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SOME FIXED POINT THEOREMS FOR (α, β)-ADMISSIBLE Z-CONTRACTION MAPPING IN METRIC-LIKE SPACES

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Abstract. The purpose of this paper is to establish some fixed point results in the setting of metric-like space by defining an (α, β)-admissible z-contraction mapping imbedded in simulation function. Our results generalize and extend several well known results in the literature of fixed point theory. A suitable example is also established to verify the validity of the results obtained.
Keywords: z-contraction mapping, fixed point, (α, β)-admissible mapping.

1. Introduction

As generalization of the standard metrics spaces, metric-like spaces were considered by Amini-Harandi [3] and proved some fixed point theorems. There after several authors have proved fixed and common fixed point theorem in metric-like space, for example see [1, 7, 5, 6, 9, 8, 10, 11, 21]. In 2012, Samet et al. [24] introduced the concept of α-contraction and α-admissible mappings and proved various fixed point theorems in complete metric spaces. Afterward, many authors obtain generalization of the result [24]. (For instance see [15, 17, 18, 19, 22]).

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Recently, Chandok[12] have introduced the notion of (α, β)-admissible mappings and obtained some fixed point results. Some of authors (For instance [13, 14]) obtained fixed point results by using the notion of (α, β)-admissible mappings and certain contractive conditions. On the other hand, Khojasteh et al [16] introduced a new class of mappings called simulation functions. In [16], they proved several fixed point theorems and shows that many results in the literature are simple consequences of their obtained results. In sequel, Argoubi et al.[4] modified the above said definition and proved some fixed point theorems with nonlinear contractions. There are many fixed point results in the setting of simulation function. (For instance [1, 14, 15, 20, 23]).

In this paper, we consider simulation functions to show the existence of fixed points of (α, β)-admissible z-contraction mapping in metric-like spaces. Our work generalizes and extends some previous results in the literature. We modify and generalize the results of Alsamir et al.[1], A. Dewangan et al.[14] and S. H, Cho[13]. Furthermore, we also give an examples to illustrate the main results.

2. Preliminaries

Let us recall some notations and definitions that we will need in the sequel. Throughout this paper we assume the symbols $\mathbb{R}$ and $\mathbb{N}$ as a set of real numbers and a set of natural numbers respectively.

Definition 2.1. [3] Let X be a non empty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a metric-like space (or a dislocated metric) on X if for any $x, y, z \in X$, the following conditions hold:

(σ_1) $\sigma(x, y) = 0 \Rightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is called metric-like space.

Then a metric-like on X satisfies all conditions of a metric except that $\sigma(x, x)$ may be positive for $x \in X$. Following [3], we have the following topological concepts.

Each metric-like σ on X generates a topology τ_σ on X, whose base is the family of open σ-balls, then for all $x \in X$ and $\epsilon > 0$

$$B_\sigma(X, \epsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \epsilon\}.$$

Now, let (X, σ) be a metric-like space. A sequence $\{x_n\}$ in the metric-like space (X, σ) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$.

Let (X, σ) be a metric-like space, and let $T : X \to X$ be a continuous mapping. Then $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} T(x_n) = T(x)$. A sequence $\{x_n\}$ is Cauchy in

(X, σ), iff $\lim_{n,m\to\infty} \sigma(x_m, x_n)$ exists and is finite. Moreover, the metric-like space (X, σ) is called complete, iff for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n\to+\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

It is clear that every metric space and partial metric space is a metric-like space but the converse is not true.

Example 2.1. Let $X = \{0, 1\}$ and

$$\sigma(x, y) = \begin{cases} 2, & \text{if x = y = 0;} \\ 1, & \text{if otherwise.} \end{cases}$$

Then (X, σ) is a metric-like space. It is neither a partial metric space ($\sigma(0, 0) \nleq \sigma(0, 1)$) nor a metric-like space ($\sigma(0, 0) = 2 \neq 0$).

Remark 2.1. A subset A of a metric-like space (X, σ) is bounded if there is a point $b \in X$ and a positive constant k such that $\sigma(a, b) \leq k$ for all $a \in A$.

