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ON THE DARBOUX ROTATION AXIS OF CURVES IN THREE-DIMENSIONAL LIE GROUPS

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Abstract. This paper presents a study on the motion of the Darboux rotation axis for curves in three-dimensional Lie groups a bi-invariant metric using the Frenet frame. Our findings reveal that the motion of the axis can be resolved into two simultaneous rotational motions. Additionally, we develop a sequence of Darboux vectors that are useful in constructing uncomplicated mechanisms. We also introduce a curve with constant precession and conduct an exhaustive analysis of its characteristics.
Keywords: Darboux axis, slant helix, a curve of constant precession, Lie groups.

1. Introduction

The investigation of the geometry of curves is a traditional problem in mathematics that has significant relevance to various areas of science and engineering, including control theory, computer graphics, and robotics, among others [8, 12, 15, 16, 31]. Specifically, the Darboux vector, also known as the Darboux axis, is a fundamental concept in differential geometry utilized to explain the behavior of curves in three-dimensional space. The Darboux vector refers to the derivative of the normal vector with respect to the arc length parameter of the curve, and the Darboux axis is the

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unit vector in the direction of the Darboux vector. Hence, the Darboux axis signifies the immediate direction of the normal vector's rate of change along the curve, and it plays a pivotal role in the analysis of curves and surfaces in differential geometry ([11]), as well as various applications such as computer graphics, robotics, and others ([3, 13, 14, 26, 28]).

In Euclidean space $\mathbb{E}^3$, let α be a regular C^3 space curve with positive curvature κ and torsion τ. Let t, n, b be the Frenet frame of the curve α. It is well known that the (corresponding) Frenet derivative equations describe a rotation with the Darboux vector $W = \tau t + \kappa b$ as a vector of rotation. This rotation is called a *Darboux rotation*. In [18], the author investigated the Darboux rotation and its decomposition into two plane rotations. Using the results of Hartl, Barthel discussed the rotation of the instantaneous Darboux axis in a little more detail, [4]. In [10], Çöken and Görgülü defined the dual Darboux rotation axis as the axis of rotation of the dual Frenet frame of the dual space curve in $\mathbb{D}^3$ and showed that it is possible to determine the direction of the rotation of the dual Darboux rotation axis using certain geometric properties of the dual space curve. In recent decades, there have been many researchers investigated the properties of the Darboux rotation axis in 3-dimensional space [1, 2, 5, 6, 7, 20, 21, 23, 29, 33, 34] and obtained some interesting conclusions.

Motivated by those ideas, in this paper, we examine the motion of the Darboux rotation axis for a curve in three-dimensional Lie groups with a bi-invariant metric using the Frenet frame. We show that the motion of the Darboux rotation axis can be decomposed into two simultaneous rotation motions, where the tangent and binormal vector fields of the curve rotate around each other with different angular speeds. Using this approach, we derive a series of Darboux vectors, which can be used to create simple mechanisms. Also, we investigate the motion of a rigid body that is rotating about a fixed axis and specifically focus on the path of a point on the rigid body that is undergoing constant precession in three-dimensional Lie groups with a bi-invariant metric. Moreover, we show that this path can be described as a curve of constant precession, and provides a detailed analysis of the properties of such curves.

2. Preliminaries

In this section, we review some basic notions of differential geometry and Lie theory that will be needed in the rest of the paper, ([17, 19]).

Let $\mathbb{G}$ be a Lie group with the Euclidean metric $\langle\ ,\rangle$ and D be the Levi-Civita connection of $\mathbb{G}$. The Lie algebra of $\mathbb{G}$, denoted by $\mathfrak{g}$, is isomorphic to the tangent space at the neutral (identity) element e of $\mathbb{G}$, i.e., $\mathfrak{g} \cong T_e\mathbb{G}$. In this case, any vector field $X \in \mathfrak{g}$ can be written as a linear combination of basis elements $\{U_1, U_2, \ldots, U_n\}$ of $\mathfrak{g}$, i.e.,

