∗-CONFORMAL CURVATURE OF CONTACT METRIC MANIFOLDS

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Abstract. We introduce the ∗-conformal curvature tensor and $^*\eta$-Einstien manifolds in contact manifolds. We investigate this tensor in the three main classes of contact manifolds: Sasakian manifolds, Kenmotsu manifolds, and cosymplectic manifolds. We prove that a manifold is η-Einstienian if and only if be $^*\eta$-Einstienian manifold.

Keywords: ∗-conformal curvature, $^*\eta$-Einstien manifolds, Sasakian manifolds, Kenmotsu manifolds, Cosymplectic manifolds.

1. Introduction

There are many similar concepts in complex geometry and contact geometry. Tachibana introduces ∗-Ricci tensor within the framework of an almost Hermitian manifold in their work [23]. Afterward, Hamada introduces the ∗-Ricci tensor for the real hypersurfaces embedded in a non-flat complex space form [16]. This notion on an almost contact metric manifold $(M, g, \eta, \xi, \varphi)$ is defined as

$$(1.1) \qquad {}^*Ric(X_1, X_2) = \frac{1}{2} \, trace\{\mathbf{X_3} \ \rightarrow K(X_1, \varphi X_2)\varphi \mathbf{X_3}\},$$

for any vector field X_1, X_2. The ∗-Ricci operator $^*L$ is characterized by the relation $g(^*L X_1, X_2) = {}^*Ric(X_1, X_2)$. With the help of the ∗-Ricci tensor, several authors have investigated ∗-Ricci soliton in contact geometry (see [14], [10], [25], [2]). In

general, the equality $^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$ does not always hold.

In a Riemannian manifold (M^{2n+1}, g), the conformal curvature tensor C is expressed as

$$
\begin{aligned}
C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &- \quad \frac{1}{2n-1}\Big(Ric(X_2, X_3)X_1 - Ric(X_1, X_3)X_2 \\
&+ \quad g(X_2, X_3)\, LX_1 - g(X_1, X_3)\, LX_2\Big) \\
&+ \quad \frac{r}{2n(2n-1)}\Big(g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big),
\end{aligned}
$$

(1.2)

where K represents the curvature tensor of (1,3) type, Ric indicates the Ricci tensor, r is the scalar curvature and L is the Ricci operator of (M, g).

The paper is organized as follows: In Section 2, we express some preliminary definitions, then we proceed to investigate $*$-conformal curvature tensor of the contact manifolds. We examine some features of $*$-conformal curvature tensor.

In Section 3, we considered the Sasakian structure. Then, having the $*$-Ricci, we determined the relationship between η-Einstien and $^*\eta$-Einstien manifold.

Theorem 1.1. *Let M^{2n+1} be a manifold with a Sasakian structure (g, η, ξ, φ). The manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is a $^*\eta$-Einstien manifold.*

Then, we investigate the $*$-conformal curvature tensor of the Sasakian manifolds. In addition, we show that ξ-conformally flat and ξ-$*$conformally flat will not co-occur in Sasakian manifolds. By the condition $^*Ric(X_1, X_2)$ and $^*r$ for a $2n+1$-dimensional Sasakian manifold, we get the following $(0, 2)$-tensor

$$
^*T(X_1, X_2) = -\frac{^*Ric(X_1, X_2)}{2n-1} + \frac{^*r\, g(X_1, X_2)}{4n(2n-1)}.
$$

We conclude that if $n > 1$, then $*$-conformal curvature tensor ans and $^*D(X_1, X_2)X_3$ do not vanish simultaneously.

In Section 4, we find some conditions for a Kenmotsu 3-manifold to have vanishing $*$-conformal curvature tensor. We show that for a special case, the $*$-conformal tensor of this manifold becomes zero as in the following Theorem.

Theorem 1.2. *If a Kenmotsu 3-manifold is of quasi-constant curvature of the form*

$$
\begin{aligned}
K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) \quad &- \quad \alpha\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\
&+ \quad g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big],
\end{aligned}
$$

where $\alpha = \frac{r}{2} + 2$, then $*$-conformal curvature tensor vanishes.

But in general, we show that on Kenmotsu manifolds, the ∗-conformal tensor cannot vanish identically. Similarly, the equivalence of η-Einstien and $^*\eta$-Einstien is also established in Kenmotsu manifolds. The same result about ∗-conformal curvature tensor and $^*D(X_1, X_2)X_3$ on the Sasakian manifold is obtained for the Kenmotsu manifold.

In the last section, we prove the ∗-conformal curvature tensor is identically zero on the 3-dimensional cosymplectic manifolds. We confirm a conformally flat cosymplectic manifold is an $^*\eta$-Einstien manifold. We prove the following theorem:

Theorem 1.3. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$. If M is a ∗-conformally flat manifold, then $^*D = 0$.*

2. Preliminaries

Definition 2.1. Consider a contact metric manifold $(M, g, \eta, \xi, \varphi)$ of dimension $2n + 1$. The ∗-conformal curvature tensor for $(M, g, \eta, \xi, \varphi)$ is expressed as

$$
\begin{aligned}
^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &- \quad \frac{1}{2n-1}\Big(^*Ric(X_2, X_3)X_1 - {}^*Ric(X_1, X_3)X_2 \\
&+ \quad g(X_2, X_3)\,^*LX_1 - g(X_1, X_3)\,^*LX_2 \Big) \\
&+ \quad \frac{^*r}{2n(2n-1)}\Big(g(X_2, X_3)X_1 - g(X_1, X_3)X_2 \Big),
\end{aligned}
$$

(2.1)

where $^*r$ represents the ∗-scalar curvature, which is the trace of the ∗-Ricci tensor.