Remark 2.2. [3] Let $X = \{0, 1\}$ such that $\sigma(x, y) = 1$ for each $x, y \in X$ and let $x_n = 1$ for $n \in \mathbb{N}$. Then it is easy to see that $x_n \to 0$ and $x_n \to 1$ and so in metric-like space, the limit of convergence sequence is not necessarily unique.

The following lemma is known and useful for the rest of paper.

Lemma 2.1. *[3] Let (X, σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$, where $x \in X$ and $\sigma(x, y) = 0$. Then for all $y \in X$ we have $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$.*

Definition 2.2. [12] Let X be a non-empty set, $T : X \to X$ and $\alpha, \beta : X \times X \to \mathbb{R}^+$. We say that T is an (α, β)-admissible mapping if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ imply that $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ for all $x, y \in X$.

Khojasteh et al.[16] introduced a new class of mappings called simulation functions and proved several fixed point theorems and established that many results in the literature are simple consequences of their obtained results.

Definition 2.3. [16] A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function if ζ satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0, \infty)$, then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.

The following unique fixed point theorem is proved by Khojasteh et al.[16].

Theorem 2.1. *Let (X, d) be a metric space and $T : X \to X$ be a z-contraction with respect to a simulation function ζ, that is*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0$$

for all $x, y \in X$. Then T has a unique fixed point.

It is worth mentioning that the Banach contraction is an example of z-contraction by defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = \lambda s - t$ for all $s, t \in [0, \infty)$, where $\lambda \in [0, 1)$.

Argoubi et al.[4] modified Definition (2.3) as follows.

Definition 2.4. [4] A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ that satisfies the following conditions:

(i) $\zeta(t, s) < s - t$ for all $s, t > 0$;

(ii) If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0, \infty)$, then $\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0$.

It is clear that any simulation function in the sense of Khojasteh et al.[16](Definition (2.3)) is also a simulation function in the sense of Argoubi et al.[4] (Definition (2.4)). The converse is not true.

Example 2.2. [4] Define a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} 1, & \text{if} \quad (s, t) = (0, 0); \\ \lambda s - t, & \text{otherwise.} \end{cases}$$

where $\lambda \in (0, 1)$. Then ζ is a simulation function in the sense of Argoubi et al.[4].

Some other examples of simulation functions in the sense of Definition (2.3) (see [2, 16, 23])are as follows:

(i) $\zeta(t, s) = cs - t$ for all $t, s \in [0, \infty)$ where $c \in [0, 1)$.

(ii) $\zeta(t, s) = s - \phi(s) - t$ for all $t, s \in [0, \infty)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a lower semi continuous function such that $\phi(t) = 0$ iff $t = 0$.

3. Main Results

Now, we are ready to prove our first result with the following definitions.

Definition 3.1. [1] Let (X, σ) be a metric-like space. Given $T : X \to X$ and $\alpha, \beta : X \times X \to \mathbb{R}^+$. Such T is said an (α, β)-admissible z-contraction with respect to ζ if

$$\zeta(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty), \sigma(x,y)) \geq 0 \tag{3.1}$$

for all $x, y \in X$, where ζ is a simulation function in the sense of Definition (2.3).

Now, we prove our first fixed point result.

Theorem 3.1. *Let (X, σ) be a complete metric-like space and $T : X \to X$ be a (α, β)-admissible z-contraction mapping with respect to a ζ simulation function if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that*

$$\zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y))) \geq 0 \tag{3.2}$$

for all $x, y \in X$, where

$$m(x,y) = \max\left\{\sigma(x,y), \frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\right\}.$$

Assume that

1. *T is (α, β)-admissible;*
2. *there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*
3. *T is σ-continuous.*

Then T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$.

