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ON SOME GENERALIZED DEFERRED STATISTICAL CONVERGENCE OF ORDER $\alpha\beta$ FOR FUZZY VARIABLE SEQUENCES IN CREDIBILITY SPACE

Ekrem Savaş[1], Ömer Kişi[2] and Mehmet Gürdal[3]

[1]**Uşak University, Department of Mathematics, Turkey**
[2]**Bartın University, Department of Mathematics, Turkey**
[3]**Süleyman Demirel University, Department of Mathematics, Turkey**

Abstract. In this paper, we investigate the concepts of deferred statistical convergence of order $\alpha\beta$ and strongly s-deferred Cesàro summability of order $\alpha\beta$ for fuzzy variable sequences in credibility space. Furthermore, the conditions of deferred statistical convergence almost surely of order $\alpha\beta$, deferred statistical convergence in credibility of order $\alpha\beta$, deferred statistical convergence in mean of order $\alpha\beta$, deferred statistical convergence in distribution of order $\alpha\beta$, and deferred statistical convergence uniformly almost surely of order $\alpha\beta$ of fuzzy variable sequences have been examined. We have proved relations between these notions.
Keywords: Statistical convergence, Deferred statistical convergence, fuzzy variable sequences.

1. Introduction

The theory of fuzzy sets was originally proposed by Zadeh through membership degree function in 1965 [55]. Fuzzy theory can be used in an exhaustive variety of real problems. For example, possibility theory has been advanced by several researchers, such as Dubois and Prade [8], Nahmias [37]. A fuzzy variable is a function from a credibility space (demonstrated with the credibility measure) to

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Corresponding Author: Ömer Kişi, Bartın University, Department of Mathematics, 74100 Bartın, Turkey | E-mail: okisi@bartin.edu.tr
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the set of real numbers. The convergence of fuzzy variables is important component of credibility theory, which can be applied into real problems in mathematical finance and engineering. Fuzzy variable, possibility distribution and membership function were presented by Kaufmann [19]. Possibility measure, which is usually defined as supremum preserving set function on the power set of a nonempty set, is a fundamental concept in possibility theory but it is not self-dual. Since a self-dual measure is absolutely required in both theory and practice, Liu and Liu [26] investigated a self-duality credibility measure. The credibility measure plays the role of possibility measure in fuzzy world because it shares some basic properties with possibility measure. Since Liu has begun the survey of credibility theoryn many specific contents have been studied (see [20, 21, 23, 24, 28, 29, 43]). Considering the fact that sequence convergence plays a key role in credibility theory, Liu [27] put forward four kinds of convergence concept for fuzzy variables: convergence in credibility, convergence almost surely, convergence in mean, convergence in distribution. In addition, based on credibility theory, some convergence properties of credibility distribution for fuzzy variables were studied by Jiang [16] and Ma [31].

Wang and Liu [50] established the relationships among convergence in mean, convergence in credibility, convergence almost uniformly, convergence in distribution, and convergence almost surely. Besides, several researchers presented convergence notions in classical measure theory, credibility theory, probability theory, and obtained the connections between them. The concerned readers may examine Chen et al. [5], Lin [25], Liu and Wang [30], Xia [51], You [52], and You et al. [54].

Statistical convergence was first presented by Fast [11] and Steinhaus [45] as a generalization of ordinary convergence for real sequences. Statistical convergence turned out to be one of the most active areas of research in the summability theory after the works of Fridy [12] and Šalát [41]. Statistical convergence has also been worked in more general abstract spaces such as the fuzzy number space [38]. More investigations in this direction and more applications of statistical convergence can be seen in [3, 14, 15, 18, 32, 33, 34, 35, 39, 40, 42, 48, 49]. Also, the readers should refer to the monographs [2], and [17], and recent papers [36] and [46] for the background on the sequence spaces.

In 1932, Agnew [1] investigated deferred Cesàro mean by modifying Cesàro mean to obtain more useful methods including stronger features which do not belong to nearly all methods. Kücükaslan and Yilmaztürk [22] came up with the idea of combining the deferred Cesàro mean and the concept of statistical convergence. This gave them the opportunity to generalize both strong s-Cesàro summability and statistical convergence with the sense of deferred Cesàro mean.

This paper is devoted to present a new kind of convergence for fuzzy variables sequences. In Section 2, some preliminary definitions and theorems related to fuzzy variables sequences, credibility space and statistical convergence are presented. In addition, in Section 3 we plan to study the notion of deferred statistical convergence of order $\alpha\beta$ for fuzzy variables and construct fundamental properties of the deferred statistical convergence of order $\alpha\beta$ in credibility.

2. Preliminaries

A set function Cr is credibility measure if it supplies the subsequent axioms: Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ (i.e., the largest algebra over Θ). Each element in $\mathcal{P}$ is called an event. For any $A \in \mathcal{P}(\Theta)$, Liu and Liu [26] presented a credibility measure $\operatorname{Cr}\{A\}$ to express the chance that fuzzy event A occurs. Li and Liu [24] proved that a set function $\operatorname{Cr}\{.\}$ a credibility measure if and only if

Axiom i. $\operatorname{Cr}\{\Theta\} = 1$;

Axiom ii. $\operatorname{Cr}\{A\} \leq \operatorname{Cr}\{B\}$ whenever $A \subset B$;

Axiom iii. Cr is self-dual, i.e., $\operatorname{Cr}\{A\} + \operatorname{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}(\Theta)$;

Axiom iv. $\operatorname{Cr}\{\cup_i A_i\} = \sup_i \operatorname{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \operatorname{Cr}\{A_i\} < 0.5$.

The triplet $(\Theta, \mathcal{P}(\Theta), \operatorname{Cr})$ is named a credibility space. A fuzzy variable is investigated by Liu and Liu [26] as function from the credibility space to the set of real numbers.

Definition 2.1. ([26]) The expected value of fuzzy variable μ is given by

$$E[\mu] := \int_0^{+\infty} \operatorname{Cr}\{\mu \geq r\}\, dr - \int_{-\infty}^{0} \operatorname{Cr}\{\mu \leq r\}\, dr$$

provided that at least one of the two integrals is finite.

If there is a $M > 0$ such that

$$\operatorname{Cr}\{\mu \leq -M\} = 0$$

and

$$\operatorname{Cr}\{\mu \leq M\} = 1,$$

then fuzzy variable μ is named as essentially bounded.

Definition 2.2. ([27]) Suppose $\phi, \phi_1, \phi_2, ...$ are fuzzy variables defined on credibility space $(\Theta, \mathcal{P}(\Theta), \operatorname{Cr})$. The sequence $\{\phi_k\}$ is said to be convergent almost surely to ϕ if and only if there exists A with $\operatorname{Cr}\{A\} = 1$ such that

$$\lim_{k\to\infty} |\phi_k(\theta) - \phi(\theta)| = 0$$

for every $\theta \in A$.

Definition 2.3. ([27]) Suppose $\phi, \phi_1, \phi_2, ...$ are fuzzy variables defined on credibility space $(\Theta, \mathcal{P}(\Theta), \operatorname{Cr})$. We say that the sequence $\{\phi_k\}$ converges in credibility to ϕ if

$$\lim_{k\to\infty} \operatorname{Cr}\{|\phi_k - \phi| \geq \rho\} = 0$$

for every $\rho > 0$.

Definition 2.4. ([27]) Suppose $\phi, \phi_1, \phi_2, ...$ are fuzzy variables with finite expected values defined on $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. We say that the sequence $\{\phi_k\}$ converges in mean to ϕ if

$$\lim_{k\to\infty} E\left[|\phi_k - \phi|\right] = 0.$$

Theorem 2.1. *(Wang and Liu [50]) If the sequence $\{\phi_k\}$ convergence in credibility to ϕ_0, then $\{\phi_k\}$ converges a.s. to ϕ_0.*

Theorem 2.2. *(Liu, [27]) If the sequence $\{\phi_k\}$ convergence in mean to ϕ_0, then $\{\phi_k\}$ converges credibility to ϕ_0.*

Theorem 2.3. *Let μ be a fuzzy variable. Then, for any given numbers $t > 0$ and $p > 0$, we have*

$$Cr\left\{|\mu| \geq t\right\} \leq \frac{E\left[|\mu|^p\right]}{t^p}. \tag{2.1}$$

Theorem 2.4. *([50]) If the sequence $\{\mu_i\}$ convergence in credibility to μ, then $\{\mu_i\}$ converges a.s. to μ.*

The idea of ordering statistical convergence was given by Gadjiev and Orhan [13], and was generalized by Bhunia et al. [4] and Çolak [6], independently.

