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ON HOMOGENEOUS 2-DIMENSIONAL FINSLER MANIFOLDS WITH ISOTROPIC FLAG CURVATURES

Akbar Tayebi[1] and Behzad Najafi[2]

[1] **Department of Mathematics, Faculty of Science**
University of Qom, Qom, Iran
[2] **Department of Mathematics and Computer Sciences**
Amirkabir University, Tehran, Iran

Abstract. We show that every Finsler surface with isotropic main scalar and isotropic flag curvature is Riemannian or relatively constant Landsberg metric. Using it, we prove that every homogeneous Finsler surface with isotropic flag curvature and isotropic main scalar is Riemannian or locally Minkowskian.
Keywords: Finsler surface, Landsberg metric, Riemannian surface.

1. Introduction

For a given Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(\Pi, y)$ is a function of tangent planes $\Pi = \text{span}\{y, v\} \subset T_xM$ and directions $y \in \Pi \setminus \{0\}$. If F is a Riemannian metric, then the flag curvature is independent of the direction and can be written as $\mathbf{K} = \mathbf{K}(\Pi)$. In this special case, $\mathbf{K}$ is called the sectional curvature of F. Also, F is said to be of scalar flag curvature if the flag curvature is a scalar function on the slit tangent space, namely $\mathbf{K} = \mathbf{K}(x, y)$. F is called of isotropic flag curvature if the flag curvature $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on the manifold M. A Riemannian metric is of scalar curvature if and only if $\mathbf{K} = \mathbf{K}(x)$ is a scalar function on M, which is a constant in dimension $n > 2$ by the Schur lemma. One

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Corresponding Author: Akbar Tayebi. E-mail addresses: akbar.tayebi@gmail.com (A. Tayebi), behzad.najafi@aut.ac.ir (B. Najafi)
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of the important problems in Finsler geometry is to study and characterize Finsler metrics of isotropic flag curvature.

In order to study the class of Finsler metrics of isotropic flag curvature, one may consider 2-dimensional Finsler metrics. In Finsler geometry, the behavior of 2-dimensional Finsler metrics is different and sometimes contradictory to the higher dimensions. For example, all 2-dimensional Finsler metrics are C-reducible, while they need not be of Randers or Kropina type. Also, Finsler surfaces are of scalar flag curvature, while these cases are not valid for higher dimensions. Due to the latter issue, Z. Shen constructed three families of Finslerian surfaces on $\mathbb{S}^2$ and $\mathbb{D}^2$ with constant flag curvature that are not projectively flat, and thus the Beltrami's famous theorem in Finsler geometry lost its validity in the world of Finslerian surfaces [14].

To study of Finsler surfaces separately, L. Berwald made a special frame for Finsler surfaces, namely Berwald's frame. In this frame, a function appears that depends of the tangent space of Finsler surface and distinguishes each metric from the other metrics. This function is known as the main scalar of the Finsler surface and denoted by $I = I(x, y)$. In [9], Matsumoto gave some geometrical meanings of the main scalar of Finsler surfaces. Very soon, Berwald discovered that the Finsler surfaces with constant main scalar are Berwald, Landsberg or Douglas surfaces [4]. Then, he characterized two-dimensional Finsler metrics with isotropic main scalar $I = I(x)$. Using this characterization, Berwald succeeded to find the classification of two-dimensional projectively flat Finsler metrics with isotropic main scalar [4]. These studies shows that the class of Finsler surfaces with isotropic main scalars has important position in Finsler geometry and deserves to more studies.

Among the class of two-dimensional Finsler metrics, homogeneous Finsler surfaces are interesting, and until now little study has been done on these spaces. Then, it is natural to study homogeneous Finsler manifolds. A Finsler manifold is called homogeneous if its group of isometries acts transitively on the manifold. In [5], Deng and Hou proved that the group of isometries $I(M, F)$ of a Finsler manifold (M, F) is a Lie transformation group of the underlying manifold which can be used to study homogeneous Finsler manifolds. In this case, M can be written as the quotient manifold $I(M, F)/H$, where H is the stabilizer subgroup at a point in M. Recently, the authors proved that there is not any unicorn among the homogeneous Finsler surfaces [18]. In this paper, we study homogeneous Finsler surfaces with isotropic main scalar $I = I(x)$ and isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$, and prove the following rigidity result.

