A NOTE ON $\alpha$-PARA KENMOTSU MANIFOLDS

Pradip Majhi

Abstract. The object of the present paper is to study 3-dimensional $\alpha$-Para Kenmotsu manifolds. First we consider $\phi$-projectively semi-symmetric 3-dimensional $\alpha$-Para Kenmotsu manifolds. We also study projectively semi-symmetric and projectively pseudosymmetric 3-dimensional $\alpha$-para Kenmotsu manifolds. Beside these 3-dimensional $\alpha$-Para Kenmotsu manifolds satisfying $P.S = 0$ is also considered.

Keywords: $\alpha$-Para Kenmotsu manifolds, curvature tensor, Euclidian space, Riemannian manifold, Ricci tensor.

1. Introduction

The projective curvature tensor is an important tensor from the differential geometric point of view. Let $M$ be a $(2n + 1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of $M$ and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $M$ is said to be locally projectively flat. For $n \geq 1$, $M$ is locally projectively flat if and only if the well known projective curvature tensor $P$ vanishes. Here $P$ is defined by [9]

\begin{equation}
P(X,Y)Z = R(X,Y)Z - \frac{1}{2n} [S(Y,Z)X - S(X,Z)Y],
\end{equation}

for all $X, Y, Z \in T(M)$, where $R$ is the curvature tensor and $S$ is the Ricci tensor. In fact $M$ is projectively flat if and only if it is of constant curvature [16]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Let $(M, g)$ be a Riemannian manifold and let $\nabla$ be the Levi-Civita connection of $(M, g)$. A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where $R$ is the Riemann curvature tensor of $(M, g)$. A Riemannian manifold $M$ is called semi-symmetric if
holds, where $R$ denotes the curvature tensor of the manifold. It is well known that the class of semi-symmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semi-symmetric Riemannian manifolds were first studied by E. Cartan, A. Lichnerowich, R. S. Couty and N. S. Sinjukov. A fundamental study on Riemannian semi-symmetric manifolds was made by Z. I. Szabó [10], E. Boeckx et al [2] and O. Kowalski [6]. A Riemannian manifold $M$ is said to be Ricci-semi-symmetric if on $M$ we have
\begin{equation}
R.S = 0,
\end{equation}
where $S$ is the Ricci tensor.

The class of Ricci-semi-symmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci-semi-symmetric manifolds were investigated by several authors.

The present paper is organized as follows: After in brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. In section 3, we consider $\phi$-projectively semi-symmetric 3-dimensional $\alpha$-Para Kenmotsu manifolds. Section 4 is devoted to study projectively semi-symmetric 3-dimensional $\alpha$-Para Kenmotsu manifolds. In section 5, we consider projectively pseudosymmetric 3-dimensional $\alpha$-para Kenmotsu manifolds. Finally, 3-dimensional $\alpha$-Para Kenmotsu manifolds satisfying $P.S = 0$ is also considered.

2. Preliminaries

2.1. Almost Paracontact Metric Manifolds

A smooth manifold $M$ of dimension $2n + 1$ is called an almost paracontact manifold ([7],[8]) equipped with the structure $(\phi, \xi, \eta)$ where $\phi$ is a tensor field of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying
\begin{equation}
\phi^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.
\end{equation}

From equation (2.1) it can easily deduced that
\begin{equation}
\phi \xi = 0, \quad \eta(\phi X) = 0 \quad \text{and} \quad \text{rank}(\phi) = 2n.
\end{equation}

If an almost paracontact manifold admits a pseudo-Riemannian metric $g$ satisfying
\begin{equation}
g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),
\end{equation}