Çolak [6] named this concept as “statistical convergence of order α, $(0 < \alpha \leq 1)$”and defined it as follows:

A sequence $x = (x_k)$ is said to be statistically convergent of order α to L if there is a real number L such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0,$$

holds, for each $\varepsilon > 0$. This concept is much more general than the concept statistical convergence.

Çolak and Bektaş [7] combined the concepts of statistical convergence of order α and λ-statistical convergence, and introduced the notion of λ-statistical convergence of order α in the following way.

A sequence $x = (x_k)$ is said to be λ-statistically convergent to L of order α provided that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : |x_k - L| \geq \varepsilon\}\right| = 0$$

for each $\varepsilon > 0$, where $\lambda = (\lambda_n) \in \Lambda, I_n = [n - \lambda_n + 1, n]$ and $0 < \alpha \leq 1$. By S_λ^α, we will denote the set of all λ-statistically convergent sequences of order α.

It can be easily seen that λ-statistical convergence of order α generalizes the two concepts above.

Another generalization of statistical convergence was brought to the literature by Et et al. [9] and Et and Şengül [10, 44] as follows.

Let $\theta = (k_w)$ be a lacunary sequence and $\alpha \in (0,1]$. The sequence $x = (x_k) \in w$ is said to be lacunary statistically convergent to L of order α, if

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$, where $\theta = (k_r)_{r=0}^{\infty}$ is a lacunary sequence such that

$$h_r = (k_r - k_{r-1}) \to \infty \quad \text{as} \quad r \to \infty$$

By S_θ^α, we will denote the set of such sequences. For $\theta = (2^r)$ we have $S_\theta^\alpha = S^\alpha$ and in the special case $\alpha = 1$ and $\theta = (2^r)$ we get $S_\theta^\alpha = S$.

Deferred Cesàro mean of a sequence was defined by Agnew [1] as follows:

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = +\infty.$$

Throughout this study, by $p = \{p(n)\}$ and $q = \{q(n)\}$ we will denote sequences of natural numbers satisfying the conditions $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = +\infty$.

Using Agnew's approach, Küçükaslan and Yilmazturk [22, 53] introduced definition of deferred density and deferred statistical convergence as follows.

Let $K \subset \mathbb{N}$ and $K_{p,q}(n) := \{k : p(n) < k \leq q(n), k \in K\}$. The deferred density of K is defined by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |K_{p,q}(n)|,$$

provided the limit exists.

A sequence $x = (x_k)$ is said to be deferred statistically convergent to L, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} |\{p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon\}| = 0,$$

holds.
By $S_{p,q}$, we will denote the set of such sequences. If $q(n) = n, p(n) = 0$, then deferred density and deferred statistical convergence are the same usual density and statistical convergence, respectively.

Temizsu et al. [47] defined the notions of deferred $\alpha-$density and deferred statistical convergence of order α as follows:

Let $K \subset \mathbb{N}$. Deferred $\alpha-$density of K is defined by

$$\delta_{p,q}^{\alpha}(K) = \lim_{n\to\infty} \frac{1}{(q(n)-p(n))^{\alpha}} \left| K_{p,q}(n) \right|,$$

provided the limit exists, $\alpha \in (0,1]$.

It is clear that

(i) If $K \subseteq M$, then $\delta_{p,q}^{\alpha}(K) \leq \delta_{p,q}^{\alpha}(M)$,

(ii) If $q(n) = n, p(n) = 0$, then $\delta_{p,q}^{\alpha}(K) = \delta^{\alpha}(K)$,

(iii) If $q(n) = n, p(n) = 0$ and $\alpha = 1$, then $\delta_{p,q}^{\alpha}(K) = \delta(K)$,

(iv) Let $\alpha, \gamma \in (0,1]$, then $\delta_{p,q}^{\alpha}(K) \leq \delta_{p,q}^{\gamma}(K)$, where $0 < \alpha \leq \gamma \leq 1$.

A sequence $x = (x_k)$ is said to be deferred statistically convergent of order α to L, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(q(n)-p(n))^{\alpha}} \left| \{ p(n) < k \leq q(n) : |x_k - L| \geq \varepsilon \} \right| = 0.$$

In this case, we write $DS_{p,q}^{\alpha} - \lim x_k = L$. By $DS_{p,q}^{\alpha}$, we will denote the set of such sequences. If $q(n) = n, p(n) = 0$, then $DS_{p,q}^{\alpha} = S^{\alpha}$, If $\alpha = 1$ then $DS_{p,q}^{\alpha} = DS_{p,q}$ and also in the special case $q(n) = n, p(n) = 0$ and $\alpha = 1$ we get $DS_{p,q}^{\alpha} = S$.

As can be seen in the example below, deferred statistical convergence of order α is well defined for $\alpha \in (0,1]$, but it is not well defined for $\alpha > 1$. To see this let (x_k) be a sequence defined as follows:

$$x_k = \begin{cases} 1, & k = 2n, \, n \in \mathbb{N} \\ 0, & k \neq 2n. \end{cases}$$

Then, both

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left| \{ p(n) < k \leq q(n) : |x_k - 1| \geq \varepsilon \} \right| \leq \lim_{n\to\infty} \frac{n}{2n^{\alpha}} = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left| \{ p(n) < k \leq q(n) : |x_k - 0| \geq \varepsilon \} \right| \leq \lim_{n\to\infty} \frac{n+1}{2n^{\alpha}} = 0,$$

holds, for $\alpha > 1$ and for $q(n) = 4n, p(n) = 3n$. So $DS_{p,q}^{\alpha} - \lim x_k = 1$ and $DS_{p,q}^{\alpha} - \lim x_k = 0$ which is impossible.

3. Main results

In this section we give the definitions of deferred statistical convergence of order $\alpha\beta$ and strongly s-deferred Cesàro summability of order $\alpha\beta$ in credibility space. The definitions and results which we give in this section more general than the above definitions and related results.

Let

$$\phi_k : \mathbb{N} \to (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$$

be a sequence of fuzzy variables for all $k \in \mathbb{N}$.

Definition 3.1. Assume the sequences $\{p(w)\}$, $\{q(w)\}$ be two sequences as above, α and β be any real numbers such that $0 < \alpha \leq \beta \leq 1$. In credibility, the sequence $\{\phi_k\}$ is called deferred statistically convergent almost surely (d.s.a.s.) of order $\alpha\beta$ to fuzzy variable ϕ_0 if and only if there exists $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{w\to\infty} \frac{1}{(q(w) - p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta} = 0,$$

for each $\rho > 0$ and all $\theta \in A$. In this case, we denote $DS^{\alpha,\beta}_{p,q}(a.s.) - \lim \phi_k = \phi_0$. By $DS^{\alpha,\beta}_{p,q}(a.s.)$, we will denote the set of such sequences. If we take $\beta = 1$ in this definition we get $DS^{\alpha,\beta}_{p,q}(a.s.) = DS^{\alpha}_{p,q}(a.s.)$. If we take $\alpha = \beta = 1$, then $DS^{\alpha,\beta}_{p,q}(a.s.) = DS_{p,q}(a.s.)$. If $q(n) = n$, $p(n) = 0$ and $\alpha = \beta = 1$ then $DS^{\alpha,\beta}_{p,q}(a.s.) = S(a.s.)$.

Example 3.1. Deferred statistical convergence of order $\alpha\beta$ almost surely is well defined for $\beta \geq \alpha$, but it is not well defined for $\beta < \alpha$. To demonstrate this, consider the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\theta_1, \theta_2, ...\}$ with $\mathrm{Cr}\{\theta_s\} = s$ for $s = 1, 2, ...$ The fuzzy variables are defined by

$$\phi_k(\theta_s) = \begin{cases} 2, & s = k = 2w \\ 0, & s = k \neq 2w \end{cases} \qquad w = 1, 2, 3 \ldots$$

If we take $q(w) = 4w$, $p(w) = 3w$, then we obtain

$$\lim_{w\to\infty} \frac{1}{w^{\alpha}} \left|\{p(w) < k \leq q(w) : |\phi_k(\theta) - 2| \geq \rho\}\right|^{\beta} \leq \lim_{w\to\infty} \frac{w^{\beta}}{2^{\beta} w^{\alpha}} = 0$$

and

$$\lim_{w\to\infty} \frac{1}{w^{\alpha}} \left|\{p(w) < k \leq q(w) : |\phi_k(\theta) - 0| \geq \rho\}\right|^{\beta} \leq \lim_{n\to\infty} \frac{(w+1)^{\beta}}{2^{\beta} w^{\alpha}} = 0,$$

for $\beta < \alpha$, for each $0 < \rho < 1$ and for all $\theta \in A$. so $DS^{\alpha,\beta}_{p,q}(a.s.) - \lim \phi_k = 2$ and $DS^{\alpha,\beta}_{p,q}(a.s.) - \lim \phi_k = 0$ which is impossible.