Theorem 1.1. *Every homogeneous Finsler surface with isotropic main scalar and isotropic flag curvature is Riemannian or locally Minkowskian.*

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent space and $TM_0 := TM - \{0\}$ the slit tangent space of M. A Finsler structure on manifold M is a function $F : TM \to [0, \infty)$ with the following properties: (i) F is C^∞ on

TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x,\lambda y)=\lambda F(x,y)$, $\forall\lambda>0$; (iii) The quadratic form $\mathbf{g}_y : T_xM\times T_xM\to\mathbb{R}$ is positive-definite on T_xM

$$\mathbf{g}_y(u,v):=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]_{s=t=0},\quad u,v\in T_xM.$$

Then, the pair (M,F) is called a Finsler manifold.

Let $x\in M$ and $F_x:=F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM\times T_xM\times T_xM\to\mathbb{R}$ by

$$\mathbf{C}_y(u,v,w):=\frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0},\quad u,v,w\in T_xM.$$

The family $\mathbf{C}:=\{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=0$ if and only if F is Riemannian.

For $y\in T_xM_0$, define $\mathbf{I}_y : T_xM\to\mathbb{R}$ by

$$\mathbf{I}_y(u):=\sum_{i=1}^{n}g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x\in M$. The family $\mathbf{I}:=\{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y)=0$ and $\mathbf{I}_{\lambda y}=\lambda^{-1}\mathbf{I}_y$, $\lambda>0$. Therefore, $\mathbf{I}_y(u):=I_i(y)u^i$, where

$$I_i:=g^{jk}C_{ijk}.$$

Let $F=F(x,y)$ be a Finsler metric on an n-dimensional manifold M. The distortion $\tau=\tau(x,y)$ on TM associated with the Busemann-Hausdorff volume form

$$dV_{BH}=\sigma(x)dx$$

is defined by

$$\tau(x,y)=\ln\frac{\sqrt{\det\big(g_{ij}(x,y)\big)}}{\sigma(x)}.$$

By definition, the distortion τ is homogeneous of degree 1 with respect to y, i.e., the following holds

$$\tau(\lambda y)=\lambda\tau(y),\qquad \lambda>0,\ y\in T_xM_0.$$

The following holds.

Lemma 2.1. ([13]) *Let F be a positive-definite Finsler metric on a manifold M. Then the following conditions are equivalent*
(a) $\tau=constant$;
(b) $\mathbf{I}=0$;
(c) $\mathbf{C}=0$;
In any case, F reduces to a Riemannian metric.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}, \quad y \in T_x M.$$

$\mathbf{G}$ is called the associated spray to (M, F).

Define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \partial/\partial x^i|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2} \mathbf{g}_y \big(\mathbf{B}_y(u, v, w), y \big).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L_{ijk}(y) u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2} y_l B^l{}_{ijk}.$$

$\mathbf{L}$ is called the Landsberg curvature and F is called a Landsberg metric if $\mathbf{L} = 0$. Also, F is called of relatively isotropic Landsberg curvature if

$$L_{ijk} = cFC_{ijk},$$

where $c = c(x)$ is a scalar function on M.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y) u^i$, where

$$J_i := g^{jk} L_{ijk}.$$

The quantity $\mathbf{J}$ is called the mean Landsberg curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J} = 0$. By definition, every Landsberg metric is a weakly Landsberg metric. F is called of relatively isotropic mean Landsberg curvature if

$$J_i = cFI_i,$$

where $c = c(x)$ is a scalar function on M.

For a non-zero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y) u^k \partial/\partial x^i$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature.

For a flag $P := \mathrm{span}\{y, u\} \subset T_xM$ with the flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(x, y, P) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \tag{2.2}$$

The flag curvature $\mathbf{K} = \mathbf{K}(x, y, P)$ is a function of tangent planes $P = \mathrm{span}\{y, v\} \subset T_xM$. This quantity tells us how curved the space is at a point. A Finsler metric F is of scalar flag curvature, if $\mathbf{K}(x, y, P) = \mathbf{K}(x, y)$ is independent of P. In this case, the flag curvature is just a scalar function on the tangent space of M.

