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ON INVARIANT CONTINUITY AND INVARIANT COMPACTNESS IN BANACH SPACES

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Abstract. In this study, we have defined the concepts of invariant continuity, invariant compactness, invariant boundedness and invariant Cauchy sequence in normed linear spaces. In general, there is no relation between continuity and invariant continuity. We have proved that if f is a linear map, then continuity of f implies invariant continuity of f. Additionally, we have shown that continuity of f and invariant continuity of f is equal under a condition. Also, we have proved that every invariant convergent sequence is invariant Cauchy. Finally, we have proved that invariant continuous image of an invariant compact space is invariant compact.
Keywords: Invariant convergence, strongly invariant convergence, invariant continuity, invariant compactness, invariant Cauchy sequence.

1. Introduction

Let l_∞ denote the Banach space of all real bounded sequences with the usual norm $\|x\| = sup_k|x_k|$. Banach [1], recognized certain nonnegative linear functionals defined on l_∞ which remain invariant under shift operators. This extended functionals is known as the Banach limits. In 1948, Lorentz [5] defined a new type of convergence known as the almost convergence. Later, Kurtz [4] introduced the concept of almost convergent sequence in a normed linear space X as follows:

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A sequence (x_k) in a normed linear space X is said to be *almost convergent* to $x \in X$ if

$$\lim_{n\to\infty} \left\| \frac{\sum_{i=1}^{n-1} x_{i+m}}{n} - x \right\| = 0, \text{uniformly in m.}$$

Raimi [12], defined the concept of invariant convergence (σ- convergence) which is generalization of almost convergence.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on l_∞ is said to be an *invariant mean or a σ-mean*, if and only if,

1. $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n (non-negative)
2. $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ (normal)
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in l_\infty$.

The mappings σ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$ denotes the mth iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

A sequence (x_k) is said to be *invariant convergent* to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} = L$$

uniformly in m [6]. In this case we write $(x_k) \longrightarrow L(V_\sigma)$ and L is called the σ-limit of (x_k).

Strongly invariant convergent sequence was defined by Mursaleen [7] as follows:

A sequence (x_k) is said to be *strongly invariant convergent* to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} |x_{\sigma^j(m)} - L| = 0$$

uniformly in m. In this case we write $(x_k) \longrightarrow L[V_\sigma]$.

It is known that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$ (see [7]).

Several authors including Dündar et al.[3], Mursaleen [6], MA Mursaleen [8], Mursaleen and Edely [9], Pancaroğlu and Nuray [11], Raimi [12], Savaş and Nuray [13], Schaefer [14], Ulusu and Nuray [15] and others have studied invariant convergent sequences.

Continuity and compactness are related to convergence. In [10], Nanda, by using the definition of almost convergence sequence, defined the concepts of almost continuity function and almost compactness in any normed linear space.

In this study, we will introduce the concepts of invariant continuous function and invariant compactness in any normed linear space. Then we will give some relations between continuity and invariant continuity. We will also prove that the invariant continuous image of an invariant compact space is invariant compact.

2. Main Results

Now, we will define the concepts of invariant convergence and invariant continuity in any normed linear space.

Definition 2.1. Let X be a normed linear space. A sequence $(x_n) \in X$ is said to be invariant convergent to $x \in X$ if

$$\lim_{n\to\infty} \left\| \frac{\sum_{j=1}^{n} x_{\sigma^j(m)}}{n} - x \right\| = 0$$

uniformly in m.

$V_\sigma(X)$ will denote the set of all invariant convergent sequences in X, that is:

$$V_\sigma(X) = \left\{ (x_k) : \lim_{n\to\infty} \left\| \frac{\sum_{j=1}^{n} x_{\sigma^j(m)}}{n} - x \right\| = 0, \text{uniformly in m} \right\}.$$

Definition 2.2. Let X and Y be normed linear spaces and $f : X \longrightarrow Y$ be a mapping. f is said to be invariant continuous at a point $x \in X$ if

$$x_k \longrightarrow x(V_\sigma(X)) \quad \text{implies} \quad f(x_k) \longrightarrow f(x)(V_\sigma(X))$$

Remark 2.1. It is easy to prove that if f and g are invariant continuous then so is $f+g$. Also if k is real number and f is invariant continuous functions, then kf is invariant continuous. Thus, the set of all invariant continuous functions is a vector space.