$$X = \sum_{i=1}^{n} x_i U_i$$

where $x_i : I \longrightarrow \mathbb{R}$ are some smooth functions. The Lie bracket of vector fields in $\mathfrak{g}$ is given by

$$[X,Y] = \sum_{i,j=1}^{n} x_i y_j \left[U_i, U_j\right],$$

where $X = \sum_{i=1}^{n} x_i U_i$ and $Y = \sum_{i=1}^{n} y_i U_i$. Moreover, for all $X, Y, Z \in \mathfrak{g}$, the following identities hold:

$$\langle X, [Y,Z] \rangle = \langle [X,Y], Z \rangle, \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X,Y], \tag{2.2}$$

where $D_X Y$ denotes the covariant derivative of Y in the direction of X.

Let $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{G}$ be a smooth curve parameterized by arc-length s in $\mathbb{G}$. Then, the covariant derivative of a vector field X along γ in $\mathfrak{g}$ is given by

$$D_T X = \dot{X} + \frac{1}{2}[T,X], \tag{2.3}$$

where $T = \dot{\gamma} = \frac{d\gamma}{ds}$ is the tangent vector field of γ and $\dot{X} = \sum_{i=1}^{n} \frac{dx_i}{ds} U_i$. It is worth noting that if X is the left-invariant vector field to the curve γ on $\mathbb{G}$, then $\dot{X} = 0$, [9].

Let α be a smooth curve parameterized by arc-length s in the three-dimensional Lie group $\mathbb{G}$ with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$. By using the relation (2.3) the curvatures κ and τ of α can be expressed as

$$\begin{aligned} \kappa &= \|D_T T\| = \left\|\dot{T}\right\|, \\ \tau &= \left\langle \dot{B}, N \right\rangle + \tau_G, \end{aligned} \tag{2.4}$$

where τ_G is given by

$$\tau_G = \frac{1}{2}\langle [T,N], B \rangle, \tag{2.5}$$

or equivalently,

$$\tau_G = \frac{1}{2\kappa^2\tau}\left\langle \ddot{T}, \left[T, \dot{T}\right] \right\rangle + \frac{1}{4\kappa^2\tau}\left\| \left[T, \dot{T}\right] \right\|^2, \tag{2.6}$$

is called Lie torsion ([9]). Therefore, the following propositions can be given:

Proposition 2.1. *Let α be an arc-length parameterized curve in three-dimensional Lie group $\mathbb{G}$ with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$. Then the following equalities hold*

$$[T,N] = \langle [B,T], N \rangle B = 2\tau_G B, \tag{2.7}$$

$$[T,B] = \langle T, [N,B] \rangle N = 2\tau_G N, \tag{2.8}$$

$$[N,B] = \langle [N,B], T \rangle T = 2\tau_G T. \tag{2.9}$$

Proposition 2.2. *For a three-dimensional Lie group $\mathbb{G}$ with bi-invariant metric, the following properties hold:*
i) If $\mathbb{G}$ is a commutative (Abel) group, then $\tau_G = 0$;
ii) If $\mathbb{G}$ is the special orthogonal group $SO(3)$, then $\tau_G = \frac{1}{2}$;
iii) If $\mathbb{G}$ is the special unitary group $S^3 \cong SU(2)$, then $\tau_G = 1$.

Definition 2.1. Let α be a parameterized curve in a three-dimensional Lie group $\mathbb{G}$ with bi-invariant metric. The curve α is called a *generalized helix* if its tangential direction makes a constant angle with a left-invariant vector field X, i.e. $\langle T, X\rangle = const.$

Generalized helices are characterized as follows (see [9] for the proof):

Theorem 2.1. *A curve in $\mathbb{G}$ is a generalized helix if and only if*

$$\frac{\tau - \tau_G}{\kappa} = constant. \tag{2.10}$$

Definition 2.2. Let a regular curve α be given in $\mathbb{G}$. If its principal normal vector makes a constant angle with a left-invariant vector field, then the curve α is called a *slant helix*, ([24]).