Remark 3.1. In view of Definition 3.1 is obvious that
(i) In credibility, if $q(w) = w$, $p(w) = 0$ and $\beta = 1$, then deferred statistical convergence almost surely of order $\alpha\beta$ is the same statistical convergence almost surely of order α. Moreover, if we take $\alpha = 1$, then deferred statistical convergence almost surely of order $\alpha\beta$ is reduced to statistical convergence almost surely.
(ii) In credibility, if we take $q(w) = k_w$ and $p(w) = k_{w-1}$, where (k_w) is a lacunary sequence and $\beta = 1$, then deferred statistical convergence almost surely of order $\alpha\beta$ is the same lacunary statistical convergence almost surely of order α. In addition, if we take $\alpha = 1$, then deferred lacunary statistical convergence almost surely of order $\alpha\beta$ coincides with lacunary statistical convergence almost surely.
(iii) In credibility, if we consider $q(w) = w$ and $p(w) = w - \lambda_w$, where (λ_w) is a nondecreasing sequence of natural numbers such that $\lambda_1 = 1$ and $\lambda_{w+1} \leq \lambda_w + 1$ and $\beta = 1$, then deferred statistical convergence almost surely of order $\alpha\beta$ coincides with definition of λ-statistical convergence of order α almost surely. In addition if we take $\alpha = 1$, then deferred statistical convergence almost surely of order $\alpha\beta$ is the same λ-statistical convergence almost surely.

Definition 3.2. The sequence $\{\phi_k\}$ is called deferred statistically convergent in credibility of order $\alpha\beta$ to fuzzy variable ϕ_0, if there exists $A \in \mathcal{P}(\Theta)$ having $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : \mathrm{Cr}\{|\phi_k - \phi_0| \geq \rho\} \geq \sigma\}\right|^{\beta} = 0,$$

for any preassigned $\sigma > 0$ and $\rho > 0$. Symbolically, we write $DS^{\alpha,\beta}_{p,q}(\mathrm{Cr}) - \lim \phi_k = \phi_0$.

Definition 3.3. Assume that $\{\phi_k\}$ is a sequence of fuzzy variables having finite expected values defined on $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. The sequence $\{\phi_k\}$ is called deferred statistically convergent in mean of order $\alpha\beta$ to fuzzy variable ϕ_0 if

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : E[|\phi_k - \phi_0|] \geq \rho\}\right|^{\beta} = 0,$$

for each $\rho > 0$. Symbolically, we indicate $DS^{\alpha,\beta}_{p,q}(E) - \lim \phi_k = \phi_0$.

Definition 3.4. Suppose $\Phi, \Phi_1, \Phi_2, \ldots$ be the credibility distribution of fuzzy variables $\phi, \phi_1, \phi_2, \ldots$, respectively. Then, we say that the sequence $\{\phi_k\}$ is called deferred statistically convergent in distribution of order $\alpha\beta$ to a fuzzy variable ϕ whose credibility distribution function is Φ if for any given $\rho > 0$, we have

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : |\Phi_k(z) - \Phi(z)| \geq \rho\}\right|^{\beta} = 0,$$

where z is any real number where Φ is continuous.

Definition 3.5. The sequence $\{\phi_k\}$ of fuzzy variables in the space $(\Theta, \mathcal{P}(\Theta), Cr)$ is called deferred statistically convergent with respect to uniformly almost surely of order $\alpha\beta$ to fuzzy variable ϕ_0 if there exists some events A_i $(i \in \mathbb{N})$ each of whose credibility measure approaches zero such that the sequence is deferred statistically converges with respect to uniformly almost surely of order $\alpha\beta$ to the same limit. In this case

$$\lim_{w\to\infty} \tfrac{1}{(q(w)-p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta} = 0,$$

for all $\theta \in \Theta - A_i$, and $\rho > 0$.

Definition 3.6. The sequence $\{\phi_k\}$ is called a deferred statistically Cauchy sequence in credibility of order $\alpha\beta$ if there is a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ and $Q = Q(\sigma)$ such that

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : \mathrm{Cr}\{|\phi_k - \phi_Q| \geq \rho\} \geq \sigma\}\right|^{\beta} = 0,$$

for each $\rho, \sigma > 0$.

Example 3.2. Let $0 < \alpha \leq \beta \leq 1$. Deferred statistical convergence a.s. of order $\alpha\beta$ does not imply deferred statistical convergence in credibility of order $\alpha\beta$. To denote this, let $\Theta = \{\theta_1, \theta_2, ...\}$, $\mathrm{Cr}\{\theta_1\} = 1$ and $\mathrm{Cr}\{\theta_s\} = (s-1)/s$ for $s = 2, 3, ...$ and the fuzzy variables are established by

$$\phi_k(\theta_s) = \begin{cases} k, & \text{if } s = k \\ 0, & \text{otherwise} \end{cases}$$

for $k = 1, 2, ...$ and $\phi_0 = 0$. By choosing arbitrary α, β we get $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim \phi_k = \phi_0$. But, for any small number $\rho > 0$ and $\sigma \in \left(0, \frac{1}{2}\right)$, the sequence $\{\phi_k\}$ is not deferred statistically convergent in credibility of order $\alpha\beta$.

Example 3.3. Conversely, deferred statistical convergence in credibility of order $\alpha\beta$ does not imply deferred statistical convergence a.s. of order $\alpha\beta$, too. For instance, $\Theta = \{\theta_1, \theta_2, ...\}$, $\mathrm{Cr}\{\theta_s\} = 1/s$ for $s = 1, 2, ...$ and the fuzzy variables are identified by

$$\phi_k(\theta_s) = \begin{cases} (s+1)/s, & \text{if } s = k, k+1, k+2, ... \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

for $k = 1, 2, ...$ and $\phi_0 = 0$. Then, for arbitrary α,β, for any small number $\rho > 0$, $\sigma \in \left[\frac{1}{2}, 1\right)$, $q(w) = 4w$ and $p(w) = 3w$,

$$\lim_{w\to\infty} \frac{1}{w^{\alpha}} \left|\{3w < k \leq 4w : \mathrm{Cr}\{|\phi_k - \phi_0| \geq \rho\} \geq \sigma\}\right|^{\beta} = 0$$

which states that $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr}) - \lim \phi_k = \phi_0$. But, it is obvious that $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim \phi_k \neq \phi_0$.

Example 3.4. Let $0 < \alpha \leq \beta \leq 1$. Deferred statistical convergence in mean of order $\alpha\beta$ does not imply deferred statistical convergence a.s of order $\alpha\beta$. Consider the fuzzy variables defined by (3.1) which does not deferred statistically converge a.s. of order $\alpha\beta$ to ϕ_0. However

$$E[|\phi_k - \phi_0|] = \frac{k+1}{2k^2} \to 0.$$

Thus, for all $\rho > 0$, for arbitrary α,β, we get

$$\lim_{w\to\infty} \frac{1}{(q(w) - p(w))^{\alpha}} \left|\{p(w) < k \leq q(w) : E[|\phi_k - \phi_0|] \geq \rho\}\right|^{\beta} = 0$$

which gives that $\{\phi_k\}$ statistically converges in mean.of order $\alpha\beta$ to ϕ_0.

Example 3.5. Deferred statistical convergence a.s. of order $\alpha\beta$ does not imply deferred statistical convergence in mean of order $\alpha\beta$, too. For instance, $\Theta = \{\theta_1, \theta_2, ...\}$, $\mathrm{Cr}\{\theta_j\} = 1/j$ for $j = 1, 2, ...$ and the fuzzy variables are itendified as

$$\phi_k(\theta_j) = \begin{cases} k, & \text{if } j = k \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

for $i = 1, 2, ...$, $\phi_0 = 0$, $q(w) = 4w$ and $p(w) = 3w$. Then, the sequence $\{\phi_k\}$ deferred statistically converges a.s.of order $\alpha\beta$ to ϕ_0. But, for any $\rho \in \left(0, \frac{1}{2}\right)$,

$$\lim_{w\to\infty} \frac{1}{w^{\alpha}} \left|\{3w < k \leq 4w : E[|\phi_k - \phi_0|] \geq \rho\}\right|^{\beta} \neq 0.$$

That is to say, the sequence $\{\phi_k\}$ does not deferred statistically converge in mean of order $\alpha\beta$ to ϕ_0.