Definition 2.2. A contact metric manifold is named $^*\eta$-Einstien if

$$(2.2) \qquad ^*Ric(X_1, X_2) = c\, g(X_1, X_2) + d\,\eta(X_1)\eta(X_2), \qquad c, d \in C^\infty(M).$$

A differentiable manifold M^{2n+1} has an almost contact structure [2] if it admits a 1-form η, a characteristic vector field ξ, and a $(1,1)$-tensor field φ, which satisfy

$$(2.3) \qquad \varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

where I indicates the identity endomorphism. Then, by (2.3), can see that

$$(2.4) \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0.$$

If an almost contact manifold M^{2n+1} admits a Riemannian metric g with the property:
$$(2.5) \qquad g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \qquad \forall X_1, X_2 \in \chi(M),$$

then $(M^{2n+1}, g, \eta, \xi, \varphi)$ is called an almost contact metric manifold. The 2-form $\Phi(X_1, X_2) = g(X_1, \varphi X_2)$ is called the fundamental 2-form on the almost contact

metric manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$. An almost contact metric manifold is called normal if the (1,2)-type torsion tensor N_φ vanishes, where $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ is the Nijenhuis tensor of φ. A normal almost contact metric manifold is called a Sasakian manifold. A Sasakian manifold is also characterized by

$$(\nabla_{X_1}\varphi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \qquad \forall X_1, X_2 \in \chi(M).$$

On a Sasakian manifold beside (2.3)-(2.5), we also have

(2.6) $\nabla_{X_1}\xi = -\varphi X_1, \qquad K(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2,$

where K denotes the curvature tensor of (1,3) type. The importance and application of Sasakian structures are in holomorphic statistical structures and are also related to string theory (see [1]).

If the 1-form η is closed and $d\Phi = 2\eta \wedge \Phi$, then the almost contact metric manifold is called almost Kenmotsu manifold. A normal almost Kenmutsu manifold is a Kenmutsu manifold, which is equivalent to:

$$(\nabla_{X_1}\varphi)X_2 = g(\varphi X_1, X_2)\xi - \eta(X_2)\varphi X_1, \qquad \forall X_1, X_2 \in \chi(M).$$

It is known that every Kenmotsu manifold is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a Kahler manifold, I is an open interval with coordinate t, and the warping function f defined by $f = ce^t$ for some positive constant c [19]. For a $(2n+1)$-dimensional Kenmotsu manifold, we have

(2.7) $\nabla_{X_1}\xi \ = \ X_1 - \eta(X_1)\xi,$
(2.8) $K(X_1, X_2)\xi \ = \ \eta(X_1)X_2 - \eta(X_2)X_1,$
(2.9) $Ric(X_1, \xi) \ = \ -2n\,\eta(X_1),$
(2.10) $K(\xi, X_1)X_2 \ = \ \eta(X_2)X_1 - g(X_1, X_2)\xi,$
(2.11) $Ric(\phi X_1, \phi X_2) \ = \ Ric(X_1, X_2) + 2n\,\eta(X_1)\eta(X_2).$

An almost contact metric manifold is termed an almost cosymplectic manifold when both the 1-form η and 2-form Φ are closed. A normal almost cosymplectic manifold is called a cosymplectic manifold [3], [15]. Every cosymplectic manifold satisfies the following:

(2.12) $\nabla_{X_1}\xi = 0, \quad K(X_1, X_2)\xi = 0, \quad Ric(X_1, \xi) = 0.$

The cosymplectic structure is a tool for time-dependent Hamiltonian mechanics. It has some applications in string theory, which shows the importance of cosymplectic manifolds.

Suppose that $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an almost contact metric manifold and $^*C$ is its $*$-conformal curvature tensor, which is defined by (2.1). A direct computation shows some symmetries of $^*C$.

Proposition 2.1. *In a contact metric manifold, the ∗-conformal curvature tensor obeys the following:*

1. $^*C(X_1, X_2)X_3 = -^*C(X_2, X_1)X_3$,

2. $^*C(X_1, X_2)X_3 + ^*C(X_2, X_3)X_1 + ^*C(X_3, X_1)X_2$
$$= -\frac{1}{2n-1}\{^*Ric(X_1, X_2)X_3 + ^*Ric(X_2, X_3)X_1 + ^*Ric(X_3, X_1)X_2$$
$$- ^*Ric(X_1, X_3)X_2 - ^*Ric(X_2, X_1)X_3 - ^*Ric(X_3, X_2)X_1\}.$$

Definition 2.3. A contact metric manifold is called ξ-conformally flat and ξ-∗conformally flat, respectively, if $C(X_1, X_2)\xi = 0$ and $^*C(X_1, X_2)\xi = 0$, respectively.