Proof. By condition (2) of this theorem there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n = x_{n+1}$ for some n, $x_n = x_{n+1} = Tx_n$. So x_n is fixed point of T and the proof is completed. From now on assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is an (α, β)-admissible mapping, we derive

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for} \quad \text{all} \quad n \geq 0. \tag{3.3}$$

Similarly,

$$\beta(x_n, x_{n+1}) \geq 1, \quad \text{for} \quad \text{all} \quad n \geq 0. \tag{3.4}$$

From (3.2)(3.3), and (3.4), we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(Tx_n, Tx_{n-1})), \psi(m(x_n, x_{n-1}))) \\ &= \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)), \psi(m(x_n, x_{n-1}))). \qquad (3.5) \end{aligned}$$

Since

$$\begin{aligned} m(x_n, x_{n-1}) &= \max\Big\{\sigma(x_n, x_{n-1}), \frac{[1+\sigma(x_n, Tx_n)]\sigma(x_{n-1}, Tx_{n-1})}{1+\sigma(x_n, x_{n-1})}\Big\} \\ &= \max\Big\{\sigma(x_n, x_{n-1}), \frac{[1+\sigma(x_n, x_{n+1})]\sigma(x_{n-1}, x_n)}{1+\sigma(x_n, x_{n-1})}\Big\} \\ &= \max\Big\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\Big\}. \qquad (3.6) \end{aligned}$$

If follows from (3.5) and (3.6) that

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)), \\ &\quad \psi(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\})) \\ &< \psi(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\}) \\ &\quad -\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)). \qquad (3.7) \end{aligned}$$

Consequently, we obtain that for all $n = 0, 1, 2, 3...$

$$\psi(\sigma(x_n, x_{n+1})) < \psi(\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\}).$$

If $\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\} = \sigma(x_n, x_{n+1})$ for some n, then

$$\psi(\sigma(x_n, x_{n+1})) < \psi(\sigma(x_n, x_{n+1})),$$

which is a contradiction.

Hence $\max\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\} = \sigma(x_n, x_{n-1})$, for all $n \geq 0$ and hence from (3.7) we get,

$$0 < \psi(\sigma(x_n, x_{n-1})) - \psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n))$$

or

$$\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)) < \psi(\sigma(x_n, x_{n-1})). \tag{3.8}$$

By using the property of ψ, we get

$$\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n) < \sigma(x_n, x_{n-1}) \tag{3.9}$$

for all $n \geq 0$. The sequence $\{\sigma(x_n, x_{n-1})\}$ is nondecreasing, so there exists $r \geq 0$ such that $\lim_{n\to\infty} \sigma(x_n, x_{n-1}) = r$. We prove that

$$\lim_{n\to\infty} \sigma(x_n, x_{n-1}) = 0. \tag{3.10}$$

Suppose that $r > 0$. By the properties of ψ, (3.5), (3.8) and (3.9) and the condition (ζ_3)

$$0 \leq \lim_{n\to\infty} \sup \zeta(\psi(\alpha(x_n, x_{n-1})\beta(x_n, x_{n-1})\sigma(x_{n+1}, x_n)), \psi(\sigma(x_n, x_{n-1}))) < 0,$$

which is a contradiction. Therefore $r = 0$. This implies that $\lim_{n\to\infty} \sigma(x_n, x_{n-1}) = 0$.

Now we will show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can assume subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that for every k,

$$\sigma(x_{n_k}, x_{m_k}) \geq \epsilon. \tag{3.11}$$

That is,

$$\sigma(x_{n_k}, x_{m_k-1}) < \epsilon. \tag{3.12}$$

By the triangular inequality and using (3.11) and (3.12), we get

$$\begin{aligned} \epsilon \leq \sigma(x_{n_k}, x_{m_k}) &\leq \sigma(x_{n_k}, x_{m_k-1}) + \sigma(x_{m_k-1}, x_{m_k}) \\ &< \epsilon + \sigma(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities and by using (3.10) and (3.11), we have

$$\lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.13}$$

Also, from the triangular inequality, we have

$$\sigma(x_{n_k}, x_{m_k}) \leq \sigma(x_{n_k}, x_{n_k+1}) + \sigma(x_{n_k+1}, x_{m_k}),$$

$$|\sigma(x_{n_k+1}, x_{m_k}) - \sigma(x_{n_k}, x_{m_k})| \leq \sigma(x_{n_k}, x_{n_k+1}).$$