Theorem 3.1. *Assume $\{\phi_k\}$ be a sequence of fuzzy variables. If the sequence $\{\phi_k\}$ deferred statistically converges in mean of order $\alpha\beta$ to a fuzzy variable ϕ_0, then $\{\phi_k\}$ deferred statistically converges in credibility of order $\alpha\beta$ to ϕ_0.*

PROOF. Let $DS_{p,q}^{\alpha,\beta}(E) - \lim \phi_k = \phi_0$. For any taken $\rho > 0$ and $\sigma > 0$, with the aid of Markov inequality, we obtain

$$\begin{aligned} &\lim_{w\to\infty} \tfrac{1}{(q(w)-p(w))^\alpha} \left|\{p(w) < k \le q(w) : \mathrm{Cr}\{|\phi_k - \phi_0| \ge \rho\} \ge \sigma\}\right|^\beta \\ &\le \lim_{w\to\infty} \tfrac{1}{(q(w)-p(w))^\alpha} \left|\left\{p(w) < k \le q(w) : \left(\tfrac{E[|\phi_k-\phi_0|]}{\rho}\right) \ge \sigma\right\}\right|^\beta . \end{aligned}$$

As a result, $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr}) - \lim \phi_k = \phi_0$.

Example 3.6. Let $0 < \alpha \le \beta \le 1$. Deferred statistical convergence in credibility of order $\alpha\beta$ does not imply deferred statistical convergence in mean of order $\alpha\beta$. Consider the fuzzy variables defined by (3.2) which does not deferred statistically converge in mean of order $\alpha\beta$ to $\phi_0 = 0$. But, for any small number $\rho > 0$, $\sigma \in \left[\frac{1}{2}, 1\right)$, for arbitrary α,β, we get

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^\alpha} \left|\{p(w) < k \le q(w) : \mathrm{Cr}\{|\phi_k - \phi_0| \ge \rho\} \ge \sigma\}\right|^\beta = 0$$

for each $\rho > 0$ and $\sigma > 0$. Namely, the sequence $\{\phi_k\}$ deferred statistically converges in credibility of order $\alpha\beta$ to ϕ_0.

Theorem 3.2. *Let $0 < \alpha \le \beta \le 1$.*
(i) If $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim \phi_k = \phi_0$ and $s \in \mathbb{C}$, then $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim s\phi_k = s\phi_0$,
(ii) If $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim \phi_k = \phi_0$ and $DS_{p,q}^{\alpha,\beta} - \lim \varpi_k = \varpi_0$, then $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim \phi_k + \varpi_k = \phi_0 + \varpi_0$.

PROOF. (i) Assume that $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim \phi_k = \phi_0$, then, for each $\rho > 0$ and for all $\theta \in A$

$$\begin{aligned} &\tfrac{1}{(q(w)-p(w))^\alpha} \left|\{p(w) < k \le q(w) : |s\phi_k(\theta) - s\phi_0(\theta)| \ge \rho\}\right|^\beta \\ &= \tfrac{1}{(q(w)-p(w))^\alpha} \left|\left\{p(w) < k \le q(w) : |\phi_k(\theta) - \phi_0(\theta)| \ge \tfrac{\rho}{|s|}\right\}\right|^\beta \end{aligned}$$

holds. This implies that $DS_{p,q}^{\alpha,\beta}(a.s.) - \lim s\phi_k = s\phi_0$.

The proof of (ii) follows from the below inequalities:

$$\begin{aligned} &\tfrac{1}{(q(w)-p(w))^\alpha} \left|\{p(w) < k \le q(w) : |(\phi_k(\theta) + \varpi_k(\theta)) - (\phi_0 + \varpi_0)| \ge \rho\}\right|^\beta \\ &\le \tfrac{1}{(q(w)-p(w))^\alpha} \left|\left\{p(w) < k \le q(w) : |\phi_k(\theta) - \phi_0(\theta)| \ge \tfrac{\rho}{2}\right\}\right|^\beta \\ &+ \tfrac{1}{(q(w)-p(w))^\alpha} \left|\left\{p(w) < k \le q(w) : |\varpi_k(\theta) - \varpi_0(\theta)| \ge \tfrac{\rho}{2}\right\}\right|^\beta . \end{aligned}$$

Suppose that c denotes the collection of all convergent fuzzy variable sequences. It is easy to see that $c \subset DS_{p,q}^{\alpha,\beta}(a.s.)$ for all $0 < \alpha \le \beta \le 1$ and the inclusion is strict for some $\alpha \le \beta$. For this, establish a sequence $\phi = \{\phi_k\}$ defined by

$$\phi_k(\theta) = \begin{cases} 3, & k = w^2 \\ 0, & k \ne w^2. \end{cases}$$

Then

$$\frac{1}{(q(w)-p(w))^{\alpha}}\left|\{p(w)<k\le q(w):|\phi_k(\theta)-0|\ge\varepsilon\}\right|^{\beta}\le\frac{(\sqrt{q(w)-p(w)}+1)^{\beta}}{(q(w)-p(w))^{\alpha}}\to 0$$

and so $DS_{p,q}^{\alpha,\beta}(a.s.)-\lim\phi_k=0$ for $\alpha>\frac{1}{2}$ and $\beta>\alpha$, but it is not convergent.

Theorem 3.3. *A fuzzy variable sequence $\{\phi_k\}$ is deferred statistically convergent in credibility of order $\alpha\beta$ iff $\{\phi_k\}$ is deferred statistically Cauchy sequence in credibility of order $\alpha\beta$.*

PROOF. Assume $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr})-\lim\phi_k=\phi$. Then, there is a $A\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\}=1$ such that

$$\lim_{w\to\infty}\frac{1}{(q(w)-p(w))^{\alpha}}\left|\{p(w)<k\le q(w):\mathrm{Cr}\{|\phi_k-\phi|\ge\rho\}\ge\sigma\}\right|^{\beta}=0,$$

for each $\rho,\sigma>0$. Select $Q\in\mathbb{N}$ such that $\mathrm{Cr}\{|\phi_Q-\phi|\ge\rho\}\ge\sigma$. Describe the sets U_1,U_2 and U_3 as follows:

$$\begin{aligned}U_1&=\{p(w)<k\le q(w):\mathrm{Cr}\{|\phi_k-\phi_Q|\ge\rho\}\ge\sigma\},\\U_2&=\{p(w)<k\le q(w):\mathrm{Cr}\{|\phi_k-\phi|\ge\rho\}\ge\sigma\},\\U_3&=\{p(w)<k\le q(w):\mathrm{Cr}\{|\phi_Q-\phi|\ge\rho\}\ge\sigma\}.\end{aligned}$$

Clearly, $U_1\subseteq U_2\cup U_3$. As a result, $\delta_{p,q}^{\alpha,\beta}(U_1)\le\delta_{p,q}^{\alpha,\beta}(U_2)+\delta_{p,q}^{\alpha,\beta}(U_3)=0$, since $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr})-\lim\phi_k=\phi$. Hence, $\{\phi_k\}$ is deferred statistically Cauchy sequence in credibility sense of order $\alpha\beta$.

Conversely, let $\{\phi_k\}$ be a deferred statistically Cauchy sequence in credibility of order $\alpha\beta$. Then, there is a $A\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\}=1$ and $Q=Q(\sigma)$ such that

$$\lim_{w\to\infty}\frac{1}{(q(w)-p(w))^{\alpha}}\left|\{p(w)<k\le q(w):\mathrm{Cr}\{|\phi_k-\phi_Q|\ge\rho\}\ge\sigma\}\right|^{\beta}=0$$

for each $\rho>0$ and $\sigma>0$. Assume in contrast that it is not deferred statistical convergence in credibility of order $\alpha\beta$. Then, there exists $A\in\mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\}=1$ such that

$$\lim_{w\to\infty}\frac{1}{(q(w)-p(w))^{\alpha}}\left|\left\{p(w)<k\le q(w):\mathrm{Cr}\left\{|\phi_k-\phi|\ge\frac{\rho}{2}\right\}\ge\frac{\sigma}{2}\right\}\right|^{\beta}\ne 0$$

for each $\rho>0$ and $\sigma>0$. Let

$$B=\left\{p(w)<k\le q(w):\mathrm{Cr}\left\{|\phi_k-\phi|\ge\frac{\rho}{2}\right\}\ge\frac{\sigma}{2}\right\}$$