3. ∗-conformal curvature tensor in Sasakian manifolds

In [14], Ghash and Patra obtained the ∗-Ricci tensor in a $(2n + 1)$-dimensional Sasakian manifold as follows

$$(3.1) \qquad ^*Ric(X_1, X_2) = Ric(X_1, X_2) - (2n - 1)\, g(X_1, X_2) - \eta(X_1)\eta(X_2).$$

Equation (3.1) provides

$$(3.2) \qquad\qquad ^*LX_1 = LX_1 - (2n - 1)\, X_1 - \eta(X_1)\xi,$$

and

$$(3.3) \qquad\qquad\qquad\qquad ^*r = r - 4n^2.$$

Theorem 3.1. *Let M^{2n+1} be a manifold with a Sasakian structure (g, η, ξ, φ). The manifold $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is a $^*\eta$-Einstien manifold.*

Proof. If $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold, then

$$(3.4) \qquad \exists c, d \in C^\infty(M), \quad Ric(X_1, X_2) = c\, g(X_1, X_2) + d\, \eta(X_1)\eta(X_2).$$

From (3.1) and (3.4), we have

$$(3.5) \qquad\qquad ^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2),$$

where $\tilde{c} = c - (2n - 1)$ and $\tilde{d} = d - 1$. Thus, $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a $^*\eta$-Einstien manifold. In this case, there are smooth scalar functions $\tilde{c}$ and $\tilde{d}$

$$(3.6) \qquad\qquad ^*Ric(X_1, X_2) = \tilde{c}\, g(X_1, X_2) + \tilde{d}\, \eta(X_1)\eta(X_2).$$

By (3.6) and (3.1), we conclude that M is a η-Einstien manifold. $\square$

A Sasakian manifold is said to be a $\phi-$recurrent manifold if there exists a nonzero $1-$form A such that

$$(3.7) \qquad \phi^2((\nabla_{X_1} K)(X_2, X_3)X_4) = A(X_1)\, K(X_2, X_3)X_4,$$

for arbitrary vector fields $X_1, X_2, X_3,$ and X_4 on the manifold M [11]. As a result, a $\phi-$recurrent Sasakian manifold is an Einstein manifold. Thus, by Theorem 3.1, it follows that every $\phi-$recurrent Sasakian manifold is a $^*\eta$-Einstein manifold.

In 1968, Yano and Sawaki [27] defined quasi-conformal curvature tensor as follows:

$$
\begin{aligned}
W(X_1, X_2)X_3 \;=\;& [-(n-2)\,d]\, C(X_1, X_2)X_3 \\
& + [c+(n-2)\,d]\, \tilde{C}(X_1, X_2)X_3,
\end{aligned}
$$
(3.8)

where c and d are arbitrary constants, C is the conformal curvature tensor, and $\tilde{C}$ given by

$$
\begin{aligned}
\tilde{C}(X_1, X_2)X_3 \;=\;& K(X_1, X_2)X_3 \\
& - \frac{r}{n(n-1)}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big],
\end{aligned}
$$
(3.9)

where K is the Riemannian curvature tensor.

A quasi-conformally flat Sasakian manifold or a quasi-conformally semi-symmetric Sasakian manifold is an η-Einstein manifold [9]. Using Theorem 3.1, we infer every quasi-conformally flat or quasi-conformally semi-symmetric Sasakian manifold is a $^*\eta$-Einstein manifold.

By using (3.1), (3.2) and (3.3), from (2.1), we get

$$
\begin{aligned}
{}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3 \;&+\; \frac{2n-2}{2n-1}\Big(g(X_2, X_3)X_1 - g(X_1, X_3)X_2\Big) \\
&+\; \frac{1}{2n-1}\Big(\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\
&+\; g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\Big).
\end{aligned}
$$
(3.10)

In Sasakian manifolds, Proposition 2.1 reduces to Proposition 3.1.

Proposition 3.1. *In a Sasakian manifold, the $*$-conformal curvature tensor obeys the following:*

$$
{}^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0.
$$

In a 3-dimensional manifold, C vanishes identically, and hence, we have:

$$
\begin{aligned}
{}^*C(X_1, X_2)X_3 \;=\;& \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 \\
& + g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi,
\end{aligned}
$$
(3.11)

In this case, (3.11) infers $^*C$ does not vanish identically. Indeed, for any non-zero vector filed $\tilde{X}$ in the kernel of η, we have

$$^*C(2\tilde{X} + \xi, \tilde{X} + \xi)\xi = \tilde{X}.$$

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a Sasakian manifold. By putting $X_3 = \xi$ in (3.10), we have

$$(3.12) \qquad ^*C(X_1, X_2)\xi = C(X_1, X_2)\xi + K(X_1, X_2)\xi.$$

Based on (3.12) and $K(X_1, X_2)\xi \neq 0$, we infer the Sasakian manifold does not become ξ-conformally flat and ξ-*conformally flat simultaneously.

Every Sasakian manifold is K-contact, but in general, every K-contact manifold is not Sasakian. For 3-dimensional manifolds, these are equivalent. In [28], the authors prove that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstien Sasakian manifold. From Theorem 3.1, we can say that a K-contact manifold is ξ-conformally flat if and only if it is a *η-Einstien Sasakian manifold.

In [8], the authors defined the $(0,2)$-tensor field T on M^{2n+1} as follows:

$$(3.13) \qquad T(X_1, X_2) = -\frac{Ric(X_1, X_2)}{2n-1} + \frac{r\, g(X_1, X_2)}{4n(2n-1)}.$$

The conformal curvature tensor is given by

$$
\begin{aligned}
C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad T(X_2, X_3) \cdot X_1 - T(X_1, X_3) \cdot X_2 \\
(3.14) \qquad\qquad &+ \quad g(X_2, X_3)\, \hat{T}(X_1) - g(X_1, X_3)\, \hat{T}(X_2),
\end{aligned}
$$

where $T(X_1, X_2) = g(\hat{T}(X_1), X_2)$. For $n > 1$, If $C = 0$, then

$$(3.15) \qquad \nabla_{X_1} T(X_2, X_3) - \nabla_{X_2} T(X_1, X_3) = 0.$$

We put $D(X_1, X_2)X_3 := \nabla_{X_1} T(X_2, X_3) - \nabla_{X_2} T(X_1, X_3)$. Now, we define $(0,2)$-tensor field $^*T$ on a Sasakian manifold M^{2n+1} as follows:

$$(3.16) \qquad ^*T(X_1, X_2) = -\frac{^*Ric(X_1, X_2)}{2n-1} + \frac{^*r\, g(X_1, X_2)}{4n(2n-1)}.$$

