-Sasakian manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection where V torse-forming vector filed and satisfied in $\bar{\nabla}_X V = fX + \omega(X)V$, then*

$$\lambda = -\frac{1}{2n+1}\left[\alpha\left(r + 2n\{2(\xi\theta) - \sigma^2 + (2n+1)\theta^2\}\right) + 2n + \rho + \beta\omega(V) + \mu|V|^2\right] - \beta f. \tag{3.19}$$

4. Example

In this section, we give an example of trans-Sasakian manifold with respect to the Schouten-van Kampen connection.

Example 4.1. Let (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and $M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}$. We consider the linearly independent vector fields

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y}, \qquad e_3 = z\frac{\partial}{\partial z}.$$

We define the metric g by $g(e_i, e_j) = 1$ if $i = j$ and $i, j \in \{1, 2, 3\}$ and otherwise $g(e_i, e_j) = 0$.

We define an almost contact structure (φ, ξ, η) on M by

$$\xi = e_3, \quad \eta(X) = g(X, e_3), \quad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for all vector field X. Note the relations $\varphi^2(X) = -X + \eta(X)\xi$, $\eta(\xi) = 1$, and $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ hold. Thus $(M, \varphi, \xi, \eta, g)$ defines an almost contact structure on M. We have

$[,]$	e_1	e_2	e_3
e_1	0	0	-e_1
e_2	0	0	-e_2
e_3	e_1	e_2	0

The Levi-Civita connection ∇ of M is determined by

$$\nabla_{e_i} e_j = \begin{pmatrix} e_3 & 0 & -e_1 \\ 0 & e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the structure (φ, ξ, η) is a trans-Sasakian structure with $\sigma = 0$ and $\theta = -1$. Now, using (2.16) we get the Schouten-van- Kampen connection on M as $\bar{\nabla}_{e_i} e_j = 0$ for $1 \leq i, j \leq 3$. Hence $\bar{S} = 0$ If we consider $V = \xi$ then $\overline{\mathcal{L}}_V g = 0$. Therefore $(g, \xi, \alpha, \beta, \mu, \rho = -\mu, \lambda = 0)$ is a generalized η-Ricci soliton on manifold M.

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- Availability of data and materials
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- Authors' contributions
  The author read and approved the final manuscript.

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