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POSITION VECTOR OF A W-CURVE IN THE 4D GALILEAN SPACE G_4

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Abstract. In this paper, the position vector of the W-curve in G_4 is given and by using the position vector we obtain some characterizations for the W-curve whose image lies on the Galilean sphere S_G^3 in G_4. Also, we characterize the unit curves with respect to the second curvature $\tau(s)$ and $\sigma(s)$.
Keywords: W-curves, 4D Galilean space G_4

1. Introduction

A Galilean space is one of the Cayley-Klein spaces and it has been largely developed by Röschel [22]. A Galilean space may be considered as the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. This limit transition corresponds to the limit transition from the special theory of relativity to the classical mechanics. On the other hand, Galilean space-time plays an important role in nonrelativistic physics. The fundamental concepts such as velocity, momentum, kinetic energy, etc. and principles such as laws of motion and conservation laws of classical physics are expressed in terms of Galilean space.

Differential geometry of the Galilean space G_3 has been largely developed in [22]. In recent years, researchers have begun to investigate curves and surfaces in the Galilean space and thereafter pseudo-Galilean space. Spherical curves in G_3 are given [18] and [3]. Bertrand curves in this space are given in [1]. It is safe to report that a good amount of research has also been done in pseudo-Galilean space by using the important paper by Divjak [5]; and thereafter classical differential geometry papers of Divjak and MilinŠipuš [6], Divjak and Milin-Šipuš [7] and Öğrenmiş and Ergüt [2]. The Frenet formulas of a curve in 4-dimensional Galilean space G_4 are given by [23]. Mannheim curves for 4-dimensional Galilean space G_4 are given in [17]. The equiform differential geometry of curves in G_4 are given in [19]. In [24], inextensible flows of curves in G_4 are investigated.

A curve α is called a W-curve (or a helix) if it has constant Frenet curvatures. W-curves in the Euclidean space E^n have been studied intensively. The simplest examples are circles as planar W-curves and helices as non-planar W-curves in E^3.

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All W-curves in the Minkowski 3-space are completely classified by Walrave in [12]. For example, the only planar spacelike W-curves are circles and hyperbolas. All spacelike W-curves in the Minkowski space-time E_1^4 are studied in [20]. The examples of null W-curves in the Minkowski space-time are given in [25]. Timelike W-curves in the same space have been studied in [13]. The position vectors of a spacelike W-curve (or a helix), i.e., a curve with constant curvatures, with spacelike, timelike and null principal normal in the Minkowski 3-space E_1^3 are given in [14]. The position vectors of a timelike and a null helix in Minkowski 3-space are studied in [15].

In this paper, we obtain position vector of a W-curve in 4-dimensional Galilean space G_4 and by using position vector we give some characterizations for W-curve whose image lies on the Galilean sphere S_G^3 in G_4. Also we characterize unit curves with respect to second curvature $\tau(s)$ and third curvature $\sigma(s)$.

2. Preliminaries

The Galilean space is a 3D complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ [11].

The study of mechanics of plane-parallel motions reduces to the study of a geometry of three dimensional space with coordinates $\{x, y, t\}$ is given by the motion formula Yaglom (1979).

$$\begin{array}{rcl} x' &=& (\cos\alpha)x + (\sin\alpha)y + (\nu\cos\beta)t + a \\ y' &=& -(\sin\alpha)x + (\cos\alpha)y + (\nu\sin\beta)t + b \\ t' &=& t + d \end{array} \tag{2.1}$$

This geometry can be called three-dimensional Galilean Geometry. Yaglom (1979) stressed that four-dimensional Galilean Geometry, which studies all properties invariant under motions of objects in space, is even more complex. Yaglom (1979) also stated this geometry could be described more precisely as the study of those properties of four-dimensional space with coordinates that are invariant under the general Galilean transformations.

$$\begin{array}{rcl} x' &=& (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)x + (\sin\beta\cos\alpha - \cos\gamma\cos\beta\sin\alpha)y \\ && +(\sin\gamma\sin\alpha)z + (v\cos\delta_1)t + a \\ y' &=& -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)x + (-\sin\beta\sin\alpha + \cos\gamma\cos\beta\cos\alpha)y \\ && +(\sin\gamma\cos\alpha)z + (v\cos\delta_2)t + b \\ z' &=& (\sin\gamma\sin\beta)x - (\sin\gamma\cos\beta)y + (\cos\gamma)z + (v\cos\delta_3)t + c \\ t' &=& t + d \end{array} \tag{2.2}$$

with $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$.