By (3.1) and (3.3), we can write (3.16) as follows

$$(3.17) \quad ^*T(X_1, X_2) = T(X_1, X_2) + \frac{n-1}{2n-1}\, g(X_1, X_2) + \frac{1}{2n-1}\, \eta(X_1)\eta(X_2).$$

Also, we define the conformal curvature tensor as follows:

$$
\begin{aligned}
^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad ^*T(X_2, X_3) \cdot X_1 - {}^*T(X_1, X_3) \cdot X_2 \\
(3.18) \qquad\qquad &+ \quad g(X_2, X_3)^*\hat{T}(X_1) - g(X_1, X_3)^*\hat{T}(X_2),
\end{aligned}
$$

where $^*T(X_1, X_2) = g(^*\hat{T}(X_1), X_2)$. So $(0,1)$-tensor field $^*\hat{T}$ is given by

$$(3.19) \qquad ^*\hat{T}(X_1) = \hat{T}(X_1) + \frac{n-1}{2n-1}\, X_1 + \frac{1}{2n-1}\, \eta(X_1)\xi.$$

By putting (3.17) and (3.19) in (3.18), we have

$$
\begin{aligned}
{}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3 \quad &+ \quad \frac{2(n-1)}{2n-1}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big] \\
&+ \quad \frac{1}{2n-1}\big[g(X_2, X_3)\xi - \eta(X_3)X_2\big]\eta(X_1) \\
&- \quad \frac{1}{2n-1}\big[g(X_1, X_3)\xi - \eta(X_3)X_1\big]\eta(X_2).
\end{aligned}
$$
(3.20)

We consider

$$
(3.21) \qquad {}^*D(X_1, X_2)X_3 := \nabla_{X_1}{}^*T(X_2, X_3) - \nabla_{X_2}{}^*T(X_1, X_3).
$$

A direct computation shows that

$$
\begin{aligned}
\nabla_{X_1}{}^*T(X_2, X_3) &= \nabla_{X_1}T(X_2, X_3) + \mu\,\nabla_{X_1}g(X_2, X_3) + \lambda\,\nabla_{X_1}(\eta(X_2)\eta(X_3)) \\
&= \nabla_{X_1}T(X_2, X_3) + \mu\,\nabla_{X_1}g(X_2, X_3) \\
&+ \lambda\,\big[(\nabla_{X_1}\eta(X_2))\,\eta(X_3) + \eta(X_2)\,(\nabla_{X_1}\eta(X_3))\big],
\end{aligned}
$$
(3.22)

and

$$
\begin{aligned}
\nabla_{X_2}{}^*T(X_1, X_3) &= \nabla_{X_2}T(X_1, X_3) + \mu\,\nabla_{X_2}g(X_1, X_3) + \lambda\,\nabla_{X_2}(\eta(X_1)\eta(X_3)) \\
&= \nabla_{X_2}T(X_1, X_3) + \mu\,\nabla_{X_2}g(X_1, X_3) \\
&+ \lambda\,\big[(\nabla_{X_2}\eta(X_1))\eta(X_3) + \eta(X_1)\,(\nabla_{X_2}\eta(X_3))\big],
\end{aligned}
$$
(3.23)

where $\mu = \frac{2n-2}{2n-1}$ and $\lambda = \frac{1}{2n-1}$. By putting (3.22) and (3.23) in (3.21), we have

$$
\begin{aligned}
{}^*D(X_1, X_2)X_3 = D(X_1, X_2)X_3 \quad &+ \quad \lambda\left\{2g(X_1, \phi X_2)\eta(X_3)\right. \\
&+ \quad \left.(\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\right\}.
\end{aligned}
$$
(3.24)

If M^{2n+1} is a conformally flat Sasakian manifold with $n > 1$, then

$$
{}^*D(X_1, X_2)X_3 = \lambda\left\{2g(X_1, \phi X_2)\eta(X_3) + (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1)\right\}.
$$
(3.25)

From (3.24), it can be concluded that, if M^{2n+1} is a Sasakian manifold of dimension greater than 3, then $D(X_1, X_2)X_3 = 0$ and ${}^*D(X_1, X_2)X_3 = 0$ do not hold simultaneously, because otherwise, we have $d\eta = 0$, which is a contradiction with the Sasakian structure.

Example 3.1. We consider the Sasakian manifold $(\mathbb{R}^3, g, \eta, \xi, \varphi)$, where the 1-form η, vector field ξ, Riemannian metric g, and $(1,1)$-tensor field φ respectively as follows

$$
\eta = \frac{1}{2}(dz - ydx), \qquad \xi = 2\frac{\partial}{\partial z}, \qquad g = \eta \otimes \eta + \frac{1}{4}\Big((dx)^2 + (dy)^2\Big),
$$

and $\varphi == dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x} + ydz \otimes \frac{\partial}{\partial y}$. Also, the vector fields are given by

$$X_1 = 2\frac{\partial}{\partial y}, \qquad X_2 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \qquad X_3 = \xi.$$