and

$$C=\{p(w)<k\le q(w):\mathrm{Cr}\{|\phi_k-\phi_Q|\ge\rho\}\ge\sigma\}.$$

Thus
$$B^c = \left\{ p(w) < k \leq q(w) : \mathrm{Cr}\left\{ |\phi_k - \phi| \geq \frac{\rho}{2} \right\} < \frac{\sigma}{2} \right\}.$$
Next we prove $B \subseteq C$. Suppose $C \subseteq B$ and $k \in B^c \cap C$. Then
$$\mathrm{Cr}\left\{ |\phi_k - \phi| \geq \frac{\rho}{2} \right\} < \frac{\sigma}{2}, \, \mathrm{Cr}\left\{ |\phi_k - \phi_Q| \geq \rho \right\} \geq \sigma.$$
Let $Q \in B^c$, we have
$$\mathrm{Cr}\left\{ |\phi_Q - \phi| \geq \frac{\rho}{2} \right\} < \frac{\sigma}{2}.$$
Hence, there is a $Q = Q(\sigma)$ such that
$$\begin{aligned} \sigma \quad & \leq \mathrm{Cr}\left\{ |\phi_k - \phi_Q| \geq \rho \right\} \\ & \leq \mathrm{Cr}\left\{ |\phi_Q - \phi| \geq \tfrac{\rho}{2} \right\} + \mathrm{Cr}\left\{ |\phi_k - \phi| \geq \tfrac{\rho}{2} \right\} \\ & < \tfrac{\sigma}{2} + \tfrac{\sigma}{2} = \sigma, \end{aligned}$$
which is not possible. Hence $B \subseteq C$. This gives that
$$\lim_{w \to \infty} \frac{1}{(q(w) - p(w))^\alpha} \left| \left\{ p(w) < k \leq q(w) : \mathrm{Cr}\left\{ |\phi_k - \phi| \geq \frac{\rho}{2} \right\} \geq \frac{\sigma}{2} \right\} \right|^\beta = 0,$$
i.e.,
$$\lim_{w \to \infty} \frac{1}{(q(w) - p(w))^\alpha} \left| \left\{ p(w) < k \leq q(w) : \mathrm{Cr}\left\{ |\phi_k - \phi| \geq \rho \right\} \geq \sigma \right\} \right|^\beta = 0.$$
As a result, the sequence $\{\phi_k\}$ is deferred statistically convergent in credibility of order $\alpha\beta$.

Deferred statistical convergence in credibility of order $\alpha\beta$ supplies some usual axioms of convergence in credibility of order $\alpha\beta$. The known axioms of convergence in credibility are the following axioms:

Consider $\phi, \phi_1, \phi_2, \ldots$ as fuzzy variables defined on credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$.

(**H**) If there exists a subset $T = \{m_1 < m_2 < \ldots\} \subseteq \mathbb{N}$ such that $\delta(T) = 1$ and $\{\phi_{m_k}\}$ converges in credibility of order $\alpha\beta$ to ϕ_0, then $\{\phi_k\}$ deferred statistically converges in credibility of order $\alpha\beta$ to ϕ_0.

(**U**) The uniqueness of limit: If $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr}) - \lim \phi_k = \phi_1$ and $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr}) - \lim \phi_k = \phi_2$, then $\phi_1 = \phi_2$ in credibility.

Theorem 3.4. *The axioms* (**U**) *and* (**H**) *are supplied by deferred statistical convergence in credibility sense of order $\alpha\beta$.*

PROOF. Assume that there is a subset $T = \{m_1 < m_2 < \ldots\} \subseteq \mathbb{N}$ such that $\delta(T) = 1$ and $\{\phi_{m_k}\}$ converges in credibility of order $\alpha\beta$ to ϕ_0, i.e., for each $\phi \in A$, any $\rho, \sigma > 0$, there is a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ and $k_0 = k_0(\rho)$ such that
$$\mathrm{Cr}\left\{ |\phi_{m_k} - \phi_0| \geq \rho \right\} < \gamma,$$

for each $k > k_0$. Let $T = \{m_{k_0+1}, m_{k_0+2}, ...\}$. Then, there is a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^\alpha} \left|\{k \in T : \mathrm{Cr}\{|\phi_k - \phi_0| \geq \rho\} \geq \sigma\}\right|^\beta = 0,$$

for each $\rho, \sigma > 0$. Hence, we obtain

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^\alpha} \left|\{p(w) < k \leq q(w) : \mathrm{Cr}\{|\phi_k - \phi_0| \geq \rho\} \geq \sigma\}\right|^\beta = 0,$$

for each $\rho, \sigma > 0$, i.e., $\{\phi_k\}$ is deferred statistically convergent in credibility of order $\alpha\beta$ to ϕ_0. So, deferred statistical convergence in credibility of order $\alpha\beta$ supplies the axiom (**H**).

Now, we demonstrate that deferred statistical convergence in credibility of order $\alpha\beta$ supplies the axiom (**U**). Assume that $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr})-\lim\phi_k = \phi_1$ and $DS_{p,q}^{\alpha,\beta}(\mathrm{Cr})-\lim\phi_k = \phi_2$. Then, there is a $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^\alpha} \left|\{k \in T : \mathrm{Cr}\{|\phi_k - \phi_1| \geq \rho\} \geq \sigma\}\right|^\beta = 0,$$

and

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^\alpha} \left|\{k \in T : \mathrm{Cr}\{|\phi_k - \phi_2| \geq \rho\} \geq \sigma\}\right|^\beta = 0,$$

for all $\rho, \sigma > 0$. Establish the sets B_1 and B_2 as follows:

$$B_1 = \{p(w) < k \leq q(w) : \mathrm{Cr}\{|\phi_k - \phi_1| \geq \rho\} \geq \sigma\},$$

and

$$B_2 = \{p(w) < k \leq q(w) : \mathrm{Cr}\{|\phi_k - \phi_2| \geq \rho\} \geq \sigma\}.$$

Now let $k \in B_1 \cup B_2$. Then, we get

$$\mathrm{Cr}\{|\phi_k - \phi_1| \geq \rho\} < \gamma,\ \mathrm{Cr}\{|\phi_k - \phi_2| \geq \rho\} < \sigma.$$

Therefore

$$\begin{aligned}\mathrm{Cr}\{|\phi_1 - \phi_2| \geq \rho\} &= \mathrm{Cr}\{|\phi_1 - \phi_k + \phi_k - \phi_2| \geq \rho\}\\ &\leq \mathrm{Cr}\left\{|\phi_k - \phi_1| \geq \tfrac{\rho}{2}\right\} + \mathrm{Cr}\left\{|\phi_k - \phi_2| \geq \tfrac{\rho}{2}\right\}\\ &< 2\sigma.\end{aligned}$$

Since $\sigma > 0$ is arbitrary, we can obtain $\mathrm{Cr}\{|\phi_1 - \phi_2| \geq \rho\} = 0$, which gives $\phi_1 = \phi_2$ in credibility of order $\alpha\beta$.

Theorem 3.5. *Suppose* $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$. *Then,* $DS_{p,q}^{\alpha_1,\beta_2}$ *(a.s.)* $\subseteq DS_{p,q}^{\alpha_2,\beta_1}$ *(a.s.) holds and the inclusion is strict for some* $\alpha_1, \alpha_2, \beta_1$ *and* β_2 *such that* $\alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$.

PROOF. If $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$ then

$$\begin{aligned} &\tfrac{1}{(q(w)-p(w))^{\alpha_2}} \left|\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta_1} \\ &\leq \tfrac{1}{(q(w)-p(w))^{\alpha_1}} \left|\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta_2} \end{aligned}$$

for all $\rho > 0$, $\theta \in A\,(\in \mathcal{P}(\Theta))$ having $\mathrm{Cr}\{A\} = 1$, and this implies that $DS_{p,q}^{\alpha_1,\beta_2}(a.s.) \subseteq DS_{p,q}^{\alpha_2,\beta_1}(a.s.)$.

To demonstrate that the inclusion is strict, take $q(w) = 3w - 1$, $p(w) = 2w - 1$ and identify a sequence $\phi = \{\phi_k\}$ by

$$\phi_k(\theta) = \begin{cases} 1, & k = w^2 \\ 0, & k \neq w^2 \end{cases}$$

Then $DS_{p,q}^{\alpha_2,\beta_1}(a.s.) - \lim \phi_k = 0$ namely, $\{\phi_k\} \in DS_{p,q}^{\alpha_2,\beta_1}(a.s.)$ for $\alpha_2 \in \left(\frac{1}{4}, 1\right]$ and $\beta_1 = \frac{1}{2}$ but $\{\phi_k\} \notin DS_{p,q}^{\alpha_1,\beta_2}(a.s.)$ for $\alpha_1 \in \left(0, \frac{1}{4}\right]$ and $\beta_2 = \frac{1}{2}$.

Corollary 3.1. *(i)* $DS_{p,q}^{\alpha_1}(a.s.) \subset DS_{p,q}^{\alpha_2}(a.s.)$ *for* $0 < \alpha_1 \leq \alpha_2 \leq 1$, $\beta_1 = \beta_2 = 1$ *and the inclusion is strict,*
(ii) $DS_{p,q}^{\alpha_1}(a.s.) = DS_{p,q}^{\alpha_2}(a.s.) \Leftrightarrow \alpha_1 = \alpha_2$.