On taking limit as $k \to \infty$ on both sides of above inequality and using (3.10) and (3.13), we get

$$\lim_{k\to\infty} \sigma(x_{n_k+1}, x_{m_k}) = \epsilon. \tag{3.14}$$

Similarly it is easy to show that

$$\lim_{k\to\infty} \sigma(x_{n_k+1}, x_{m_k+1}) = \lim_{k\to\infty} \sigma(x_{n_k}, x_{m_k+1}) = \epsilon. \tag{3.15}$$

Moreover, since T is an (α, β)-admissible mapping, we have

$$\alpha(x_{n_k}, x_{m_k}) \geq 1 \quad \text{and} \quad \beta(x_{n_k}, x_{m_k}) \geq 1. \tag{3.16}$$

We deduce that

$$\begin{aligned} m(x_{n_k}, x_{m_k}) &= \max\left\{\sigma(x_{n_k}, x_{m_k}), \frac{[1+\sigma(x_{n_k}, Tx_{n_k})]\sigma(x_{m_k}, Tx_{m_k})}{1+\sigma(x_{n_k}, x_{m_k})}\right\} \\ &= \max\left\{\sigma(x_{n_k}, x_{m_k}), \frac{[1+\sigma(x_{n_k}, x_{n_k+1})]\sigma(x_{m_k}, x_{m_k+1})}{1+\sigma(x_{n_k}, x_{m_k})}\right\}. \end{aligned}$$

Taking $k \to \infty$ and using (3.10), (3.13) and (3.14), we obtain

$$\lim_{k\to\infty} \psi(m(x_{n_k}, x_{m_k})) = \epsilon. \tag{3.17}$$

By the fact T is an (α, β)-admissible z-contraction with respect to ζ, together with (3.13), (3.16) and (ζ_3), we get

$$\begin{aligned} 0 \leq \lim_{k\to\infty} \sup \zeta(\psi(\alpha(x_{n_k}, x_{m_k})\beta(x_{n_k}, x_{m_k})\sigma(x_{n_k+1}, x_{m_k+1})), \\ \psi(m(x_{n_k}, x_{m_k}))) < 0, \end{aligned}$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Owing to the fact that (X, σ) is a complete metric-like space, there exists some $u \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0 \tag{3.18}$$

Moreover, the continuity of T implies that

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \lim_{n\to\infty} \sigma(Tx_n, Tu) = \sigma(Tu, Tu) = 0. \tag{3.19}$$

Using Lemma 2.1 and (3.19), we have

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \sigma(u, Tu). \tag{3.20}$$

Continuing (3.19) and (3.20), we deduce that $\sigma(Tu, u) = \sigma(Tu, Tu)$. That is $Tu = u$. To prove the uniqueness of the fixed point, suppose that there exists $w \in X$ such that $Tw = w$ and $w \neq u$. Then

$$0 \leq \zeta(\psi(\alpha(u, w)\beta(u, w)\sigma(Tu, Tw)), \psi(m(u, w))) \tag{3.21}$$

where

$$m(u, w) = \max\left\{\sigma(u, w), \frac{[1+\sigma(u, Tu)]\sigma(w, Tw)}{1+\sigma(u, w)}\right\}$$

$$m(u, w) = \sigma(u, w) \tag{3.22}$$

from (3.21), (3.22) and (ζ_2) we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(u, w)\beta(u, w)\sigma(Tu, Tw)), \psi(\sigma(u, w))) \\ &< \psi(\sigma(u, w)) - \psi(\alpha(u, w)\beta(u, w)\sigma(Tu, Tw)). \end{aligned} \tag{3.23}$$

By using the property of ψ, we have

$$0 < \sigma(u, w) - \alpha(u, w)\beta(u, w)\sigma(Tu, Tw) \leq 0.$$

Which is a contradiction, so $u = w$. □

Theorem (3.1) remains true if we drop the continuity hypothesis by the following property:

(H): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all n, then there exists a subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k+1}) \geq 1$ and $\beta(x_{n_k}, x_{n_k+1}) \geq 1$ for all $k \in \mathbb{N}$ and $\alpha(x, Tx) \geq 1$ and $\beta(x, Tx) \geq 1$.