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$R.R = 0$ holds, where $R$ denotes the curvature tensor of the manifold. It is well known that the class of semi-symmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. Semi-symmetric Riemannian manifolds were first studied by E. Cartan, A. Lichnerowich, R. S. Couty and N. S. Sinjukov. A fundamental study on Riemannian semi-symmetric manifolds was made by Z. I. Szabó [10], E. Boeckx et al [2] and O. Kowalski [6]. A Riemannian manifold $M$ is said to be Ricci-semi-symmetric if on $M$ we have
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The present paper is organized as follows: After in brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. In section 3, we consider $\phi$-projectively semi-symmetric 3-dimensional $\alpha$-Para Kenmotsu manifolds. Section 4 is devoted to study projectively semi-symmetric 3-dimensional $\alpha$-Para Kenmotsu manifolds. In section 5, we consider projectively pseudosymmetric 3-dimensional $\alpha$-para Kenmotsu manifolds. Finally, 3-dimensional $\alpha$-Para Kenmotsu manifolds satisfying $P.S = 0$ is also considered.
where signature of $g$ is $(n + 1, n)$ for any vector fields $X, Y \in \chi(M)$, (where $\chi(M)$ is the set of all differential vector fields on $M$) then the manifold is called almost paracontact metric manifold.

An almost paracontact structure is said to be a contact structure if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric $g$ [17]. For an almost paracontact metric manifold, there always exists a special kind of local pseudo orthonormal $\phi$ basis $\{X_i, \phi X_i, \xi\}$, $X_i$’s and $\xi$ are space-like vector fields and $\phi X_i$’s are time-like. Thus, an almost paracontact metric manifold is an odd dimensional manifold.

### 2.2. Normal Almost Paracontact Metric Manifolds

An almost paracontact metric manifold is said to be normal if the induced almost paracomplex structure $J$ on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$
J(X, f \frac{d}{dt}) = (\phi X + f \xi, \eta(X) \frac{d}{dt})
$$

is integrable where $X$ is tangent to $M$, $t$ is the coordinate of $\mathbb{R}$ and $f$ is a smooth function on $M^{2n+1} \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor $N_\phi$ defined by $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$.

**Proposition 2.1.** [14] For a 3-dimensional almost paracontact metric manifold $M$, the following conditions are mutually equivalent

(a) $M$ is normal,

(b) there exist differential functions $\alpha, \beta$ on $M$ such that

$$(\nabla_X \phi)Y = \beta\{g(X, Y)\xi - \eta(Y)X\} + \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

(c) there exist differential functions $\alpha, \beta$ on $M$ such that

$$\nabla_X \xi = \alpha\{X - \eta(X)\xi\} + \beta\phi X,$$

where $\nabla$ is the Levi-Civita connection of the pseudo-Riemannian metric $g$ and $\alpha, \beta$ are given by

$$2\alpha = \text{Trace}\{X \to \nabla_X \xi\}, 2\beta = \text{Trace}\{X \to \phi \nabla_X \xi\}.$$

**Definition 2.1.** A 3-dimensional normal almost paracontact metric manifold $M$ is said to be

1. paracosymplectic if $\alpha = \beta = 0$ [4],

2. $\alpha$-para Kenmotsu if $\alpha$ is a non-zero constant and $\beta = 0$ [13], in particular para Kenmotsu if $\alpha = 1$ [1],
3. quasi-para Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ [5],
4. $\beta$-para Sasakian if and only if $\alpha = 0$ and $\beta$ is a non-zero constant, in particular para Sasakian if $\beta = -1$ [17].

In a 3-dimensional $\alpha$-para Kenmotsu manifold, the following results hold [11]:

$$R(X,Y)Z = \left(\frac{r}{2} + 2\alpha^2\right)g(Y,Z)X - g(X,Z)Y$$

$$-\left(\frac{r}{2} + 3\alpha^2\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi$$

$$+\left(\frac{r}{2} + 3\alpha^2\right)[\eta(X)Y - \eta(Y)X]\eta(Z),$$

(2.5)

where $r$ is the scalar curvature of the manifold and $g$, pseudo-metric.

