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SOME NEW SAIGO FRACTIONAL INTEGRAL INEQUALITIES IN QUANTUM CALCULUS

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Abstract. In this article, the Saigo fractional $q-$integral operator is used, to establish new classes of fractional q-integral inequalities using two parameters of deformation q_1 *and* q_2.
Keywords: Saigo fractional integral operators, Saigo fractional $q-$integral operators, $q-$integral inequalities, integral inequalities.

1. Introduction

Integral inequalities involving fractional calculus operators and fractional $q-$integral calculus operators have extensively been studied by several researchers. By applying the fractional integral operators and fractional q-integral operators, many researchers have obtained a lot of fractional integral inequalities and fractional $q-$integral inequalities and applications. For more details, we refer to [1, 2, 3, 6, 7, 8, 9, 11, 12, 13, 14, 15, 19, 20, 21] and the references therein. Dahmani [9] gave new classes of integral inequalities of fractional order using the Riemann-Liouville fractional integrals. In [9, 11] Dahmani *et al* and Brahim *et al.* [4] established some new fractional integral inequalities by using fractional $q-$integral operators. Also in [5] V. L. Chinchane *et al.* obtained some integral inequalities for the Hadamard fractional integral operators. Recently, Purohit *et al.* [15] and Yang [21] investigated some other integral inequalities involving the Saigo fractional integral operators and also established the $q-$extensions of the main results. In the literature, few results were obtained on some fractional integral inequalities using Saigo fractional $q-$integral operators, see [15, 21]. Motivated by the results presented in [9, 10, 11], we prove some new fractional $q-$integral inequalities using Saigo fractional $q-$integral operator of the two parameters of deformation q_1 and q_2.

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2. Factional $q-$calculus

In this section, we give some necessary definitions and mathematical preliminaries of fractional $q-$calculus. More details, one can consult [1, 2, 16, 17, 18].

Definition 2.1. A real valued function $f(t)$, is said to be in the space $\mathbb{C}_{\upsilon}(0,\infty)$, $\upsilon \in \mathbb{R}$, if there exists a real number $p > \upsilon$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in \mathbb{C}(0,\infty)$.

Definition 2.2. A function $f(t); t > 0$ is said to be in the space $\mathbb{C}_{\upsilon}^{n}, n \in R$, if $f^{(n)} \in \mathbb{C}_{\upsilon}$.

For any complex number $\alpha \in \mathbb{C}$, we define

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}, q \neq 1; [n]_q! = [n]_q [n-1]_q ... [2]_q [1]_q, n \in \mathbb{N}, \tag{2.1}$$

and

$$\left([\vartheta]_q\right)_n = [\vartheta]_q [\vartheta+1]_q ... [\vartheta+n-1]_q, \ n \in \mathbb{N}, \ \vartheta \in \mathbb{C}, \tag{2.2}$$

with $[0]_q! = 1$ and the $q-$shifted factorial is defined for as a product of n factors by

$$(\alpha; q)_n = 1, n = 0; \ (\alpha; q)_n = (1-\alpha)(1-\alpha q) ... \left(1-\alpha q^{n-1}\right), \ n \in \mathbb{N}, \tag{2.3}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha}; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \ n > 0, \tag{2.4}$$

where the $q-$gamma function is defined by

$$\Gamma_q(z) = \frac{(q;q)_{\infty}(1-q)^{1-z}}{(q^z;q)_{\infty}}, \ 0 < q < 1. \tag{2.5}$$

We note that

$$\Gamma_q(1+z) = \frac{(1-q)^z \Gamma_q(z)}{1-q}, \tag{2.6}$$

and if $|q| < 1$, the definition (2.3) remains meaningful for $n = \infty$, as a convergent infinite product given by

$$(\alpha; q)_{\infty} = \prod_{i=0}^{\infty} \left(1-\alpha q^i\right). \tag{2.7}$$

Also, the $q-$binomial expansion is given by

$$(\tau-\rho)_{\epsilon}=\tau^{\epsilon}\left(\frac{-\rho}{\tau};q\right)_{\varepsilon}=\tau^{\epsilon}\prod_{i=0}^{\infty}\left(\frac{1-\left(\frac{\rho}{\tau}\right)q^{i}}{1-\left(\frac{\rho}{\tau}\right)q^{\epsilon+i}}\right). \tag{2.8}$$

Let $t_0 \in \mathbb{R}$, then we define a specific time scale

$$T_{t_0}=\left\{t;t=t_0q^{n},\ n\in\mathbb{N}\right\}\cup\{0\},\ 0<q<1, \tag{2.9}$$

The Jackson's $q-$derivative and $q-$integral of a function f defined on T_{t_0} are, respectively, given by

$$D_{q,t}\left[f\left(t\right)\right]=\frac{f\left(t\right)-f\left(qt\right)}{t\left(1-q\right)},\ t\neq 0,\ q\neq 1, \tag{2.10}$$

and

$$\int_0^t f\left(x\right)dx=t\left(1-q\right)\sum_{j=0}^{\infty}q^{j}f\left(tq^{j}\right). \tag{2.11}$$

Definition 2.3. The Riemann-Liouville fractional $q-$integral operator of a function $f(t)$ of order α is given by

$$I_q^{\alpha}\left[f\left(t\right)\right]=\frac{t^{\alpha-1}}{\Gamma_q\left(\alpha\right)}\int_0^t\left(\frac{qx}{t};q\right)_{\alpha-1}f\left(x\right)d_qx,\ \alpha>0,\ 0<q<1, \tag{2.12}$$

where

$$\left(a;q\right)_{\alpha}=\frac{\left(a;q\right)_{\infty}}{\left(aq^{\alpha};q\right)_{\infty}},\ \alpha\in\mathbb{R}. \tag{2.13}$$