In affine coordinates, the inner product of two vectors $a = (a_1, a_2, a_3, a_4)$ and

$b = (b_1, b_2, b_3, b_4)$ is defined by

$$\langle a, b\rangle_G = \begin{cases} a_1 b_1 & ,\text{if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2 b_2 + a_3 b_3 + a_4 b_4 & ,\text{if } a_1 = 0 \text{ and } b_1 = 0. \end{cases} \tag{2.3}$$

For the vectors $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ the Galilean cross product in G_4 is defined as follows:

$$(a \wedge b \wedge c)_G = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \tag{2.4}$$

where e_i are the standard basis vectors.

A curve α is an arbitrary curve in a 4-dimensional Galilean space G_4 defined by

$$\alpha(t) = (x(t), y(t), z(t), w(t)) \tag{2.5}$$

where $x(t), y(t), z(t), w(t)$ are smooth functions. Let $\alpha : I \subset\rightarrow G_4$, $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arc length s in G_4.

For the curve α, the Frenet formulas are given in the following form

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{2.6}$$

where T, N_1, N_2, N_3 are mutually orthogonal vector fields satisfying equations (2.7)

$$\langle T, T\rangle_G = \langle N_1, N_1\rangle_G = \langle N_2, N_2\rangle_G = \langle N_3, N_3\rangle_G = 1$$
$$\langle T, N_1\rangle_G = \langle T, N_2\rangle_G = \langle T, N_3\rangle_G = \langle N_1, N_2\rangle_G = \langle N_1, N_3\rangle_G = \langle N_2, N_3\rangle_G = 0.$$

The Galilean sphere of the space G_4 is defined by

$$S_G^3(m, r) = \{\varphi - m \in G_4 : \langle \varphi - m, \varphi - m\rangle_G = \mp r^2\}. \tag{2.8}$$

See [23].

3. Position Vector of a W-curve in G_4

Let $\alpha = \alpha(s)$ be unit speed W-curve in G_4, with non-zero curvatures κ, τ and σ. Then the position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N_1(s) + \gamma(s)N_2(s) + \upsilon(s)N_3(s) \tag{3.1}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$, $\gamma(s)$ and $\upsilon(s)$. These functions are called component functions (or simply components) of the position vector.

Then differentiating (3.1) with respect to s and using the corresponding Frenet equations (2.6), we obtain

$$\begin{aligned} \lambda' - 1 &= 0, \\ \lambda\kappa + \mu' - \gamma\tau &= 0, \\ \mu\tau + \gamma' - \upsilon\sigma &= 0, \\ \gamma\sigma + \upsilon' &= 0. \end{aligned} \tag{3.2}$$

From the first third equations in (3.2) we get

$$\lambda(s) = s + c_1, \quad c_1 \in \mathbb{R}, \tag{3.3}$$

$$\gamma(s) = \frac{(s+c_1)\kappa + \mu'}{\tau} \tag{3.4}$$

and

$$\upsilon(s) = \frac{\mu'' + \mu\tau^2 + \kappa}{\sigma\tau}. \tag{3.5}$$

By using (3.4) and (3.5) in the last equation in (3.2) we easily obtain the differential equation

$$\mu''' + A\mu' + \sigma^2\kappa(s + c_1) = 0. \tag{3.6}$$

where $A = \sigma^2 + \tau^2$. The solution of the previous equation is

$$\mu(s) = c_2 + c_3\cos(\sqrt{A}s) + c_4\sin(\sqrt{A}s) - \frac{\sigma^2\kappa}{A}\left(\frac{s^2}{2} - c_1 s\right) \tag{3.7}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Then using (3.7) in (3.4) and (3.5) we get

(3.8)

$$\begin{aligned} \gamma(s) &= \tfrac{\kappa\tau}{A}(s+c_1) + \tfrac{\sqrt{A}}{\tau}\left(-c_3\sin\left(\sqrt{A}s\right) + c_4\cos\left(\sqrt{A}s\right)\right) \\ \upsilon(s) &= \tfrac{\tau}{\sigma}\left(c_2 + \tfrac{\kappa}{A}\right) - \tfrac{\tau\sigma\kappa}{A}\left(\tfrac{s^2}{2} - c_1 s\right) - \tfrac{\sigma}{\tau}\left(c_3\cos\left(\sqrt{A}s\right) + c_4\sin\left(\sqrt{A}s\right)\right) \end{aligned}$$