So, we have

$$\varphi X_1 = X_2, \qquad \varphi X_2 = -X_1, \qquad \varphi\xi = 0.$$

We know that, $\mathbb{R}^3$ is a conformally flat manifold, then $C = 0$. By (3.10) and $C(X_1, X_2)X_3 = 0$, we have $^*C(X_1, X_2)X_3 = -y\,X_1$. Therefore, for this 3-dimensional Sasakian manifold, the tensor $^*C$ will not be zero. On the other hand, we know that since $C(X_1, X_2)X_3 = 0$, then $D(X_1, X_2)X_3 = 0$. Therefore, having (3.25), we calculate the tensor $^*D$ as follows:

$$^*D(X_1, X_2)X_3 = -2.$$

4.   ∗-conformal curvature tensor in Kenmotsu manifolds

In [25], the author proves that in a Kenmotsu 3-manifold the ∗-Ricci tensor is given by

(4.1) $$^*Ric(X_1, X_2) \quad = \quad (\frac{r}{2} + 2)g(\varphi X_1, \varphi X_2),$$

(4.2) $$^*r \quad = \quad r + 4,$$

(4.3) $$^*LX_1 \quad = \quad (\frac{r}{2} + 2)\big[X_1 - \eta(X_1)\xi\big].$$

By substituting (4.1), (4.2), and (4.3) into (2.1) yields

$$^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad - \quad (\frac{r}{2} + 2)\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big]$$
$$+ \quad (\frac{r}{2} + 2)\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2$$
(4.4) $$+ \quad g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big].$$

Definition 4.1. [18] If the curvature tensor K of an almost contact metric manifold obeys the subsequent condition, then is called quasi-constant curvature:

$$K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) \quad + \quad \beta\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2$$
(4.5) $$+ \quad g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big],$$

where $(X_1 \wedge X_2)(X_3) := g(X_2, X_3)X_1 - g(X_1, X_3)X_2$, α and β are smooth functions.

By some calculation, one concludes that the following holds.

Theorem 4.1. *If a Kenmotsu 3-manifold is of quasi-constant curvature of the form*

$$K(X_1, X_2)X_3 = \alpha(X_1 \wedge X_2)(X_3) \quad - \quad \alpha\big[\eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2$$
(4.6) $$+ \quad g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi\big],$$

where $\alpha = \frac{r}{2} + 2$, then ∗-conformal curvature tensor vanishes.

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a Kenmostu manifold. By [21], we have

$$(4.7) \quad {}^*Ric(X_1, X_2) \quad = \quad Ric(X_1, X_2) + (2n-1)g(X_1, X_2) + \eta(X_1)\eta(X_2),$$

$$(4.8) \qquad\qquad {}^*r \quad = \quad r + 4n^2,$$

$$(4.9) \qquad\qquad {}^*LX_1 \quad = \quad LX_1 + (2n-1)X_1 + \eta(X_1)\xi.$$

By putting $X_2 = \xi$ in (4.7) and from (2.9), we have

$$(4.10) \qquad\qquad\qquad {}^*Ric(X_1, \xi) = 0,$$

from (2.11) and (4.7), we have

$$(4.11) \qquad\qquad\qquad {}^*Ric(\phi X_1, \phi X_2) = {}^*Ric(X_1, X_2).$$

Theorem 4.2. *Suppose M^{2n+1} is a manifold and (g, η, ξ, φ) is a Kenmotsu structure on M. The M is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. In [5], the contact metric structure is said to be η-Einstein if

$$(4.12) \qquad\qquad L = c\,I + d\,\eta \otimes \xi, \qquad c, d \in C^\infty(M).$$

Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a η-Einstein Kenmotsu manifold. By (4.9) and (4.12), we have

$$(4.13) \qquad\qquad\qquad {}^*L = \tilde{c}\,I + \tilde{d}\,\eta \otimes \xi,$$

where $\tilde{c} = c + (2n-1)$ and $\tilde{d} = c + 1$.

Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a ${}^*\eta$-Einstein Kenmotsu manifold, then there are smooth functions $\tilde{c}$, and $\tilde{d}$ such that

$$(4.14) \qquad\qquad {}^*Ric(X_1, X_2) = \tilde{c}\,g(X_1, X_2) + \tilde{d}\,\eta(X_1)\eta(X_2).$$

By (4.14) and (4.7), we have

$$(4.15) \qquad\qquad Ric(X_1, X_2) = c\,g(X_1, X_2) + d\,\eta(X_1)\eta(X_2),$$

where $c = \tilde{c} - (2n-1)$ and $d = \tilde{d} - 1$. $\square$

By substituting (4.7), (4.8), and (4.9) into (2.1) yields

$$
\begin{aligned}
{}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3 \quad &- \quad \frac{2n-2}{2n-1}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big] \\
&- \quad \frac{1}{2n-1}\big[g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi \\
(4.16) \qquad &+ \quad \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2\big].
\end{aligned}
$$

By putting $X_3 = \xi$ in (4.16), we obtain

$$(4.17) \qquad\qquad {}^*C(X_1, X_2)\xi = C(X_1, X_2)\xi + K(X_1, X_2)\xi.$$

From (4.17), we conclude that if $C(X_1, X_2)\xi = 0$ then $^*C(X_1, X_2)\xi \neq 0$. In other words, the Kenmotsu manifold cannot be ξ-conformally flat and ξ-∗conformally flat simultaneously.

In the Kenmotsu manifold, (2) results in $^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$. By Proposition 2.1 and $^*Ric(X_1, X_2) = {}^*Ric(X_2, X_1)$, the ∗-conformal curvature tensor satisfies in Bianchi type identity, which leads to the next proposition.