Theorem 3.2. *Let (X, σ) be a complete metric-like space and let T be a self-mapping on X satisfying the following conditions:*

(1) T is (α, β)-admissible;

(2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;

(3) (H) holds;

(4) T is an (α, β)-admissible z-contraction mapping with respect to a ζ simulation function if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that

$$\zeta(\psi(\alpha(x, y)\beta(x, y)\sigma(Tx, Ty)), \psi(m(x, y))) \geq 0,$$

for all $x, y \in X$, where

$$m(x, y) = \max\Big\{\sigma(x, y), \frac{[1 + \sigma(x, Tx)]\sigma(y, Ty)}{1 + \sigma(x, y)}\Big\}.$$

Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

Proof. Following the proof of Theorem (3.1), we construct a sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$, which converges to some $u \in X$. From definition of (α, β)-admissible mapping and (H), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k+1}) \geq 1$ and $\beta(x_{n_k}, x_{n_k+1}) \geq 1$ for all $k \in \mathbb{N}$. Thus applying condition (3.2) for all k, we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(Tx_{n_k}, Tu)), \psi(m(x_{n_k}, u))) \\ &= \zeta(\psi(\alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu)), \psi(m(x_{n_k}, u))) \end{aligned}$$

$$< \psi(m(x_{n_k}, u)) - \psi(\alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu)). \tag{3.24}$$

By suing the property ψ, we have

$$0 < m(x_{n_k}, u) - \alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu). \tag{3.25}$$

Also from (3.22) and (3.25), we get

$$0 < \sigma(x_{n_k}, u) - \alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(x_{n_k+1}, Tu) \tag{3.26}$$

which is equivalent to

$$\sigma(x_{n_k+1}, Tu) = \sigma(Tx_{n_k}, Tu) \leq \alpha(x_{n_k}, u)\beta(x_{n_k}, u)\sigma(Tx_{n_k}, Tu)$$

$$\leq \sigma(x_{n_k}, u). \tag{3.27}$$

Letting $k \to \infty$ in the above, we have $\sigma(u, Tu) = 0$. Using similar arguments as above, we can show that u is a fixed point of T. The uniqueness of the fixed point of T is obtained by similar arguments as these given in the proof of Theorem (3.1)

Now, we apply Theorem (3.1) to obtain the following result which is known as Banach type. □

Corollary 3.1. *Let (x, σ) be a complete metric-like space and let T be a self-mapping on X satisfying the following conditions:*

(1) *T is (α, β)-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*

(3) *T is σ-continuous;*

(4) *$\psi(\alpha(x, y)\beta(x, y)\sigma(Tx, Ty)) \leq \lambda(\psi(m(x, y)))$, for all $x, y \in X$ and $\lambda \in [0, 1)$ and also $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) \leq t, \psi(0) = 0$.*

Then T has a unique fixed point $u \in X$ with $\sigma(u, u) = 0$.

Proof. Following the lines of Theorem (3.1), by taking as a σ-simulation function, $\zeta(t, s) = \lambda s - t$. □

Corollary 3.2. *Let (X,σ) be a complete metric-like space and T be a self-mapping on X satisfying the following conditions:*

(1) *T is (α,β)-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*

(3) *T is σ-continuous;*

(4) *there exists a lower semi continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma^{-1} = \{0\}$ such that*

$$\alpha(x,y)\beta(x,y)\sigma(Tx,Ty) \leq m(x,y) - \gamma(m(x,y))$$

for all $x, y \in X$. Then T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$.