$$S(X,Y) = \left(\frac{r}{2} + \alpha^2\right)g(X,Y) - \left(\frac{r}{2} + 3\alpha^2\right)\eta(X)\eta(Y).$$

(2.6)

$$S(X,\xi) = -2\alpha^2\eta(X),$$

(2.7)

$$R(X,Y)\xi = -\alpha^2[\eta(Y)X - \eta(X)Y],$$

(2.8)

$$(\nabla_X\eta)Y = \alpha[g(X,Y) - \eta(X)\eta(Y)],$$

(2.9)

$$(\nabla_X\phi)Y = \alpha[g(\phi X,Y)\xi - \eta(Y)\phi X],$$

(2.10)

$$\nabla_X\xi = \alpha[X - \eta(X)\xi],$$

(2.11)

for all vector fields $X,Y,Z$ and $W \in \chi(M)$.

Now, we state two Theorems which will be used in the next sections.

**Theorem 2.1.** [15] A 3-dimension Riemannian manifold is Einstein if and only if it is manifold of constant curvature.

**Theorem 2.2.** [15] A Riemannian manifold is projectively flat if and only if the manifold is of constant curvature.

3. $\phi$-projectively semisymmetric 3-dimensional $\alpha$-para Kenmotsu manifolds

Let $M$ be a 3-dimensional $\alpha$-para Kenmotsu manifold. Therefore $P(X,Y)\phi = 0$ turns into

$$(P(X,Y)\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0,$$

for any vector fields $X, Y$ and $Z$.

Now, in view of (1.1), (2.5) we have

$$P(X,Y)\phi Z = \left(\frac{r}{2} + 2\alpha^2\right)[g(Y,\phi Z)X - g(X,\phi Z)Y]$$

$$-\left(\frac{r}{2} + 3\alpha^2\right)[g(Y,\phi Z)\eta(X) - g(X,\phi Z)\eta(Y)]\xi$$

$$-\frac{1}{2}\left(\frac{r}{2} + \alpha^2\right)g(Y,\phi Z)X - \left(\frac{r}{2} + \alpha^2\right)g(X,\phi Z)Y.$$

(3.2)
Similarly, we obtain
\( \phi(P(X, Y)Z) = \phi[R(X, Y)\phi Z - \frac{1}{2}(S(Y, Z)X - S(X, Z)Y)] \).

By virtue of (3.2) and (3.3), we get from (3.1)
\[ \phi[R(X, Y)\phi Z - \frac{1}{2}(S(Y, Z)X - S(X, Z)Y)] = (r + 2a^2)g(Y, \phi Z)X - g(Y, \phi Z)Y \]
\[ - (\frac{r}{2} + 3a^2)g(Y, \phi Z)\eta(X) - g(X, \phi Z)\eta(Y)\xi \]
(3.4)
\[ - \frac{1}{2}((\frac{r}{2} + a^2)g(Y, \phi Z)X - (\frac{r}{2} + a^2)g(X, \phi Z)Y). \]

Putting \( X = \xi \) in (3.4) we have
\[ \phi[R(\xi, Y)\phi Z - \frac{1}{2}(S(Y, Z)\xi - S(\xi, Z)Y)] = (\frac{r}{2} + 2a^2)g(\xi, \phi Z)\xi - (\frac{r}{2} + 3a^2)g(\xi, \phi Z)\xi \]
(3.5)
\[ - \frac{1}{2}((\frac{r}{2} + a^2)g(\xi, \phi Z)\xi). \]

Using (2.7), (2.8) in (3.5) yields
\[ (\frac{r}{2} + 2a^2)g(\xi, \phi Z)\xi - (\frac{r}{2} + 3a^2)g(\xi, \phi Z)\xi \]
(3.6)
\[ - \frac{1}{2}((\frac{r}{2} + a^2)g(\xi, \phi Z)\xi) = 0, \]
which implies
(3.7)
\[ \frac{1}{2}((\frac{r}{2} + 3a^2)g(\xi, \phi Z)\xi = 0, \]

Since \( g(\xi, \phi Z) \neq 0 \), in (3.7) taking inner product with \( \xi \) we have
(3.8)
\[ r = -6a^2. \]

Substituting (3.8) in (2.6) we obtain
(3.9)
\[ S(X, Y) = -2a^2g(X, Y). \]

Therefore the manifold is an Einstein manifold.