Theorem 3.6. *Suppose* $\lim_w \left(\frac{q(w)-p(w)}{w}\right)^{\alpha_2} > 0$. *If* $q(w) < w$ *and* $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, *then* $S^{\alpha_1,\beta_2}(a.s.) \subset DS_{p,q}^{\alpha_2,\beta_1}(a.s.)$.

PROOF. For a given $\rho > 0$, and for all $\theta \in A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ we get

$$\{k \leq w : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\} \supseteq \{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\},$$

therefore

$$\begin{aligned} &\tfrac{1}{w^{\alpha_1}} \left|\{k \leq w : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta_2} \\ &\geq \tfrac{1}{w^{\alpha_2}} \left|\{p(n) < k \leq q(n) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta_1} \\ &= \tfrac{(q(w)-p(w))^{\alpha_2}}{w^{\alpha_2}} \cdot \tfrac{1}{(q(w)-p(w))^{\alpha_2}} \left|\{p(n) < k \leq q(n) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}\right|^{\beta_1} \end{aligned}$$

If we consider the fact that $\lim_w \left(\frac{q(w)-p(w)}{w}\right)^{\alpha_2} > 0$, we obtain

$$S^{\alpha_1,\beta_2}(a.s.) - \lim \phi_k = \phi_0 \Rightarrow DS_{p,q}^{\alpha_2,\beta_1}(a.s.) - \lim \phi_k = \phi_0.$$

Definition 3.7. Assume $\{p(w)\}$ and $\{q(w)\}$ be given as above, α and β be any real numbers such that $0 < \alpha \leq \beta \leq 1$, $s \in \mathbb{R}^+$, and $\{\phi_k\}$ be a fuzzy variable sequence. In credibility, a fuzzy variable sequence $\{\phi_k\}$ is called to be strongly s-deferred Cesàro summable of order $\alpha\beta$ to fuzzy variable ϕ_0 provided that

$$\lim_{w\to\infty} \frac{1}{(q(w)-p(w))^{\alpha}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s\right)^{\beta} = 0,$$

for all $\theta \in A$, and this is demonstrated by $Dw_s^{\alpha,\beta}[p,q]-\lim \phi_k = \phi_0$. By $Dw_s^{\alpha,\beta}[p,q]$, we will indicate the set of such sequences. If we take in this definition $\alpha=\beta=1$, we get $Dw_s^{\alpha,\beta}[p,q] = Dw_s[p,q]$. If $q(w)=w$, $p(w)=0$ and $\alpha=\beta=1$ then we have $Dw_s^{\alpha,\beta}[p,q] = w_s$, and also we have $Dw_{s,0}^{\alpha,\beta}[p,q]$ in case $\phi_0 = 0$.

Theorem 3.7. *Suppose* $0<\alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2 \le 1$, $s \in \mathbb{R}^+$, *and* $\{\phi_k\}$ *be a fuzzy variable sequence. Then,* $Dw_s^{\alpha_1,\beta_2}[p,q] \subseteq Dw_s^{\alpha_2,\beta_1}[p,q]$ *and the inclusion is strict for some* $\alpha_1, \alpha_2, \beta_1$ *and* β_2 *such that* $\alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2$.

PROOF. Assume $\phi = \{\phi_k\} \in Dw_s^{\alpha_1,\beta_2}[p,q]$. Then, given $\alpha_1, \alpha_2, \beta_1$ and β_2 such that $0<\alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2 \le 1$ and $s \in \mathbb{R}^+$. We can write

$$\begin{aligned}&\frac{1}{(q(w)-p(w))^{\alpha_2}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s \right)^{\beta_1} \\ &\qquad \le \frac{1}{(q(n)-p(n))^{\alpha_1}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s \right)^{\beta_2}\end{aligned}$$

and this yields that $Dw_s^{\alpha_1,\beta_2}[p,q] \subseteq Dw_s^{\alpha_2,\beta_1}[p,q]$.

To denote that the inclusion is strict, take $\phi = \{\phi_k\}$ as

$$\phi_k(\theta) = \begin{cases} 1, & k = w^2 \\ 0, & k \ne w^2. \end{cases}$$

Then, for $\alpha_2 \in \left(\frac{1}{2}, 1\right]$ and $\beta_1 = 1$

$$\begin{aligned}&\frac{1}{(q(w)-p(w))^{\alpha_2}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - 0|^s \right)^{\beta_1} = \frac{1}{(q(w)-p(w))^{\alpha_2}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta)|^s \right)^{\beta_1} \\ &\le \frac{(\sqrt{q(w)-p(w)}+1)^{\beta_1}}{(q(w)-p(w))^{\alpha_2}} \to 0.\end{aligned}$$

namely, $\phi \in Dw_s^{\alpha_2,\beta_1}[p,q]$. But for $\alpha_1 \in \left(0, \frac{1}{2}\right)$ and $\beta_2 = 1$

$$\frac{(\sqrt{q(w)-p(w)}-1)^{\beta_2}}{(q(w)-p(w))^{\alpha_1}} \le \frac{1}{(q(w)-p(w))^{\alpha_1}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - 0|^s \right)^{\beta_2} \to \infty$$

i.e., $\phi \notin Dw_s^{\alpha_1,\beta_2}[p,q]$.

Corollary 3.2. *Let* $0<\alpha_1 \le \alpha_2 \le \beta_1 \le \beta_2 \le 1$ *and* $s \in \mathbb{R}^+$, *and* $\{\phi_k\}$ *be a fuzzy variable sequence. Then*
(i) $Dw_s^{\alpha_1}[p,q] \subseteq Dw_s^{\alpha_2}[p,q]$ *iff* $\beta_1 = \beta_2 = 1$,
(ii) $Dw_s^{\alpha_1}[p,q] \subseteq Dw_s[p,q]$ *for all* $\alpha_2 = \beta_1 = \beta_2 = 1$ *and* $1 < s < \infty$.

Theorem 3.8. *Let $0 < \alpha \leq \beta \leq 1$ and $0 < r < s < \infty$, and $\{\phi_k\}$ be a fuzzy variable sequence. Then $Dw_s^{\alpha,\beta}[p,q] \subseteq Dw_r^{\alpha,\beta}[p,q]$.*

PROOF. Omitted.

Theorem 3.9. *Let $\{\phi_k\}$ be a fuzzy variable sequence, $\alpha_1, \alpha_2, \beta_1$ and β_2 be fixed real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$ and $s \in [1,\infty)$. Then, $Dw_s^{\alpha_1,\beta_2}[p,q] \subseteq DS_{p,q}^{\alpha_2,\beta_1}$.*

PROOF. For any fuzzy variable sequence $\{\phi_k\}$ and $\rho > 0$, we have

$$\begin{aligned}
&\left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s\right)^{\beta_2} \\
&= \left(\sum_{\substack{p(w)+1 \\ |\phi_k(\theta)-\phi_0(\theta)|\geq\rho}}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s + \sum_{\substack{p(w)+1 \\ |\phi_k(\theta)-\phi_0(\theta)|<\rho}}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s\right)^{\beta_2} \\
&\geq \left(\sum_{\substack{p(w)+1 \\ |\phi_k(\theta)-\phi_0(\theta)|\geq\rho}}^{q(w)} |x_k - L|^s\right) \\
&\geq |\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}|^{\beta_1} \rho^{s\beta_1}.
\end{aligned}$$

and so that

$$\begin{aligned}
&\frac{1}{(q(w)-p(w))^{\alpha_1}} \left(\sum_{p(w)+1}^{q(w)} |\phi_k(\theta) - \phi_0(\theta)|^s\right)^{\beta_2} \\
&\geq \frac{1}{(q(w)-p(w))^{\alpha_1}} |\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}|^{\beta_1} \cdot \rho^{s\beta_1} \\
&\geq \frac{1}{(q(w)-p(w))^{\alpha_2}} |\{p(w) < k \leq q(w) : |\phi_k(\theta) - \phi_0(\theta)| \geq \rho\}|^{\beta_1} \cdot \rho^{s\beta_1}.
\end{aligned}$$

Corollary 3.3. *(i) Let $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $\beta_1 = \beta_2 = 1$ be fixed real numbers and $s \in [1,\infty)$. Then $Dw_s^{\alpha_1}[p,q] \subseteq DS_{p,q}^{\alpha_2}$.*
(ii) Let $0 < \alpha_1 \leq 1, \alpha_2 = \beta_1 = \beta_2 = 1$ and $s \in \mathbb{R}^+$. Then $Dw_s^{\alpha_1}[p,q] \subset DS_{p,q}$.