Proposition 4.1. *In a Kenmotsu manifold, the ∗-conformal curvature tensor obeys the relation:*

$$^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0.$$

Let us define

$$C(X_1, X_2, X_3, X_4) := g\big(C(X_1, X_2)X_3, X_4\big), \qquad \forall X_1, X_2, X_3, X_4 \in \chi(M).$$

By substituting (4.7) into (2.1), we have

$$
\begin{aligned}
^*C(X_1, X_2, X_3, X_4) \quad = \quad & C(X_1, X_2, X_3, X_4) \\
& - \frac{2n-2}{2n-1}\big[g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\big] \\
& - \frac{1}{2n-1}\big[g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) \\
& + \ g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3)\big].
\end{aligned}
$$

(4.18)

Proposition 4.2. *For a Kenmotsu manifold, the ∗-conformal tensor cannot vanish identically.*

Proof. One can see that

$$(4.19) \qquad C(X_1, X_2, X_3, X_4) = -C(X_1, X_2, X_4, X_3).$$

Suppose that $^*C$ vanishes identically. Therefore, by (4.18) and (4.19), we have

$$
\begin{aligned}
2(2n-2) \quad \Big[\quad & g(X_2, X_3)g(X_1, X_4) - g(X_1, X_3)g(X_2, X_4)\Big] \\
+ \ & 2\big[g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) \\
+ \ & g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3)\big] = 0.
\end{aligned}
$$

(4.20)

Putting $X_3 = X_1 = \xi$ into (4.20) implies that

$$(4.21) \qquad (2n-1)\Big(g(X_2, X_4) - \eta(X_2)\eta(X_4)\Big) = 0.$$

Since $2n - 1$ is an odd number, we have

$$(4.22) \qquad g(X_2, X_4) - \eta(X_2)\eta(X_4) = 0, \qquad \forall X_2, X_4 \in \chi(M),$$

which is impossible. $\square$

Using Propositions 4.1 and 4.2, one concludes that a Kenmotsu 3-manifold cannot be of quasi-constant curvature of the form (4.6).

Now, we consider $(0, 2)$-tensor field ${}^*T$ on Kenmotsu manifold M^{2n+1} as follows:

$$(4.23) \qquad {}^*T(X_1, X_2) = -\frac{{}^*Ric(X_1, X_2)}{2n - 1} + \frac{{}^*r\, g(X_1, X_2)}{4n(2n - 1)}.$$

By (4.8) and (4.7), we can write (4.23) as follows:

$$(4.24)\ {}^*T(X_1, X_2) = T(X_1, X_2) + \frac{(1 - n)}{(2n - 1)}\, g(X_1, X_2) + \frac{-1}{2n - 1}\, \eta(X_1)\eta(X_2).$$

Also, we define the conformal curvature tensor as follows:

$$
\begin{aligned}
{}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad {}^*T(X_2, X_3) \cdot X_1 - {}^*T(X_1, X_3) \cdot X_2 \\
(4.25) \qquad\qquad &+ \quad g(X_2, X_3){}^*\hat{T}(X_1) - g(X_1, X_3){}^*\hat{T}(X_2),
\end{aligned}
$$

where ${}^*T(X_1, X_2) = g({}^*\hat{T}(X_1), X_2)$. So ${}^*\hat{T}$ is given by

$$(4.26) \qquad {}^*\hat{T}(X_1) = \hat{T}(X_1) + \frac{(1 - n)}{(2n - 1)}\, X_1 + \frac{-1}{2n - 1}\, \eta(X_1)\xi.$$

By putting (4.24) and (4.26) in (4.25), we have

$$
\begin{aligned}
{}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3 \quad &+ \quad \frac{2(1 - n)}{(2n - 1)}\big[g(X_2, X_3)X_1 - g(X_1, X_3)X_2\big] \\
&+ \quad \Big(\frac{-1}{2n - 1}\Big)\big[g(X_2, X_3)\xi - \eta(X_3)X_2\big]\eta(X_1) \\
(4.27) \qquad\qquad &- \quad \Big(\frac{-1}{2n - 1}\Big)\big[g(X_1, X_3)\xi - \eta(X_3)X_1\big]\eta(X_2).
\end{aligned}
$$

We consider

$$(4.28) \qquad {}^*D(X_1, X_2)X_3 := \nabla_{X_1}{}^*T(X_2, X_3) - \nabla_{X_2}{}^*T(X_1, X_3).$$

Now, we consider can we conclude ${}^*D(X_1, X_2)X_3 = 0$ if ${}^*C(X_1, X_2)X_3 = 0$. So

$$
\begin{aligned}
\nabla_{X_1}{}^*T(X_2, X_3) \quad &= \quad \nabla_{X_1}T(X_2, X_3) + \mu\, \nabla_{X_1}g(X_2, X_3) + \lambda\, \nabla_{X_1}(\eta(X_2)\eta(X_3)) \\
&= \quad \nabla_{X_1}T(X_2, X_3) + \mu\, \nabla_{X_1}g(X_2, X_3) \\
(4.29) \qquad\qquad &+ \quad \lambda\,\big[(\nabla_{X_1}\eta(X_2))\,\eta(X_3) + \eta(X_2)\,(\nabla_{X_1}\eta(X_3))\big],
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{X_2}{}^*T(X_1, X_3) \quad &= \quad \nabla_{X_2}T(X_1, X_3) + \mu\, \nabla_{X_2}g(X_1, X_3) + \lambda\, \nabla_{X_2}(\eta(X_1)\eta(X_3)) \\
&= \quad \nabla_{X_2}T(X_1, X_3) + \mu\, \nabla_{X_2}g(X_1, X_3) \\
(4.30) \qquad\qquad &+ \quad \lambda\,\big[(\nabla_{X_2}\eta(X_1))\eta(X_3) + \eta(X_1)\,(\nabla_{X_2}\eta(X_3))\big],
\end{aligned}
$$

where $\mu = \frac{2(1-n)}{(2n-1)}$ and $\lambda = \frac{-1}{2n-1}$. By putting (4.29) and (4.30) in (4.28), we have

$$
\begin{aligned}
{}^*D(X_1, X_2)X_3 &= D(X_1, X_2)X_3 \\
&\quad + \lambda \left\{ (\nabla_{X_1}\eta)(X_3)\eta(X_2) - (\nabla_{X_2}\eta)(X_3)\eta(X_1) \right\}.
\end{aligned}
$$
(4.31)