Thus we find the position vector as;

$$\begin{aligned} \alpha(s) = {} & (s+c_1)T(s) + \Big[c_2 + c_3\cos(\sqrt{A}s) + c_4\sin(\sqrt{A}s) \\ & -\frac{\sigma^2\kappa}{A}\left(\frac{s^2}{2} - sc_1\right)\Big] N_1(s) + \Big[\frac{\kappa\tau}{A}(s+c_1) \\ & +\frac{\sqrt{A}}{\tau}\left(-c_3\sin\left(\sqrt{A}s\right) + c_4\cos\left(\sqrt{A}s\right)\right)\Big] N_2(s) \\ & + \Big[\frac{\tau}{\sigma}\left(c_2 + \frac{\kappa}{A}\right) - \frac{\tau\sigma\kappa}{A}\left(\frac{s^2}{2} - c_1 s\right) \\ & -\frac{\sigma}{\tau}\left(c_3\cos\left(\sqrt{A}s\right) + c_4\sin\left(\sqrt{A}s\right)\right)\Big] N_3(s) \end{aligned} \tag{3.9}$$

Theorem 3.1. *Let $\alpha = \alpha(s)$ be a unit W-curve in G_4 with $\kappa(s) \neq 0$, $\tau(s) \neq 0$ and $\sigma(s) \neq 0$ for each $s \in \mathbb{R}$. Then position vector of the curve is given by the equation (3.9).*

Next, the following theorems characterize unit speed curves with respect to second curvature $\tau(s)$ and third curvature $\sigma(s)$.

Theorem 3.2. *Let $\alpha = \alpha(s)$ be a unit curve in G_4 with non-zero curvature κ. Then α has $\tau = 0$ if and only if α lies fully in a 2-dimensional hyperplane of G_4, spanned by $\{T, N_1\}$.*

Proof. If α has $\tau(s) = 0$ then by using the Frenet equations we obtain $\alpha' = T$, $\alpha'' = \kappa N_1$. Next, all higher order derivates of α are in the direction of the vector $\alpha^{''}$, so by using the MacLaurin expansion for α given by

$$\alpha(s) = \alpha(0) + \dot{\alpha}(0)s + \ddot{\alpha}(0)\frac{s^2}{2!} + ..., \tag{3.10}$$

we conclude that α lies fully in a hyperplane of G_4, spanned by $\{T, N_1\}$.

Conversely assume that α satisfies the assumptions of the theorem and lies fully in a hyperplane π of G_4. Then there exist points $p, q \in G_4$, such that α satisfies the equation of π given by $\langle \alpha(s) - p, q \rangle_G = 0$, where $q \in \pi^\perp$. Differentiating the last equation yields $\langle T, q \rangle_G = \langle N_1, q \rangle_G = 0$. Next, differentiation of the equation $\langle N_1, q \rangle_G = 0$ gives $\tau \langle N_2, q \rangle_G = 0$. Since N_2 is the unit vector perpendicular to $\{T, N_1\}$, it follows $\langle N_2, q \rangle_G \neq 0$. Therefore $\tau = 0$. $\square$

Theorem 3.3. *Let $\alpha = \alpha(s)$ be a unit curve in G_4 with non-zero curvatures κ and τ. Then α has $\sigma = 0$ if and only if α lies fully in a 3-dimensional hyperplane of G_4, spanned by $\{T, N_1, N_2\}$.*

Proof. If α has $\sigma(s) = 0$ then by using the Frenet equations we obtain $\alpha' = T$, $\alpha'' = \kappa N_1$, $\alpha''' = \kappa' N_1 + \kappa\tau N_2$. Next, all higher order derivates of α are combinations of vector $\alpha^{''}$ and $\alpha^{'''}$, so by using the MacLaurin expansion for α given by

$$\alpha(s) = \alpha(0) + \dot{\alpha}(0)s + \ddot{\alpha}(0)\frac{s^2}{2!} + ..., \tag{3.11}$$

we conclude that α lies fully in a hyperplane of G_4, spanned by $\{T, N_1, N_2\}$.