The pulled-back bundle π^*TM admits a unique linear connection, called the Berwald connection. Let (M, F) be an n-dimensional Finsler manifold. Let $\{e_j\}$ be a local frame for π^*TM, $\{\omega^i, \omega^{n+i}\}$ be the corresponding local coframe for $T^*(TM_0)$ and $\{\omega^i_j\}$ be the set of local Berwald connection forms with respect to $\{e_j\}$. In local coordinate system, the Berwald connection determined by following

$$d\omega^i = \omega^j \wedge \omega^i_j, \tag{2.3}$$

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{2.4}$$

where

$$\begin{aligned} \omega^i &:= dx^i, \\ \omega^{n+k} &:= dy^k + y^j \omega^k{}_j. \end{aligned}$$

Thus

$$g_{ij|k} = -2L_{ijk}, \qquad g_{ij,k} = 2C_{ijk}.$$

For a tensor $\mathbf{T} = T_{i\cdots k}\omega^i \otimes \cdots \otimes \omega^k$, we have

$$T_{i\cdots k\cdot m} = \frac{\partial T_{i\cdots k}}{\partial y^m}.$$

For a non-zero vector $y \in T_xM$, the tensor $\mathbf{T}$ induces a multi-linear form

$$\mathbf{T}_y(u, \cdots, w) := T_{i\cdots k}(x, y)u^i \cdots w^k$$

on T_xM. Let $\sigma(t)$ denote the geodesic with $\dot\sigma(0)=y$. We have

$$\frac{d}{dt}\Big[\mathbf{T}_{\dot\sigma(t)}\Big(U(t),\cdots,W(t)\Big)\Big]=T_{i\cdots k|m}(\sigma(t),\dot\sigma(t))\dot\sigma^m(t)U^i(t)\cdots W^k(t),$$

where $U(t)=U^i(t)\partial/\partial x^i|_{\sigma(t)},\cdots,W(t)=W^k(t)\partial/\partial x^k|_{\sigma(t)}$ are linearly parallel vector fields along σ. Thus the Landsberg curvature is given by

$$L_{ijk}=C_{ijk|m}y^m. \tag{2.5}$$

3. Proof of Theorem 1.1

It is well known that for any Minkowskian plane $(V,\mathcal{F})$ and any vector $v\in V$ with $\mathcal{F}(v)\neq 0$, there is a non-zero vector $w\in V$ such that is orthogonal to v with respect to the fundamental tensor raised by Minkowski functional $\mathcal{F}$. The special and useful Berwald frame was founded and developed method by Berwald in order to study of two-dimensional Finsler spaces [4]. It works under the assumption that the fundamental tensor is positive-definite.

Let (M,F) be a two-dimensional Finsler manifold. It is easy to see that for every $\mathbf{y}\in T_xM$, $x\in M$, there is a vector $\mathbf{y}^\perp\in T_xM_0$ such that

$$\mathbf{g}(\mathbf{y},\mathbf{y}^\perp)=0,\qquad \mathbf{g}(\mathbf{y}^\perp,\mathbf{y}^\perp)=F(\mathbf{y}).$$

The pair $\{\mathbf{y},\mathbf{y}^\perp\}$ is called the Berwald frame at $\mathbf{y}$.

Based on the Berwald frame, the Cartan torsion can be determined by a scalar function on slit tangent bundle. Let us define

$$\mathcal{I}(\mathbf{y}):=\frac{\mathbf{C}_\mathbf{y}(\mathbf{y}^\perp,\mathbf{y}^\perp,\mathbf{y}^\perp)}{F(\mathbf{y})}=\mathcal{I}(\mathbf{y}^\perp).$$

One can see that $\mathcal{I}(\lambda\mathbf{y})=\mathcal{I}(\mathbf{y})$ holds for $\forall\lambda>0$ and $\forall\mathbf{y}\in T_xM_0$. We call $\mathcal{I}$ the main scalar of Finsler metric F.