We can introduce four types of continuity:

$$\lim_{n\to\infty} x_n = x \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) = f(x) \tag{2.1}$$

$$\lim_{n\to\infty} x_n = x \quad \text{implies} \quad \sigma - \lim_{n\to\infty} f(x_n) = f(x) \tag{2.2}$$

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(2.1) is continuity and (2.4) is invariant continuity. We have

$$(2.3) \Rightarrow (2.1) \Rightarrow (2.2)$$

$$(2.3) \Rightarrow (2.4) \Rightarrow (2.2)$$

In general, there is no relation between continuity and invariant continuity. In [2], the following Lemma 2.1 and Theorem 2.1 were proved for almost continuity. We will prove similar lemmas for invariant continuity.

Lemma 2.1. *Let X and Y be normed linear spaces. If $f : X \longrightarrow Y$ is invariant continuous at $x_0 \in X$, then it is continuous at x_0.*

Proof. Firstly, we prove that the function f is bounded at x_0, i.e., there exists an $a > 0$ such that f is bounded on the interval $(x_0 - a, x_0 + a)$. To prove this it suffices to show that if $(x_n) \longrightarrow x_0$, then the sequence $f(x_n)_{n=1}^{\infty}$is bounded.

Let $(x_n) \longrightarrow x_0$. Then $(x_n) \longrightarrow x_0(V_\sigma(X))$ and by the assumption of Lemma, we have $f(x_n) \longrightarrow f(x_0)(V_\sigma(X))$. Hence $(f(x_n))_{n=1}^{\infty}$ as an invariant convergent sequence is bounded. Now, we can prove the continuity of the function f at the point x_0. Suppose that f is discontinuous at x_0. Since it is bounded on an interval $(x_0 - a, x_0 + a)$, there exists a sequence (y_n) of elements of $(x_0 - a, x_0 + a)$ such that $(y_n) \longrightarrow x_0$ and $f(y_n) \longrightarrow b \neq f(x_0)$. From this we get $f(y_n) \longrightarrow bV_\sigma(X))$. On the other hand from $(y_n) \longrightarrow x_0$ we have $(y_n) \longrightarrow x_0(V_\sigma(X))$ and so by the assumption of Lemma we get

$$f(y_n) \longrightarrow f(x_0) \neq b(V_\sigma(X)).$$

This contradicts $f(y_n) \longrightarrow b(V_\sigma(X))$. Hence, f is continuous and the proof is completed. □

Theorem 2.1. *Let X and Y be normed linear spaces. If $f : X \longrightarrow Y$ is invariant continuous at a point $x_0 \in X$, then f is a linear function.*

Proof. We will prove the theorem in two stages that special and general. First of all we will prove the following special case. Let $g : X \longrightarrow Y$ is invariant continuous at the point 0 and $g(0) = 0$. Let a, b, c be real numbers such that $a + b + c = 0$. Construct the sequence

$$(x_n)_{n=1}^{\infty} = a, b, c, a, b, c, ...$$

We show that this sequence is invariant convergent to 0. Let $\sigma(m) = m + 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} = 0$$

uniformly in m. Hence, (x_n) is invariant convergent to 0. According to the assumption of theorem $g(x_n) \longrightarrow g(0) = 0(V_\sigma(X))$, i.e.

$$g(x_n) = g(a), g(b), g(c), g(a), g(b), g(c), ...$$

is invariant convergent to 0. Additonally, a direct calculation shows that

$$\sigma - \lim g(x_n) = \frac{g(a) + g(b) + g(c)}{3}$$

Hence $g(a) + g(b) + g(c) = 0$. Since $c = -a - b$, we get $g(-a - b) = -g(a) - g(b)$. Putting $b = 0$ we have $g(-a) = -g(a)$.