Conversely, assume that α satisfies the assumptions of the theorem and lies fully in a hyperplane π of G_4. Then there exist points $p, q \in G_4$, such that α satisfies the equation of π given by $\langle \alpha(s) - p, q \rangle_G = 0$, where $q \in \pi^\perp$. Differentiating the last equation yields $\langle T, q \rangle_G = \langle N_1, q \rangle_G = \langle N_2, q \rangle_G = 0$. Next, differentiation of the equation $\langle N_2, q \rangle_G = 0$ gives $\sigma \langle N_3, q \rangle_G = 0$. Since N_3 is the unit vector perpendicular to $\{T, N_1, N_2\}$, it $\langle N_3, q \rangle_G \neq 0$. Therefore $\sigma = 0$. $\square$

4. W-curves on Galilean sphere S^3_G in G_4

In this section we give some characterizations for W-curve whose image lies on a Galilean sphere S^3_G.

Theorem 4.1. *Let $\alpha = \alpha(s)$ be a unit W- curve in G_4 with non-zero curvatures $\kappa(s)$, $\tau(s)$ and $\sigma(s)$. Then α lies on Galilean sphere S^3_G for each $s \in I \subset \mathbb{R}$ if and only if*

(4.1)

$$s + c_1 = 0,$$

$$c_2 + c_3 \cos(\sqrt{A}s) + c_4 \sin(\sqrt{A}s) - \tfrac{\sigma^2\kappa}{A}\left(\tfrac{s^2}{2} - sc_1\right) = -\tfrac{1}{\kappa},$$

$$\tfrac{\kappa\tau}{A}(s + c_1) + \tfrac{\sqrt{A}}{\tau}\left(-c_3 \sin\left(\sqrt{A}s\right) + c_4 \cos\left(\sqrt{A}s\right)\right) = 0,$$

$$\tfrac{\tau}{\sigma}\left(c_2 + \tfrac{\kappa}{A}\right) - \tfrac{\tau\sigma\kappa}{A}\left(\tfrac{s^2}{2} - c_1 s\right) - \tfrac{\sigma}{\tau}\left(c_3 \cos\left(\sqrt{A}s\right) + c_4 \sin\left(\sqrt{A}s\right)\right) = -\tfrac{\tau}{\kappa\sigma}$$

where $A = \sigma^2 + \tau^2$ and $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Proof. Let us first suppose that α lies on a Galilean sphere S^3_G with radius r

$$\langle \alpha, \alpha \rangle_G = \mp r^2 \tag{4.2}$$

for every $s \in I \subset \mathbb{R}$. Differentiation in s gives

$$\langle T, \alpha \rangle_G = 0. \tag{4.3}$$

By a new differentiation, we find that

$$\langle N_1, \alpha \rangle_G = -\frac{1}{\kappa} \tag{4.4}$$

Then one more differentiation in s gives

$$\langle N_2, \alpha \rangle_G = 0 \tag{4.5}$$

and

$$\langle N_3, \alpha \rangle_G = -\frac{\tau}{\kappa\sigma}. \tag{4.6}$$

By using Eqs. (4.3), (4.4), (4.5) and (4.6) in Eq. (3.9), we find equations in (4.1). Conversely, we assume that equations in (4.1) hold for each $s \in I \subset \mathbb{R}$ then from Eq. (3.9) we find the position vector of the curve $\alpha - m = -\frac{1}{\kappa}N_1 - \frac{\tau}{\kappa\sigma}N_3$ which satisfies the equation $\langle \alpha, \alpha \rangle_G = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\tau}{\kappa\sigma}\right)^2 = r^2$ which means that the curve lies in S^3_G. $\square$

Corollary 4.1. *Let $\alpha(s)$ be a unit W-curve in G_4 with $\kappa(s) \neq 0$, $\tau(s) \neq 0$ and $\sigma(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. If α is a Galilean spherical curve then the radius of S^3_G is $r = \sqrt{\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\tau}{\kappa\sigma}\right)^2}$.*

Corollary 4.2. *Let $\alpha(s)$ be a unit W-curve in G_4 with $\kappa(s) \neq 0$, $\tau(s) \neq 0$ and $\sigma(s) \neq 0$ for each $s \in I \subset \mathbb{R}$. If α is a Galilean spherical curve then α lies fully in a 2-dimensional hyperplane of G_4, spanned by $\{N_1, N_3\}$.*

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