Slant helices are characterized as follows (see [24, 25] for the proof):

Theorem 2.2. *A curve is a slant helix in $\mathbb{G}$ if and only if*

$$\frac{\kappa\left(H^2+1\right)^{\frac{3}{2}}}{\dot{H}} = const, \tag{2.11}$$

where $H = \frac{\tau-\tau_G}{\kappa}$ is called the harmonic curvature function of the curve.

Remark 2.1. Yampolsky et al. extended the above definitions to three-dimensional Lie groups with left-invariant metric and introduced the first, second, and third kinds of helices ([30]). They obtained the description of these helices in terms of new geometric invariants of the curve and generalized the corresponding descriptions for helices in three-dimensional Lie groups with bi-invariant metric. Also, Yoon investigated $AW(k)-$type curves within the Lie group $\mathbb{G}$, utilizing a bi-invariant metric and established a characterization of general helices by means of $AW(k)-$type curves ([32]). Moreover, Mak studied Frenet curves and their natural mate and conjugate mate in a $3-$dimensional Lie group with bi-invariant metric ([22]).

3. On the Darboux axis in three-dimensional Lie groups

In this section, we will introduce the concept of the Darboux rotation axis in a three-dimensional Lie group $\mathbb{G}$ equipped with a bi-invariant metric $\langle , \rangle$. Additionally, we will give a description of some special curves, such as general helices and slant helices, in terms of new geometric invariants of the curve

From Proposition 2.1 and the relation (2.3), we can be expressed the Frenet formulas of the curve α as

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau-\tau_G \\ 0 & -(\tau-\tau_G) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{3.1}$$

which implies that the Darboux axis rotates in the plane spanned by T and B at a rate proportional to the curvature κ and that the rate of rotation is also affected by $\tau - \tau_G$. Hence, the Frenet frame $\{T, N, B\}$ of a curve α makes an instantaneous helix motion in three-dimensional Lie group $\mathbb{G}$ and there exists an axis of frame's rotation called a *Darboux vector* (*centrode*), whose direction is given by

$$W = (\tau - \tau_G)\,T + \kappa B, \tag{3.2}$$

satisfies the *Darboux equations*

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = W \times \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Therefore, the vectors T and B rotate with κ and $(\tau - \tau_G)$ angular speeds around the vectors B and T, respectively (i.e. $D_T T = (\kappa B) \times T$ and $D_T B = ((\tau - \tau_G)\,T)$ $\times B$). The norm of Darboux vector is

$$\|W\| = \sqrt{\kappa^2 + (\tau - \tau_G)^2},$$

and so the unit vector in the $W-$direction is obtained

$$\omega = \frac{W}{\|W\|} = \frac{(\tau - \tau_G)\,T + \kappa B}{\sqrt{\kappa^2 + (\tau - \tau_G)^2}}. \tag{3.3}$$

Taking a derivative the relation (3.3) and using the Frenet formulas give by (3.1), we get

$$\dot{\omega} = \left(\frac{\kappa\,(\tau - \tau_G)' - (\tau - \tau_G)\,\kappa'}{\kappa^2 + (\tau - \tau_G)^2} \right) \frac{\kappa T - (\tau - \tau_G)\,B}{\sqrt{\kappa^2 + (\tau - \tau_G)^2}}. \tag{3.4}$$

Putting $\frac{(\tau-\tau_G)\kappa' - \kappa(\tau-\tau_G)'}{\kappa^2 + (\tau-\tau_G)^2} = \delta$, the relation (3.4) can be written as

$$\dot{\omega} = \frac{\delta}{\|W\|}\dot{N} \tag{3.5}$$

Since ω is orthogonal to both vector $\dot{\omega}$ and N, the unit vector v in the $\dot{\omega}-$direction is given by

$$v = \frac{\dot{\omega}}{\|\dot{\omega}\|} = \frac{1}{\|W\|}\dot{N} \tag{3.6}$$

Therefore, we get a new Frenet trihedron $\{N, v, \omega\}$ of the curve α in $\mathbb{G}$. In this situation, the Frenet equations are obtained as:

$$\begin{aligned} \dot{N} &= \|W\| \, v, \\ \dot{v} &= \delta\omega - \|W\| \, N, \\ \dot{\omega} &= -\delta v, \end{aligned} \tag{3.7}$$

where

$$\|W\| = \sqrt{\kappa^2 + (\tau - \tau_G)^2} > 0, \tag{3.8}$$

and

$$\delta = \frac{(\tau - \tau_G)\,\kappa' - \kappa\,(\tau - \tau_G)'}{\kappa^2 + (\tau - \tau_G)^2} = -\frac{\left(\frac{\tau - \tau_G}{\kappa}\right)'}{1 + \left(\frac{\tau - \tau_G}{\kappa}\right)^2}, \tag{3.9}$$

are curvatures of the curve α in terms of the moving orthonormal frame $\{N, v, \omega\}$. Moreover, from the relations (3.8) and (3.9) we have

$$\frac{\delta}{\|W\|} = -\frac{\left(\frac{\tau - \tau_G}{\kappa}\right)'}{\kappa\left[1 + \left(\frac{\tau - \tau_G}{\kappa}\right)^2\right]^{3/2}}. \tag{3.10}$$

or since the harmonic curvature function $H = \frac{\tau - \tau_G}{\kappa}$ of the curve,

$$\frac{\delta}{\|W\|} = -\frac{H'}{\kappa\,(1 + H^2)^{3/2}}.$$

Using Theorem 2.2, we can give the following corollary.

Corollary 3.1. *Let $\alpha : I \subseteq R \to \mathbb{G}$ be a unit speed curve in three-dimensional Lie group $\mathbb{G}$ with the moving orthonormal frame. Then, α is a slant helix if and only if the ratio $\frac{\delta}{\|W\|}$ given by (3.10) is a constant function.*

Now, with the help of the equations (3.7), we can obtain the Darboux vector W_1 as follows

$$W_1 = \delta N + \|W\| \, \omega,$$

and from the relation (3.3) we get

$$W_1 = \delta N + W.$$

Also, the momentum rotation vector is expressed as follows:

$$\begin{bmatrix} \dot{N} \\ \dot{v} \\ \dot{\omega} \end{bmatrix} = W_1 \times \begin{bmatrix} N \\ v \\ \omega \end{bmatrix}.$$

Here, the rotation motion of this axis can be separated into two rotation motions, that is, N rotates with the angular speed $\|W\|$ around the vector ω, and ω rotates with the angular speed δ around the vector N.

Remark 3.1. When continued in a similar way, the rotation motion of the Darboux axis is done in a consecutive manner, and so the series of Darboux vectors can be obtained.

4. Curves of constant precession in three-dimensional Lie groups

According to Scofield in [27], the Euclidean curves of constant precession are defined as the curves, whose Darboux vectors make a constant angle with a fixed direction and rotate about it with a constant speed. Therefore, the curves of constant precession are a special kind of helices.

In this section, we introduce curves of constant precession in three-dimensional Lie groups with bi-invariant metrics, and some characterizations are obtained.

Definition 4.1. Let $\alpha : I \subseteq R \to \mathbb{G}$ be a unit speed curve in the three-dimensional Lie group $\mathbb{G}$ with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$ and W be the Darboux vector of α. If the Darboux vector W revolves around a left-invariant vector field d with constant angle (i.e., $\langle W, d\rangle = const.$) and constant speed (i.e., $\|W\| = const.$), then α is called a curve of constant precession.

Lemma 4.1. *Let $\alpha : I \subseteq R \to \mathbb{G}$ be a unit speed curve of constant precession in the three-dimensional Lie group $\mathbb{G}$ with the Frenet frame appartus $\{T, N, B, \kappa, \tau\}$. If we consider the left-invariant vector field $d(s) = W(s) + bN(s)$ arbitrary constants $a > 0$, b and $c = \sqrt{a^2 + b^2}$, then the following are equivalent:*
(i) $\|W\| = a$;
(ii) $\cos\varphi = \frac{a}{c}$, *where φ is the angle between d and W;*
(iii) $\|D_T N\| = a$;
(iv) $\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{b}{c}$;
(v) $\|d\| = c$.