Let x, y be arbitrary. Put $c = x + y$ $a = -x$ $b = -y$ then we get $g(x + y) = -g(-x) - g(-y) = g(x) + g(y)$. Hence the function g satisfies the Cauchy functional equation. It is continuous at 0 that Lemma 2.1. On the basis of well-known knowledge on Cauchy equation we get $g(x) = ax$ for $x \in X$, a being a constant.

Now we will prove the general case. Let $f : X \longrightarrow Y$ is invariant continuous at $x_0 \in X$. We introduce new coordinates $x' = x - x_0$, $y' = y - f(x_0)$. Put $g(x') = f(x) - f(x_0)$. Since g has the form $g(x') = ax'$, $f(x) - f(x_0) = a(x - x_0) = ax - ax_0$, $f(x) = ax + (f(x_0) - ax_0) = ax + b$ and f is linear. $\square$

Remark 2.2. It follows from Theorem 2.1 that Lemma 2.1 cannot be conversed.

Theorem 2.2. *If f is a linear map, then continuity of f implies invariant continuity of f.*

Proof. For linear maps continuity implies invariant continuity. Let $(x_k) \longrightarrow x_0$ and f is continuous. So,

$$(x_k) \longrightarrow x_0 \quad \text{implies} \quad f(x_k) \longrightarrow f(x_0).$$

Since f is continuous and linear, f is bounded thus for $M > 0$ we can write

$$\|f(x_n) - f(x_0)\| = \|f(x_n - x_0)\| \leq M \|x_n - x_0\| .$$

Hence, we have

$$\left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) - f(x_0) \right\| = \left\| f(\frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} - x_0) \right\| \leq M \left\| \frac{1}{n} \sum_{j=1}^{n} x_{\sigma^j(m)} - x_0 \right\|$$

for each m. Thus f is invariant continuous and so continuity implies invariant continuity for linear maps.

But, the situation is changing for nonlinear map. For this, let us take the example of Teorem 1 in [10]. Let us consider the nonlinear map $f : L^2[0,1] \longrightarrow [0,1]$ defined by

$$[fx](s) = \int_0^1 x^2(t)\, \mathrm{d}t$$

Let (x_k) be a sequence which converges to x in $L^2[0,1]$. We have

$$\begin{aligned}
\|f(x_k) - f(x)\|^2 &= \int_0^1 \left(\int_0^1 (x_k^2(t) - x^2(t))\mathrm{d}t \right)^2 \mathrm{d}s \\
&\leq \int_0^1 \left(\int_0^1 (x_k(t) + x(t))^2 \mathrm{d}t \right) \left(\int_0^1 (x_k(t) - x(t))^2 \mathrm{d}t \right) \mathrm{d}s \\
&\leq N \|x_k - x\|^2
\end{aligned}$$

where $N = \sup_k \|x_k + x\|^2$. The continiuty of f follows from the above inequality.

Observe that if $x_k = sink\pi t$ and $\sigma(m) = m+1$ then,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} x_{\sigma^j(m)} = \lim_{n\to\infty} \frac{sin(m+1)\pi t + ... + sin(m+n)\pi t}{n} = 0$$

uniformly in m. So $(x_k) \longrightarrow 0(V_\sigma)$. But

$$\|f(x_k) - f(0)\| = \int_0^1 (sink\pi t)^2 \mathrm{dt} = \int_0^1 (\frac{1}{2} - \frac{cos2k\pi t}{2})\mathrm{dt} = \frac{1}{2}$$

for all k and so $f(x_k) \nrightarrow 0$. Thus f is not invariant continuous and this completes the proof. $\square$

Theorem 2.3. *Let X and Y be normed linear spaces and $f : X \longrightarrow Y$. Continuity of f and invariant continuity of f are equivalent under the condition that*

$$\left\| \frac{f(x_{\sigma(m)}) + f(x_{\sigma^2(m)}) + ... + f(x_{\sigma^j(m)})}{n} - f(x_k) \right\| \to 0 \quad (n \longrightarrow \infty)$$

uniformly in m.

Proof. Let f be a continuous function as well as the condition holds. Let $(x_k) \to x$. Then

$$\|f(x_k) - f(x)\| \to 0$$

and

$$\left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x_k) \right\| \to 0 \quad (n \longrightarrow \infty)$$

unifromly in m. We can write,

$$\left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x) \right\| \le \left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x_k)\| + \|f(x_k) - f(x) \right\|$$

and

$$\left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x) \right\| \to 0 \quad (n \longrightarrow \infty)$$

unifromly in m. Hence f is invariant continuous.