Even if $\{\phi_k\} \in \ell_\infty$, the converse of Theorem 3.9 and Corollary 3.3 does not supply. To demonstrate this we have to establish a sequence that bounded and deferred statistically convergent almost surely of order $\alpha_2\beta_1$, but need not to be strongly s-deferred Cesàro summable of order $\alpha_1\beta_2$. To denote this assume $p(w) = 0$ and $q(w) = w$ for all $w \in \mathbb{N}$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = 1$ and $\phi = \{\phi_k\}$ be defined as follow:

$$\phi_k(\theta) = \begin{cases} \frac{1}{\sqrt{k}}, & k \neq u^3 \\ 1, & k = u^3. \end{cases}$$

It is obvious that $\phi \in \ell_\infty \cap DS^{\alpha}_{p,q}$ for $\alpha \in \left(\frac{1}{3}, 1\right]$. Recall that the inequality $\sum\limits_{k=1}^{w} \frac{1}{\sqrt{k}} > \sqrt{w}$ is supplied for $w \geq 2$. Consider

$$L_w = \left\{p(w) < k \leq q(w) : k \neq u^3,\, u = 1, 2, 3, \ldots\right\}$$

and take $s = 1$. Since

$$\begin{aligned}\sum_{p(w)+1}^{q(w)} \left|\phi_k\left(\theta\right)\right|^s = \sum_{k=1}^{w} \left|\phi_k\left(\theta\right)\right| &= \sum_{k\in L_w}^{w} \left|\phi_k\left(\theta\right)\right| + \sum_{k\notin L_w}^{w} \left|\phi_k\left(\theta\right)\right| \\ &= \sum_{k\in L_w}^{w} \frac{1}{\sqrt{k}} + \sum_{k\notin L_w}^{w} 1 > \sum_{k=1}^{w} \frac{1}{\sqrt{k}} > \sqrt{w}\end{aligned}$$

we obtain

$$\frac{1}{(q(w)-p(w))^{\alpha}} \sum_{p(w)+1}^{q(w)} \left|\phi_k\left(\theta\right)\right|^s = \frac{1}{w^{\alpha}} \sum_{k=1}^{w} \left|\phi_k\left(\theta\right)\right| > \frac{1}{w^{\alpha}} \sqrt{w} = \frac{1}{w^{\alpha-\frac{1}{2}}} \to \infty$$

as $w \to \infty$ for $\alpha \in \left(0, \frac{1}{2}\right)$. So $\phi \notin Dw^{\alpha}_{s}[p,q]$ for $\alpha \in \left(0, \frac{1}{2}\right)$ and therefore $\phi \in DS^{\alpha}_{p,q} - Dw^{\alpha}_{s}[p,q]$ for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

Theorem 3.10. *Let $\{\phi_k\}$ be a fuzzy variable sequence, $\{p(w)\}, \{q(w)\}, \{p'(w)\}$ and $\{q'(w)\}$ be sequences of nonnegative integers such that*

$$p'(w) < p(w) < q(w) < q'(w) \text{ for all } w \in \mathbb{N} \tag{3.3}$$

and $\alpha_1, \alpha_2, \beta_1, \beta_2$ be fixed real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, then
(i) If

$$\lim_{w\to\infty} \inf \frac{(q(w)-p(w))^{\alpha_1}}{(q'(w)-p'(w))^{\alpha_2}} > 0 \tag{3.4}$$

then $DS^{\alpha_2,\beta_2}_{p',q'} \subseteq DS^{\alpha_1,\beta_1}_{p,q}$.
(ii) If

$$\lim_{n\to\infty} \frac{q'(w)-p'(w)}{(q(w)-p(w))^{\alpha_2}} = 1 \tag{3.5}$$

then $DS^{\alpha_1,\beta_2}_{p,q} \subseteq DS^{\alpha_2,\beta_1}_{p',q'}$.

PROOF. Suppose (3.4) be satisfied. For given $\rho > 0$ we get

$$\begin{aligned}&\{p'(w) < k \leq q'(w) : \left|\phi_k\left(\theta\right) - \phi_0\left(\theta\right)\right| \geq \rho\} \\ &\qquad \supseteq \{p(w) < k \leq q(w) : \left|\phi_k\left(\theta\right) - \phi_0\left(\theta\right)\right| \geq \rho\},\end{aligned}$$

and so

$$\begin{aligned}&\frac{1}{(q'(n)-p'(n))^{\alpha_2}}\left|\{p'(w)<k\leq q'(w):|\phi_k(\theta)-\phi_0(\theta)|\geq\rho\}\right|^{\beta_2}\\ \geq&\frac{(q(w)-p(w))^{\alpha_1}}{(q'(w)-p'(w))^{\alpha_2}}\frac{1}{(q(w)-p(w))^{\alpha_1}}\left|\{p(w)<k\leq q(w):|\phi_k(\theta)-\phi_0(\theta)|\geq\rho\}\right|^{\beta_1}.\end{aligned}$$

As a result $DS_{p',q'}^{\alpha_2,\beta_2}\subseteq DS_{p,q}^{\alpha_1,\beta_1}$.

(ii) Omitted.

Theorem 3.11. *Suppose $\phi=\{\phi_k\}$ be a fuzzy variable sequence, $\{p(n)\},\{q(n)\},\{p'(n)\}$ and $\{q'(n)\}$ be four sequences of nonnegative integers given as in (3.3) and $\alpha_1,\alpha_2,\beta_1,\beta_2$ be fixed real numbers such that $0<\alpha_1\leq\alpha_2\leq\beta_1\leq\beta_2\leq 1$. Then*
(i) Let (3.4) satisfies, if a sequence is strongly $Dw_s^{\alpha_2,\beta_2}[p',q']$-convergent to ϕ_0, then it is $DS_{p,q}^{\alpha_1,\beta_1}$-convergent to ϕ_0,
(ii) Let (3.5) satisfies and $\phi\in\ell_\infty$, if a sequence is $DS_{p,q}^{\alpha_1,\beta_2}$-convergent to ϕ_0 then it is strongly $Dw_s^{\alpha_2,\beta_1}[p',q']$-convergent to ϕ_0.

PROOF. (i) Omitted.
(ii) Suppose that $DS_{p,q}^{\alpha_1,\beta_2}-\lim\phi_k=\phi_0$ and $\{\phi_k\}\in\ell_\infty$. Then, there is some $Q>0$ such that $|\phi_k(\theta)-\phi_0(\theta)|<Q$ for all k and for each $\theta\in A$, then for each $\rho>0$ we write

$$\begin{aligned}&\frac{1}{(q'(w)-p'(w))^{\alpha_2}}\left(\sum_{p'(w)+1}^{q'(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_1}\\
\leq&\frac{1}{(q'(w)-p'(w))^{\alpha_2}}\left(\sum_{q(w)-p(w)+1}^{q'(w)-p'(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_1}\\
&+\frac{1}{(q'(w)-p'(w))^{\alpha_2}}\left(\sum_{p(w)+1}^{q(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_1}\\
\leq&\frac{\big(q'(w)-p'(w)\big)-(q(n)-p(n))}{(q'(w)-p'(w))^{\alpha_2}}Q^{s\beta_1}+\frac{1}{(q'(w)-p'(w))^{\alpha_2}}\left(\sum_{p(w)+1}^{q(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_1}\\
\leq&\frac{\big(q'(w)-p'(w)\big)-(q(w)-p(w))^{\alpha_2}}{(q'(w)-p'(w))^{\alpha_2}}Q^{s\beta_1}+\frac{1}{(q'(w)-p'(w))^{\alpha_2}}\left(\sum_{p(w)+1}^{q(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_1}\\
\leq&\left(\frac{q'(w)-p'(w)}{(q(w)-p(w))^{\alpha_2}}-1\right)Q^{s\beta_1}+\frac{1}{(q'(w)-p'(w))^{\alpha_2}}\left(\sum_{\substack{p(w)+1\\|\phi_k(\theta)-\phi_0(\theta)|\geq\rho}}^{q(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_2}\\
&+\frac{1}{(q(w)-p(w))^{\alpha_2}}\left(\sum_{\substack{p(w)+1\\|\phi_k(\theta)-\phi_0(\theta)|<\rho}}^{q(w)}|\phi_k(\theta)-\phi_0(\theta)|^s\right)^{\beta_2}\\
\leq&\left(\frac{q'(w)-p'(w)}{(q(w)-p(w))^{\alpha_2}}-1\right)Q^{s\beta_1}\\
&+\frac{Q^{s\beta_2}}{(q(w)-p(w))^{\alpha_1}}\left|\{p(w)<k\leq q(w):|\phi_k(\theta)-\phi_0(\theta)|\geq\rho\}\right|^{\beta_2}\\
&+\frac{q'(w)-p'(w)}{(q(w)-p(w))^{\alpha_2}}\rho^{s\beta_2}.\end{aligned}$$

Utilizing (3.5), we get $Dw_s^{\alpha_2,\beta_1}[p',q']-\lim \phi_k = \phi_0$, whenever $DS_{p,q}^{\alpha_1,\beta_2}-\lim \phi_k = \phi_0$.