It is known [15] that a 3-dimension Riemanian manifold is Einstein if and only if it is manifold of constant curvature. Again a Riemanian manifold is projectively flat if and only if the manifold is of constant curvature.

By the above discussion we have the following:
Theorem 3.1. In a 3-dimensional $\alpha$-para Kenmotsu manifold $M$, the following conditions are equivalent:

(a) $\phi$-projectively semi-symmetric,
(b) the scalar curvature $r = -6\alpha^2$,
(c) the manifold $M$ is of constant curvature,
(d) $M$ is an Einstein manifold.

4. projectively semi-symmetric 3-dimensional $\alpha$-para Kenmotsu manifolds

In view of (1.1), the projective curvature tensor is given by

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n} [S(Y, Z)X - S(X, Z)Y].
\]

Now from the above equation with the help of (2.5), (2.6) we have

\[
P(U, V)\xi = 0,
\]

for any vector fields $U$, $V$. We suppose that a 3-dimensional $\alpha$-para Kenmotsu manifold is projectively semi-symmetric, that is,

\[
(R(X, Y)P)(U, V) = 0.
\]

This implies

\[
\]

Using $Y = U = W = \xi$ in (4.4) we have

\[
R(X, \xi)P(\xi, V)\xi - P(R(X, \xi)\xi, V)\xi - P(\xi, R(X, \xi)V)\xi + P(\xi, V)R(X, \xi)\xi = 0.
\]

Therefore from (4.2) and (4.5) we get

\[
-P(\xi, V)R(X, \xi)\xi = 0.
\]

In view of (2.5), (2.6), (2.8) we obtain from (4.6)

\[
-\alpha^2 P(\xi, V)X = 0.
\]

Since $\alpha \neq 0$, the above equation implies

\[
P(\xi, V)X = 0.
\]
Therefore

\[ R(\xi, V)X - \frac{1}{2}[S(V, X)\xi - S(\xi, X)V] = 0. \]

(4.9)

Applying (2.5), (2.6), (2.8) in (4.9) we get

\[ \frac{1}{2}r + 3\alpha^2 g(\phi X, \phi V) = 0. \]

(4.10)

Since \(g(\phi X, \phi V) \neq 0\), we have

\[ r = -6\alpha^2. \]

(4.11)

Substituting (4.11) in (2.6) we obtain

\[ S(X, Y) = -2\alpha^2 g(X, Y). \]

(4.12)

Thus the manifold is an Einstein manifold.

In view of Theorems 2.1 and 2.2 we have the following:

**Theorem 4.1.** In a 3-dimensional \(\alpha\)-para Kenmotsu manifold \(M\), the following conditions are equivalent:

(a) projectively semi-symmetric,

(b) the scalar curvature \(r = -6\alpha^2\),

(c) the manifold \(M\) is of constant curvature,

(d) \(M\) is an Einstein manifold.

5. **Projectively pseudosymmetric 3-dimensional \(\alpha\)-para Kenmotsu manifolds**

A Riemannian manifold is said to be projectively pseudosymmetric [12] if at every point of the manifold the following relation holds


(5.1)

for any vector fields \(X, Y, U, V, W\); where \(L_R\) is some function of \(M\). The endomorphism \(X \wedge Y\) is defined by

\[ (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \]

(5.2)

A Riemannian manifold is said to be projectively pseudosymmetric if it satisfies the condition


(5.3)
where \( L_P(\neq \alpha^2) \) is some function on \( M \).