Theorem 4.3. *Let M be a $2n+1$-dimension manifold with $n > 1$ and (g, η, ξ, φ) is a Kenmotsu structure on M. Then $D(X_1, X_2)X_3 = 0$ and ${}^*D(X_1, X_2)X_3 = 0$ do not hold at the same time.*

Proof. From (4.31), it is easily proved. □

Example 4.1. We consider the Kenmotsu manifold $(\mathbb{R}^3 - (0,0,0), g, \eta, \xi, \varphi)$, where the 1-form η, vector field ξ, Riemannian metric g, and $(1,1)$-tensor field φ respectively as follows

$$
\eta = -\frac{1}{z}dz, \qquad \xi = -z\frac{\partial}{\partial z}, \qquad g = (dx)^2 + (dy)^2 + (dz)^2,
$$

and $\varphi = dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$. Also, the vector fields are given by

$$
X_1 = z\frac{\partial}{\partial x}, \qquad X_2 = z\frac{\partial}{\partial y}, \qquad X_3 = \xi.
$$

So, we have

$$
\varphi X_1 = -X_2, \qquad \varphi X_2 = X_1, \qquad \varphi \xi = 0.
$$

By conformally flat manifold $\mathbb{R}^3$, we have $C = 0$. By (4.16) and $C = 0$, then ${}^*C(X_1, X_2)X_3 = 0$. We know that since $C(X_1, X_2)X_3 = 0$, then $D(X_1, X_2)X_3 = 0$. Therefore, having (4.31), ${}^*D(X_1, X_2)X_3 = 0$.

5. ∗-conformal curvature of the cosymplectic manifolds

Let (g, η, ξ, φ) be a cosymplectic structure on M^{2n+1}. In [17], it is proved that for a cosymplectic manifold

$$
{}^*Ric(X_1, X_2) = Ric(X_1, X_2),
$$
(5.1)

and

$$
{}^*r = r.
$$
(5.2)

Theorem 5.1. *Suppose $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a cosymplectic manifold. Then M is an η-Einstien manifold if and only if it is a ${}^*\eta$-Einstien manifold.*

Proof. It is easy to conclude from (5.1) that for the cosymplectic manifold, the η-Einstien manifold and ${}^*\eta$-Einstien manifold are equivalent. □

Substituting (5.1) and (5.2) into (2.1) yields

$$
{}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3.
$$
(5.3)

Proposition 5.1. *In a cosymplectic manifold, the $*$-conformal curvature tensor obeys the relation:*

$$^*C(X_1, X_2)X_3 + {}^*C(X_2, X_3)X_1 + {}^*C(X_3, X_1)X_2 = 0.$$

The following results are obtained from (5.3).

Corollary 5.1. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a cosymplectic manifold. Then M is a conformally flat if and only if it is a $*$-conformally flat.*

Corollary 5.2. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a cosymplectic manifold. Then M is a ξ-conformally flat if and only if it is a ξ-$*$conformally flat.*

The conformal curvature tensor is zero in dimension 3. Thus we have:

Proposition 5.2. *For a 3-dimensional cosymplectic manifold, $^*C$ is identically zero.*

We consider $(0, 2)$-tensor field $^*T$ on cosymplectic manifold M^{2n+1} as follows:

$$(5.4) \qquad {}^*T(X_1, X_2) = -\frac{{}^*Ric(X_1, X_2)}{2n - 1} + \frac{{}^*r\ g(X_1, X_2)}{4n(2n - 1)}.$$

By (5.1) and (5.2), we can

$$(5.5) \qquad {}^*T(X_1, X_2) = T(X_1, X_2).$$

Also, define the conformal curvature tensor as follows:

$$(5.6) \qquad \begin{aligned} {}^*C(X_1, X_2)X_3 = K(X_1, X_2)X_3 \quad &+ \quad {}^*T(X_2, X_3) \cdot X_1 - {}^*T(X_1, X_3) \cdot X_2 \\ &+ \quad g(X_2, X_3){}^*\hat{T}(X_1) - g(X_1, X_3){}^*\hat{T}(X_2), \end{aligned}$$

where $^*T(X_1, X_2) = g(^*\hat{T}(X_1), X_2)$. So $(0, 1)$-tensor field $^*\hat{T}$ is given by

$$(5.7) \qquad {}^*\hat{T}(X_1) = \hat{T}(X_1).$$

By putting (5.5) and (5.7) in (5.6), we have

$$(5.8) \qquad {}^*C(X_1, X_2)X_3 = C(X_1, X_2)X_3.$$

We consider

$$(5.9) \qquad {}^*D(X_1, X_2)X_3 := \nabla_{X_1}{}^*T(X_2, X_3) - \nabla_{X_2}{}^*T(X_1, X_3).$$

On the other hand, we have

$$(5.10) \qquad \nabla_{X_1}{}^*T(X_2, X_3) = \nabla_{X_1}T(X_2, X_3),$$

and

$$\nabla_{X_2}{}^*T(X_1, X_3) = \nabla_{X_2}T(X_1, X_3).$$ (5.11)

By putting (5.10) and (5.11) in (5.9), we have

$${}^*D(X_1, X_2)X_3 = D(X_1, X_2)X_3.$$ (5.12)

We know that if $C = 0$ for a $2n + 1$-dimension cosymplectic manifold with $n \geqslant 1$, then $D = 0$. Now, if we assume ${}^*C = 0$, then according to (5.12), the following theorem is obtained.