Proof. Following the proof of Theorem (3.1), it sufficient to take $\zeta(t,s) = s - \gamma(s) - t$. □

If we consider in Theorem (3.1), $\alpha(x,y) = \beta(x,y) = 1$ for all $x, y \in X$, we have:

Corollary 3.3. *Let (X,σ) be a complete metric-like space and let T be a self-mapping on X. Suppose that there exists a σ-simulation function ζ such that*

$$\zeta(\psi(\sigma(Tx,Ty)), \psi(m(x,y))) \geq 0$$

for all $x, y \in X$. Then T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$.

We present the following illustrated example.

Example 3.1. Let $X = [0,\infty), \sigma(x,y) = x + y$ for all $x, y \in X$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{x}{4}, & \text{if } \ 0 \leq x \leq 1, \\ 4x, & \text{otherwise.} \end{cases}$$

consider $\zeta(t,s) = \lambda s - t$, where $0 \leq 1/4 < \lambda < 1$.

We define two mappings $\alpha, \beta : X \times X \to \mathbb{R}^+$ as

$$\alpha(x,y) = \begin{cases} \frac{5}{3}, & \text{if } \ 0 \leq x, y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta(x,y) = \begin{cases} \frac{3}{2}, & \text{if } \ 0 \leq x, y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\psi(t) = t$ for all $t \geq 0$. We shall prove that Corollary 3.1 can be applied. Clearly (X,σ) is a complete metric-like space. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$. Since $x, y \in [0,1]$ and so $Tx \in [0,1], Ty \in [0,1]$ and $\alpha(Tx,Ty) = 1$ and $\beta(Tx,Ty) = 1$. Hence T is (α,β)-admissible. Condition (2) is satisfied with $x_0 = 1$. Condition (3.2) is also satisfied with $x_n = T^n x_1 = 1/n$.

If $0 \leq x \leq 1$, then $\alpha(x,y) = 5/3$ and $\beta(x,y) = 3/2$.
Now

$$\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y)) = \alpha(x,y)\beta(x,y)\sigma(Tx,Ty), m(x,y)$$

where

$$\begin{aligned} m(x,y) &= \max\left\{x+y, \frac{[1+x+Tx](y+Ty)}{1+x+y}\right\} \\ &= \max\left\{x+y, \frac{[1+x+x/4](y+y/4)}{1+x+y}\right\} \\ &= \max\left\{x+y, \frac{[4+5x](5y)}{16(1+x+y)}\right\} = \{x+y\} \end{aligned}$$

$$\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y)) = \alpha(x,y)\beta(x,y)\sigma(Tx,Ty), x+y$$

$$\begin{aligned} \zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y))) &= \zeta(\alpha(x,y)\beta(x,y) \\ &\quad \sigma(Tx,Ty), x+y) \\ &= \lambda(x+y) - \\ &\quad \alpha(x,y)\beta(x,y)\sigma(Tx,Ty) \\ &= \frac{3}{4}(x+y) - \\ &\quad \left(\frac{5}{3}\right)\left(\frac{3}{2}\right)\left(\frac{x}{4}+\frac{y}{4}\right) \\ &= \frac{3}{4}(x+y) - \frac{5}{8}(x+y) \\ &= \left(\frac{3}{4}-\frac{5}{8}\right)(x+y) \\ &= \frac{1}{8}(x+y) \geq 0. \end{aligned}$$

If $0 \leq x \leq 1$ and $y > 1$, then $\zeta(\psi(\alpha(x,y)\beta(x,y)\sigma(Tx,Ty)), \psi(m(x,y))) \geq 0$. Since $\alpha(x,y) = \beta(x,y) = 0$. Consequently, all assumptions of Corollary 3.1 are satisfied and hence T has a unique fixed point which is $u = 0$

4. Conclusion

In this attempt, we studied (α,β)-admissible z-contraction mappings imbedded in simulation function and proved some fixed point theorems in metric-like spaces. Our results are generalized and extended forms of recent results in the literature. Finally, we have illustrated an example in support of our obtained results.

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