Let $(x_k) \to x$ and f be invariant continuous at $x \in X$. Let the condition hold.Then

$$\left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x) \right\| \to 0 \quad (n \longrightarrow \infty)$$

unifromly in m. We have,

$$\|f(x_k) - f(x)\| \le \left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x_k) \right\| + \left\| \frac{\sum_{j=1}^n f(x_{\sigma^j(m)})}{n} - f(x) \right\|$$

and hence f is continuous. $\square$

Definition 2.3. Let X and Y be normed linear spaces. A function $f : X \longrightarrow Y$ is said to be invariant bounded if there is a constant $M \geq 0$ such that for all n and m,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\| \leq M$$

Theorem 2.4. *Boundednes and invariant boundedness of functions are equivalent.*

Proof. Observe that

$$\|f(x_k)\| = \sup_{1,m} \left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\| \leq \sup_{n,m} \left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\|.$$

Also,

$$\sup_{n,m} \left\| \frac{1}{n} \sum_{j=1}^{n} f(x_{\sigma^j(m)}) \right\| \leq \sup_{n} \frac{\sup_k \|f(x_k)\|}{n} \sum_{j=1}^{n} 1 = \sup_k \|f(x_k)\|$$

The result follows from the above two inequalities. □

Definition 2.4. A sequence (x_k) in X is said to be invariant Cauchy sequence if

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} \right\| \longrightarrow 0, \quad n, p \longrightarrow \infty$$

uniformly in m.

Theorem 2.5. *Let (x_k) be invariant Cauchy sequence. If*

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x_n \right\| \longrightarrow 0$$

uniformly in m, then it is Cauhy and vice-versa.

Proof. Let (x_k) be invariant Cauchy sequence and condition hold. We have

$$\|x_k - x_n\| \leq \left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x_n \right\|$$

$$+ \left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{k+1} \sum_{j=0}^{k} x_{\sigma^j(m)} \right\| + \left\| \frac{1}{k+1} \sum_{j=0}^{k} x_{\sigma^j(m)} - x_k \right\|$$

so (x_k) is Cauchy sequence.
Conversely, let (x_k) be Cauchy sequence and condition hold. Then

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} \right\| \leq$$

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x_n \right\| + \|x_n - x_p\| + \left\| \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} - x_p \right\|$$

so (x_k) is invariant Cauchy sequence. The proof is completed. □

Theorem 2.6. *Every invariant convergent sequence is invariant Cauchy sequence.*

Proof. Let the sequence (x_k) be invariant convergent to x. Then we can write

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} \right\| \leq$$

$$\left\| \frac{1}{n+1} \sum_{j=0}^{n} x_{\sigma^j(m)} - x \right\| + \left\| \frac{1}{p+1} \sum_{j=0}^{p} x_{\sigma^j(m)} - x \right\|.$$

So (x_k) is invariant Cauchy sequence. □

Definition 2.5. A Banach space X is said to be invariant compact if every sequence in X has an invariant convergent subsequence.

Theorem 2.7. *Invariant continuous image of an invariant compact space is invariant compact.*

Proof. Let X and Y be normed linear spaces, K an invariant compact subspace of X and let $f : X \longrightarrow Y$ be invariant continuous. We have to show that $f(K) = \{f(x) : x \in K\}$ is also invariant compact.

Let $\{f(x_k)\}$, be a sequence in $f(K)$. Then (x_k) is a sequence in K. Since K is invariant compact, there is a subsequence (x_{k_n}) which is invariant convergent to $x \in X$. Observe that $\{f(x_{k_n})\}$ is a subsequence of $f(x_k)$. Since f is invariant continuous,

$$x_{k_n} \longrightarrow x(V_\sigma(X)) \quad \text{implies} \quad f(x_{k_n}) \longrightarrow f(x)(V_\sigma(X)).$$

Thus, $f(K)$ is invariant compact and the proof is completed. □

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