Theorem 5.2. *Let $(M^{2n+1}, g, \eta, \xi, \varphi)$ be a $2n+1$-dimension cosymplectic manifold with $n \geqslant 1$. If M is a ∗-conformally flat manifold, then ${}^*D = 0$.*

REFERENCES

1. S. Amari and H. Nagaoka: *Methods of information geometry.* Amer. Math. Soc. **191** (2000).

2. D. E. Blair: *Riemannian geometry of contact and symplectic manifolds.* Springer Science and Business Media (2010).

3. D. E. Blair: *The theory of quasi-Sasakian structures.* J. Diff. Geom. **1** (1967), 331–381.

4. D. E. Blair: *Two remarks on contact metric manifolds.* Tohoku Math. J. **29** (1977), 319–324.

5. D. E. Blair, T. Koufogiorgos and R. Sharma: *A classification of 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$.* Kodai. J. Math, **13 (3)** (1990), 391–401.

6. M. C. Chaki and B. Gupta: *On conformally symmetric spaces.* Indian J. Math. **5**, (1963) 113–122.

7. B. Y. Chen and K. Yano: *Hypersurfaces of conformally flat spaces.* Tensor (N. S) **26** (1972), 318–322.

8. B. Chen and K. Yano: *Special conformally flat spaces and canal hypersurfaces.* Tohoku. J. Math. **25 (2)** (1973), 177–184.

9. U. C. De, J. B. Jun and A. K. Gazi: *Sasakian manifolds with quasi-conformal curvature tensor.* Bull. Korean Math. Soc. **45 (2)** (2008), 313–319.

10. U. C. De, M. Majhi and Y. J. Suh: *∗-Ricci soliton on Sasakian 3-manifolds.* Publ. Math. Debrecen **93** (2018), 241–252.

11. U. C. De, A. A. Shaikh and S. Biswas: *On φ-recurrent Sasakian manifolds.* Novi Sad J. Math. **33 (2)** (2003), 43–48.

12. A. Derdzinski and W. Roter: *On Conformally Symmetric Manifolds with Metrics of Indices 0 and 1.* Tensor N. S. **31** (1977) 255–259.

13. M. S. El Naschie: *Gödel universe, dualities and high energy particles in E-infinity.* Chaos, Solitons & Fractals, **25 (3)** (2005), 759–764.

14. A. Ghosh and D. S. Patra: *∗-Ricci Soliton within the framework of Sasakian and (k, μ)-contact manifold.* Int. J. Geom. methods modern Phys. **15** 1850120 (2018).

15. S. I GOLDBERG and K. YANO: *Integrebility of almost cosymplectic structures.* Pacific J. Math. **31** (1969), 373–382.

16. T. HAMADA: *Real hypersurfaces of complex space forms in terms of Ricci ∗-tensor.* Tokyo J. Math. **25** (2002) 473–483.

17. A. HASEEB, D. G. PRAKASHA and H. HARISH: *∗-Conformal η-Ricci solotons on α-cosymplectic manifolds.* International Journal of Analysis and Applications **12 (2)** (2021), 165–179.

18. S. IANUS and D. SMARANDA: *Some remarkable structures on the product of an almost contact metric manifold with the real line.* Soc. Sti. Mat., Univ. Timisoara, 1977.

19. K. KENMOTSU: *A class of almost contact Riemannian manifolds.* Tohoku Math. J. **24** (1972), 93–103.

20. H. N. NICKERSON: *On conformally symmetric spaces.* Geometriae Dedicata **18 (1)** (1985), 87–99.

21. D. S. PATRA, A. ALI and F. MOFARREH: *Geometry of almost contact metrics as almost ∗-Ricci solitons.* arXiv e-prints (2021): arXiv-2101.

22. W. SLOSARSKA: *On some property of conformally symmetric manifold admitting a semi-symmetric metric connection.* Demonstratio Math. **17 (4)** (1984), 813–816.

23. S. TACHIBANA: *On almost-analytic vectors in almost Kahlerian manifolds.* Tohoku Math. J. **11** (1959), 247–265.

24. S. TANNO: *Note on infinitesimal transformations over contact manifolds.* Tohoku Mathematical Journal, Second Series, **14 (4)** (1962), 416–430.

25. Y. WANG: *Contact 3-manifolds and ∗-Ricci soliton.* Kodai Math. J. **43** (2020), 256–267.

26. K. YANO: *On semi-symmetric metric connections.* Rev. Roumaine Math. Pures Appl. **15** (1970) 1579–1586.

27. K. YANO and S. SAWAKI; *Riemannian manifolds admitting a conformal transformation group.* Journal of Differential Geometry **2 (2)** (1968), 161–184.

28. G. ZHEN, J. L. COBRERIZO, L. M. FERANDEZ and M. FERNADEZ: *On ξ-conformally flat contact metric manifolds.* Indian J. Pure. Appl. Math. **28** (1997), 725–734.