In most of literature of Finsler geometry, the special notation (ℓ,m) was used instead of $\{\mathbf{y},\mathbf{y}^\perp\}$. By considering this notation, for a scalar $T=T(x,y)$, we define the horizontal scalar derivatives $(T_{|1},T_{|2})$ and vertical scalar derivatives $(T_{,1},T_{,2})$ as follows

$$T_{|i}:=T_{|1}\ell_i+T_{|2}m_i,\quad FT_{,i}:=T_{,1}\ell_i+T_{,2}m_i,$$

where

$$T_{|i}:=\frac{\partial T}{\partial x^i}-G^j_{\ i}\frac{\partial T}{\partial y^j},\qquad FT_{,i}:=F\frac{\partial T}{\partial y^i}$$

denote the horizontal and vertical derivations with respect to the Berwald connection of F and

$$G^j_{\ i}:=\frac{\partial G^i}{\partial y^j}.$$

In order to prove Theorem 1.1, we need to know the special form of Berwald curvature of Finsler surface. We remark that the following identity holds

$$B^p_{jkl} = g^{ip}\Big\{C_{ijl|k} + C_{ikl|j} - C_{jkl|i} + L_{ijk,l}\Big\}. \tag{3.1}$$

See (10.19) at page 145 in [13]. On the other hand, the Cartan torsion of a Finsler surface (M, F) has no components in the direction ℓ^i, i.e., $C_{ijk}y^i = 0$. Then it can be written in the Berwald frame (ℓ, m) as follows

$$FC_{ijk} = \mathcal{I}m_i m_j m_k. \tag{3.2}$$

Taking a horizontal derivation of (3.2) implies that

$$FC_{ijk|s} = \big(\mathcal{I}_{|1}\ell_s + \mathcal{I}_{|2}m_s\big)m_i m_j m_k. \tag{3.3}$$

Contracting (3.3) with y^s yields

$$FL_{ijk} = \mathcal{I}_{|1}m_i m_j m_k. \tag{3.4}$$

By putting (3.3) and (3.4) in (3.1), we get

$$FB^i{}_{jkl} = \Big\{-2\mathcal{I}_{|1}\ell^i + \big(\mathcal{I}_{|1,2} + \mathcal{I}_{|2}\big)m^i\Big\}m_j m_k m_l. \tag{3.5}$$

Let us put

$$\mathcal{I}_2 := \mathcal{I}_{|1,2} + \mathcal{I}_{|2}.$$

Thus the Berwald curvature of Finsler surfaces is given by

$$B^i{}_{jkl} = \frac{1}{F}\Big(\mathcal{I}_2 m^i - 2\mathcal{I}_{|1}\ell^i\Big)m_j m_k m_l. \tag{3.6}$$

By (3.2) and (3.6), we have

$$B^i{}_{jkl} = -\frac{2\mathcal{I}_{,1}}{\mathcal{I}}C_{jkl}\ell^i + \frac{\mathcal{I}_2}{3F}\Big\{h_{jk}h^i_l + h_{kl}h^i_j + h_{lj}h^i_k\Big\}, \tag{3.7}$$

where $\mathbf{h} = h_{ij}dx^i dx^j$ denotes the angular metric. Then for a Finsler surface, the Berwald curvature can be written as follows

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda\Big(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}\Big), \tag{3.8}$$

where

$$\mu := -\frac{2}{\mathcal{I}}\mathcal{I}_{|1}, \qquad \lambda := \frac{1}{3}\mathcal{I}_2.$$

Proposition 3.1. *Every non-Riemannian Finsler surface with isotropic main scalar and isotropic flag curvature is a relatively constant Landsberg metric.*

Proof. A 2-dimensional Finsler metrics F is of scalar curvature $\mathbf{K}=\mathbf{K}(x,y)$. This is equivalent to the following identity:

$$R^i{}_k=\mathbf{K}F^2\ h^i_k. \tag{3.9}$$

The following hold

$$L_{ijk|m}y^m=-\frac{1}{3}F^2\Big\{\mathbf{K}_{.i}h_{jk}+\mathbf{K}_{.j}h_{ik}+\mathbf{K}_{.k}h_{ij}+3\mathbf{K}C_{ijk}\Big\} \tag{3.10}$$