Proof. Using the relations (3.1) and (3.2) we get $\|W\| = \sqrt{(\tau - \tau_G)^2 + \kappa^2} = \left\|\dot{N}\right\| = a$, and also $\|d\| = \sqrt{a^2 + b^2} = c$. Then, it is clear that **(i)**, **(iii)** and **(v)** hold. Moreover, since $\langle W, d\rangle = \|W\| \|d\| \cos\varphi$, from the relation (3.2) and parts (i) and (v) of the Lemma 4.1 we get $\cos\varphi = \frac{a}{c}$. Therefore, **(ii)** is proved.
Since the vector field d is lying in the plane $Span\{W, N\}$ and W is orthogonal to N, the angle between d and N is $\frac{\pi}{2} - \varphi$. Using $c^2 = a^2 + b^2$ and part (v) of the Lemma, we prove **(iv)**. These complete the proof. □

Corollary 4.1. *Let $\alpha : I \subseteq R \to \mathbb{G}$ be a unit speed curve of constant precession in the three-dimensional Lie group $\mathbb{G}$ with the Frenet frame appartus $\{T, N, B, \kappa, \tau\}$. Then, the followings hold:*
1) $\dot{d} = 0$;
2) $\left\|\dot{W}\right\| = \left\|b\dot{N}\right\|$.

Corollary 4.2. *Let $\alpha : I \subseteq R \to \mathbb{G}$ be a unit speed curve in the three-dimensional Lie group $\mathbb{G}$ with the Frenet frame appartus $\{T, N, B, \kappa, \tau\}$. Then, α is a curve of constant precession if and only if α is a slant helix*

Proof. Assume that α is a curve of constant precession. From Lemma 4.1, we get $\langle N, d\rangle = \|N\| \|d\| \cos\left(\frac{\pi}{2} - \varphi\right) = b = const.$, and so the curve α is a slant helix.
Conversely, assume that α is a slant helix. Then, using Lemma 4.1, we can say that α is a curve of constant precession. □

Theorem 4.1. *Let $\alpha : I \subseteq R \to \mathbb{G}$ be a unit speed curve of constant precession in the three-dimensional Lie group $\mathbb{G}$ with the Frenet frame appartus $\{T, N, B, \kappa, \tau\}$. Then, the curvature and the torsion of the curve α are*

$$\kappa = -a\sin(bs), \quad \tau = a\cos(bs) + \tau_G,$$

respectively.

Proof. Let α be a curve of constant precession in $\mathbb{G}$. Then there is a left-invariant vector field $d = W + bN$ such that $\dot{d} = 0$. From the relations (2.3), (3.1) and (3.2) we obtain

$$\left((\tau - \tau_G)' - \kappa b\right) T + \left(\kappa' + b(\tau - \tau_G)\right) B = 0. \tag{4.1}$$

and since T and B are nonzero vector fields, the following system can be written

$$\begin{cases} (\tau - \tau_G)' - \kappa b = 0, \\ \kappa' + b(\tau - \tau_G) = 0. \end{cases} \tag{4.2}$$

Also, from the relations in (4.2), we get

$$(\tau - \tau_G)^2 + \kappa^2 = C, \tag{4.3}$$

where C is the constant of integration. From Lemma 4.1, $a^2 = C$ and so we have

$$\begin{aligned} \kappa &= a\sin\theta, \\ (\tau - \tau_G) &= a\cos\theta, \end{aligned} \tag{4.4}$$

where $\theta = \theta(s)$ is some function of s.
Putting (4.2) into the second equation of (4.4), we get $\theta' = -b$, and so $\theta = -bs$. Therefore, the curvature and the torsion of the curve α can be written as

$$\begin{aligned} \kappa &= -a\sin(bs), \\ \tau &= a\cos(bs) + \tau_G. \end{aligned}$$

□

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