4. Conclusion

In this paper, within framework credibility theory, we presented several notions of statistical convergence of fuzzy variable sequences. The convergence of fuzzy variable sequences such as the notion of convergence in credibility, convergence in distribution, convergence in mean, and convergence uniformly virtually certainly via postponed Cesàro mean were examined using fuzzy variables. We investigated the connections between these concepts. Significant results on deferred statistical convergence for fuzzy variable sequences were thoroughly investigated. The findings of this study are more general and a natural extension of the conventional convergence of fuzzy variable sequences.

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REFERENCES

1. R. P. Agnew: *On deferred Cesàro means*, Ann. Math. **33** (3) (1932), 413–421.
2. F. Başar: *Summability Theory and its Applications*, Second edition, CRC Press/Taylor & Francis Group, 2022.
3. C. Belen and S. A. Mohiuddine: *Generalized weighted statistical convergence and application*, Appl. Math. Comput. **219** (2013), 9821–9826.
4. S. Bhunia, P. Das and S. K. Pal: *Restricting statistical convergence*, Acta Math. Hungar. **134** (1-2) (2012), 153–161.
5. X. Chen, Y. Ning and X. Wang: *Convergence of complex uncertain sequence*, J. Intell. Fuzzy Systems **30** (6) (2016), 3357–3366.
6. R. Çolak: *Statistical convergence of order α*, In: Modern Methods in Analysis and Its Applications, Anamaya Publisher, New Delhi, India, (2010), 121–129.
7. R. Çolak and C. A. Bektaş: *λ-Statistical convergence of order α*, Acta Math. Sin. Engl. Ser. **31** (3) (2011), 953–959.
8. D. Dubois and H. Prade: *Possibility theory: An approach to computerized processing of uncertainty*, New York, Plenum, 1998.
9. M. Et, H. S. Kandemir and M. Cinar: *Generalized deferred statistical convergence*, In: Approximation Theory, Sequence Spaces and Applications, Springer, (2022), 35–52.
10. M. Et and H. Şengül: *Some Cesàro-type summability spaces of order α and lacunary statistical convergence of order α*, Filomat **28** (8) (2014), 1593–1602.
11. H. Fast: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

12. J. A. Fridy: *On statistical convergence*, Analysis **5** (1985), 301–313.
13. A. D. Gadjiev and C. Orhan: *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32** (1) (2002), 129–138.
14. M. Gürdal, A. Şahiner and I. Açik: *Approximation theory in 2-Banach spaces*, Nonlinear Anal. **71** (5-6) (2009), 1654–1661.
15. B. Hazarika, A. Alotaibi and S. A. Mohiuddine: *Statistical convergence in measure for double sequences of fuzzy-valued functions*, Soft Comput. **24** (2020), 6613–6622.
16. Q. Jiang: *Some remarks on convergence in cedibility distribution of fuzzy variable*, In: Proceedings of a Conference on Intelligence Science and Information Engineering, Wuhan, China, 2011, pp. 446–449.
17. U. Kadak and F. Başar: *Power series with real or fuzzy coefficients*, Filomat **25** (3) (2012), 519–528.
18. U. Kadak and S. A. Mohiuddine: *Generalized statistically almost convergence based on the difference operator which includes the (p,q)-Gamma function and related approximation theorems*, Results Math. **73** (9) (2018), 1–31.
19. A. Kaufmann: *Introduction to the theory of fuzzy subsets*, New York: Academic Press, 1975.
20. Ö. Kişi, M. Gürdal and E. Savaş: *On lacunary convergence in credibility space*, Facta Univ. Math. Inform. **37** (4) (2022), 683–708.
21. Ö. Kişi, M. Gürdal and E. Savaş: *On deferred statistical convergence of fuzzy variables*, Appl. Appl. Math. **17** (2) (2022), 366–385.
22. M. Küçükaslan and M. Yilmaztürk: *On deferred statistical convergence of sequences*, Kyungpook Math. J. **56** (2) (2016), 357–366.
23. H. Kwakernaak: *Fuzzy random variables-I: definition and theorem*, Inf. Sci. **15** (1) (1978), 1–29.
24. X. Li and B. Liu: *A sufficient and necessary condition for credibility measures*, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems **14** (2006), 527–535.
25. X. Lin: *Characteristics of convex function*, Journal of Guangxi University for Nationalities (Natural Science Edition) **6** (4) (2000), 250–253.
26. B. Liu and Y. K. Liu: *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Trans. Fuzzy Syst. **10** (4) (2002), 445–450.
27. B. Liu: *Inequalities and convergence concepts of fuzzy and rough variables*, Fuzzy Optim. Decis. Mak. **2** (2) (2003), 87–100.
28. B. Liu: *Uncertainty theory*, Springer, Berlin, Heidelberg, Second Edition, 2007.
29. B. Liu: *A survey of credibility theory*, Fuzzy Optim. Decis. Mak. **5** (4) (2006), 387–408.
30. Y. K. Liu and S. M. Wang: *Theory of fuzzy random optimization, China Agricultural University Press*, Beijing, p. 280, 2006.
31. S. Ma: *The convergences properties of the credibility distribution sequence of fuzzy variables*, J. Modern Math. Frontier. **3** (1) (2014), 24–27.
32. S. A. Mohiuddine, H. Şevli and M. Cancan: *Statistical convergence in fuzzy 2-normed space*, J. Comput. Anal. Appl. **12** (4) (2010), 787–798.

33. S. A. Mohiuddine, A. Asiri and B. Hazarika: *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst. **48** (5) (2019), 492–506.

34. S. A. Mohiuddine, B. Hazarika and A. Alotaibi: *On statistical convergence of double sequences of fuzzy valued functions*, J. Intell. Fuzzy Syst. **32** (2017), 4331–4342.

35. S. A. Mohiuddine and B. A. S. Alamri: *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM **113** (3) (2019), 1955–1973.

36. M. Mursaleen and F. Başar: *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press, Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton, London, New York, 2020.

37. S. Nahmias: *Fuzzy variables*, Fuzzy Sets Syst. **1** (1978), 97–110.

38. F. Nuray and E. Savaş: *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca **45** (1995), 269–273.

39. D. Rath and B. C. Tripathy: *On statistically convergent and statistical Cauchy sequences*, Indian J. Pure Appl. Math. **25** (4) (1994), 381–386.

40. D. Rath and B. C. Tripathy: *Matrix maps on sequence spaces associated with sets of integers*, Indian J. Pure Appl. Math. **27** (2) (1996), 197–206.

41. T. Šalát: *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (2) (1980), 139–150.

42. E. Savaş and M. Gürdal: *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, Scienceasia **41** (4) (2015), 289–294

43. E. Savaş, Ö. Kişi and M. Gürdal: *On statistical convergence in credibility space*, Numer. Funct. Anal. Optim. **43** (8) (2022), 987–1008.

44. H. Şengül and M. Et: *On lacunary statistical convergence of order α*, Acta Math. Sci. Ser. B Engl. Ed. **34** (2) (2014), 473–482.

45. H. Steinhaus: *Sur la convergence ordinarie et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.

46. Ö. Talo and F. Başar: *Quasilinearity of the classical sets of sequences of fuzzy numbers and some related results*, Taiwanese J. Math. **14** (5) (2010), 1799–1819.

47. F. Temizsu, M. Et, and M. Cinar: *Δ^m-deferred statistical convergence of order α*, Filomat **30** (3) (2016), 667–673.

48. B. C. Tripathy: *Matrix transformations between some classes of sequences*, J. Math. Anal. **206** (1997), 448–450.

49. B. C. Tripathy and M. Sen: *On generalized statistically convergent sequences*, Indian J. Pure Appl. Math. **32** (11) (2001), 1689–1694.

50. G. Wang and B. Liu: *New theorems for fuzzy sequence convergence*, In: Proceedings of a Conference on Information and Management Science, Chengdu, China, 2003, pp. 100–105.

51. Y. Xia: *Convergence of uncertain sequences*, M.S. Thesis, Suzhou University of Science and Technology, 2011.

52. C. You: *On the convergence of uncertain sequences*, Math. Comput. Model. **49** (3-4) (2009), 482–487.

53. M. Yilmaztürk and M. Küçükaslan: *On strongly deferred Cesàro summability and deferred statistical convergence of the sequences*, Bitlis Eren Univ. J. Sci. Technol. **3** (2011), 22–25.
54. C. You, R. Zhang and K. Su: *On the convergence of fuzzy variables*, J. Intell. Fuzzy Systems **36** (2) (2019), 1663–1670.
55. L. A. Zadeh: *Fuzzy set*, Inf. Control. **8** (3) (1965), 338–353.