Let us suppose that 3-dimensional \( \alpha \)-para Kenmotsu manifold \( M \) satisfies the condition

\[
(R(X, Y).P)(U, V)W = L_P((X \wedge Y).P)(U, V)W. \tag{5.4}
\]

Putting \( Y = W = U = \xi \) (5.4) we have

\[
(R(X, \xi).P)(\xi, V)\xi = L_P((X \wedge \xi).P)(\xi, V)\xi, \tag{5.5}
\]

Now

\[
L_P((X \wedge \xi).P)(\xi, V)\xi = L_P((X \wedge \xi)P(\xi, V)\xi - P((X \wedge \xi)\xi, V)\xi - P(\xi, V)(X \wedge \xi)\xi. \tag{5.6}
\]

Using (4.2) in (5.6), we get

\[
L_P((X \wedge \xi).P)(\xi, V)\xi = P(\xi, V)(X \wedge \xi)\xi - P(\xi, V)(X \wedge \xi)\xi. \tag{5.7}
\]

In view of (4.7), (5.7) we have from (5.5)

\[
-\alpha^2 P(\xi, V)X = -L_P(\xi, V)X. \tag{5.8}
\]

Therefore

\[
(L_P - \alpha^2)P(\xi, V)X = 0. \tag{5.9}
\]

By assumption \( L_P \neq \alpha^2 \) and hence

\[
P(\xi, V)X = 0. \tag{5.10}
\]

The above equation same as (4.8), hence it follows that

\[
r = -6\alpha^2. \tag{5.11}
\]

Substituting (5.11) in (2.6) we obtain

\[
S(X, Y) = -2\alpha^2 g(X, Y). \tag{5.12}
\]

Thus the manifold is an Einstein manifold.

In view of Theorems 2.1 and 2.2 we can state the following:

**Theorem 5.1.** In a 3-dimensional \( \alpha \)-para Kenmotsu manifold \( M \), the following conditions are equivalent:

(a) Projectively pseudosymmetric,

(b) the scalar curvature \( r = -6\alpha^2 \),

(c) the manifold \( M \) is of constant curvature,

(d) \( M \) is an Einstein manifold.
6. 3-dimensional $\alpha$-Para Kenmotsu manifolds satisfying $P.S = 0$

In this section we study 3-dimensional $\alpha$-Para Kenmotsu manifolds satisfying $P.S = 0$. Therefore we have

\[(6.1) \quad (P(X, Y) \cdot S)(U, V) = 0.\]

This implies

\[(6.2) \quad S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0.\]

Putting $U = \xi$ in (6.2) we have

\[(6.3) \quad S(P(X, Y)\xi, V) + S(\xi, P(X, Y)V) = 0.\]

Using (4.2) in (6.3), we get

\[(6.4) \quad S(\xi, P(X, Y)V) = 0.\]

Therefore from (2.7) and (6.4) we obtain

\[(6.5) \quad -2\alpha^2 g(P(X, Y)V, \xi) = 0.\]

This implies

\[(6.6) \quad g(R(X, Y)V, \xi) - \frac{1}{2}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)] = 0.\]

Using (2.5) in (6.6) we have

\[(6.7) \quad \eta(Y)[\alpha^2 g(X, V) - \frac{1}{2}S(X, V)] = \eta(X)[\alpha^2 g(Y, V) + \frac{1}{2}S(Y, V)].\]

Putting $Y = \xi$ in (6.7), we get

\[(6.8) \quad S(X, V) = 2\alpha^2 g(X, V).\]

Thus the manifold is an Einstein manifold.

In view of Theorems 2.1 and 2.2 we have the following:

**Theorem 6.1.** In a 3-dimensional $\alpha$-para Kenmotsu manifold $M$, the following conditions are equivalent:

(a) $P.S = 0$,

(b) the scalar curvature $r = -6\alpha^2$,

(c) the manifold $M$ is of constant curvature,

(d) $M$ is an Einstein manifold.
REFERENCES


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