and

$$J_{k|m}y^m=-F^2\Big\{\mathbf{K}_{.k}+\mathbf{K}I_k\Big\}. \tag{3.11}$$

Contracting (3.8) with y_i implies that

$$L_{jkl}+\frac{1}{2}\mu FC_{jkl}=0. \tag{3.12}$$

Taking a trace of (3.12) implies that

$$J_i=-\frac{1}{2}\mu FI_i. \tag{3.13}$$

By taking a horizontal derivation of (3.13) along the Finslerian geodesics yields

$$J_{i|s}y^s=-\frac{F}{4}\Big(2\mu_{x^k}y^k-\mu^2F\Big)I_i. \tag{3.14}$$

By (3.11), (3.14) and $I_k=\tau_{.k}$, we get

$$\mathbf{K}_{y^i}+\frac{1}{4}\Big(4\mathbf{K}+\mu^2(x)-\frac{2}{F}\mu_{x^k}y^k\Big)\tau_{y^i}=0. \tag{3.15}$$

Now, suppose that $\mathbf{K}=\mathbf{K}(x)$ is a scalar function on M. Then (3.15) simplifies to

$$\Big(4\mathbf{K}+\mu^2-\frac{2}{F}\mu_0\Big)\tau_{y^i}=0. \tag{3.16}$$

where $\mu_0:=\mu_{x^k}y^k$.

Now, we claim that $\mu(x)=c$ is a constant. If this is false, then there is an open subset $\mathcal{U}$ such that $d\mu(x)\neq 0$ for any $x\in\mathcal{U}$. Clearly, at any $x\in\mathcal{U}$,

$$\mathbf{K}(x)\neq\frac{1}{4}\Big(-\mu(x)^2+\frac{2\mu_0(x)}{F(x,y)}\Big)$$

for almost all $y\in T_xM$. By (3.16), $\tau_{.i}=I_i=0$. Thus F is Riemannian on $\mathcal{U}$ by Deicke's theorem. This contradicts with the assumption. Then $\mu=constant$. $\square$

Proposition 3.2. *Let (M, F) be a Finsler surface. Suppose that F has isotropic main scalar and isotropic flag curvature. Then for any geodesic $\gamma = \gamma(t)$ and any parallel vector field $X = X(t)$ along γ, the following function*

$$\mathbf{C}(t) = \mathbf{C}_{\dot{\gamma}}\big(X(t), X(t), X(t)\big), \tag{3.17}$$

satisfies the following equation

$$\mathbf{C}(t) = \exp\Big(-\frac{1}{2}\mu t\Big)\mathbf{C}(0). \tag{3.18}$$

Proof. By definition, we have

$$\mathbf{L}_y(u, v, w) + \frac{1}{2}\mu F\mathbf{C}_y(u, v, w) = 0. \tag{3.19}$$

where $\mu = constant$. Let us define

$$\mathbf{L}(t) = \mathbf{L}_{\dot{\gamma}}\big(X(t), X(t), X(t)\big). \tag{3.20}$$

From the definition of $\mathbf{L}_y$, we have

$$\mathbf{L}(t) = \mathbf{C}^{'}(t). \tag{3.21}$$

Then, (3.19) can be written as follows

$$\mathbf{C}^{'}(t) = -\frac{1}{2}\mu\mathbf{C}(t). \tag{3.22}$$

Integration (3.22) gives (3.18). □

Proof of Theorem 1.1: The proof has two main cases as follows:

Case 1: If $\mu = 0$, then F is a Landsberg metric. In [18], we proved that every homogeneous Landsberg surface is Riemannian or locally Mikowskian.

Case 2: If $\mu \neq 0$. In this case, we have (3.18). In [17], it is proved that every homogeneous Finsler manifold is complete. By definition, every two points of a homogeneous Finsler manifold (M, F) map to each other under an isometry. This causes the norm of an invariant tensor under the isometries of a homogeneous Finsler manifold is a constant function on M, and consequently, it has a bounded norm. Using this fact, we showed that for a homogeneous Finsler manifold (M, F), every invariant tensor under the isometries of F has a bounded norm with respect to it [16]. Then letting $t \to -\infty$ in (3.18) and using $||\mathbf{C}|| < \infty$ implies that $\mathbf{C}(0) = 0$ and then $\mathbf{C}(t) = 0$. Then F reduces to a Riemannian metric. □

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