Definition 2.4. For $\alpha>0$ and $\eta>0$, the basic analogue of the Kober fractional integral operator is given by

$$I_q^{\alpha,\eta}\left[f\left(t\right)\right]=\frac{t^{-\eta-1}}{\Gamma_q\left(\alpha\right)}\int_0^t\left(\frac{qx}{t};q\right)_{\alpha-1}x^{\eta}f\left(x\right)d_qx,\ 0<q<1. \tag{2.14}$$

Definition 2.5. For $\alpha>0,\beta\in\mathbb{R}$ a basic analogue of the Saigo's fractional integral operator is given for $\left|\frac{x}{t}\right|<1$ by

$$\begin{aligned} I_q^{\alpha,\beta,\eta}\left[f\left(t\right)\right] &= \frac{t^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q\left(\alpha\right)}\int_0^t\left(\frac{qx}{t};q\right)_{\alpha-1} \\ &\quad\times\Pi_{q,\frac{q^{\alpha+1}x}{t}}\left({}_2\Omega_1\left[q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right]\right)f\left(x\right)d_qx, \end{aligned} \tag{2.15}$$

where η is any non-negative integer, and the function ${}_2\Omega_1\left(.\right)$ and the $q-$translation operator occurring in the right-hand side of (2.15) are, respectively, defined by

$$\left({}_2\Omega_1\left[a,b;c;q,t\right]\right)=\sum_{n=0}^{\infty}\frac{\left(a;q\right)_n\left(b,q\right)_n}{\left(c;q\right)_n\left(q,q\right)_n}t^{n},\ \ |q|<1,|t|<1, \tag{2.16}$$

and

$$\Pi_{q,x}\left(f\left(t\right)\right)=\sum_{-\infty}^{\infty}A_nt^n\left(\frac{x}{t};q\right)_n, \tag{2.17}$$

where $(A_n)_{n\in\mathbb{Z}}$ ($\mathbb{Z}=0,\pm1,\pm2,...$) is any bounded sequence of real or complex numbers.

For $f(t)=t^{\varpi}$ in (2.15), we get the known formula

$$I_q^{\alpha,\beta,\eta}\left[t^{\varpi}\right]=\frac{\Gamma_q\left(\varpi+1\right)\Gamma_q\left(\varpi+1-\beta+\eta\right)}{\Gamma_q\left(\varpi+1-\beta\right)\Gamma_q\left(\varpi+1+\alpha+\eta\right)}t^{\varpi-\beta}, \tag{2.18}$$

for all $t>0$, $\min\left(\varpi,\varpi-\beta+\eta\right)>1$, $0<q<1$.

3. Saigo fractional $q-$integral inequalities

In this section, we prove some $q-$integral inequalities concerning the Saigo fractional $q-$integral operators.

Theorem 3.1. *Suppose that f is a positive, continuous and decreasing function on T_{t_0}. Then for all $t>0, 0<q_1,q_2<1$, we have*

$$\begin{aligned} &I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}f^{\delta}\left(t\right)\right]I_{q_2}^{\alpha,\beta,\eta}\left[f^{\theta}\left(t\right)\right]+I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}\left(t\right)\right]I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma}f^{\delta}\left(t\right)\right]\\ \leq\;& I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}\left(t\right)\right]I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma}f^{\theta}\left(t\right)\right]+I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}f^{\theta}\left(t\right)\right]I_{q_2}^{\alpha,\beta,\eta}\left[f^{\delta}\left(t\right)\right]. \end{aligned} \tag{3.1}$$

where $\delta\geq\theta>0,\sigma>0,\alpha>\max\left(0,-\beta\right),\beta<1,\beta-1<\eta<0.$

Proof. Consider

$$\begin{aligned} F_q\left(t,x\right) &= \frac{t^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q\left(\alpha\right)}\left(qx/t;q\right)_{\alpha-1}\\ &\quad\times\Pi_{q,\frac{q^{\alpha+1}x}{t}}\left({}_2\Omega_1\left[q^{\alpha+\beta},q^{-\eta};q^{\alpha};q,q\right]\right). \end{aligned} \tag{3.2}$$

We note that the function $F_q\left(t,x\right)$ remains positive for all values of $x\in\left(0,t\right)$, $t>0$ and under the conditions imposed with Theorem 3.1.

Since the function f is positive, continuous and decreasing on T_{t_0}, then for all $\delta\geq\theta>0,\sigma>0$, $x,y\in\left(0,t\right)$, $t>0$, we can write

$$\left(f^{\delta-\theta}\left(x\right)-f^{\delta-\theta}\left(y\right)\right)\left(y^{\sigma}-x^{\sigma}\right)\geq0, \tag{3.3}$$

which implies that

$$x^{\sigma}f^{\delta-\theta}\left(x\right)+y^{\sigma}f^{\delta-\theta}\left(y\right)\leq y^{\sigma}f^{\delta-\theta}\left(x\right)+x^{\sigma}f^{\delta-\theta}\left(y\right). \tag{3.4}$$

Multiplying both sides of (3.4) by $F_{q_1}(t,x)\, f^{\theta}(x)$ and integrating the resulting inequality with respect to x from 0 to t, we get

$$\begin{aligned} &I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\delta}(t)\right] + y^{\sigma} f^{\delta-\theta}(y)\, I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] \\ \leq\ & y^{\sigma} I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] + f^{\delta-\theta}(y)\, I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\theta}(t)\right]. \end{aligned} \tag{3.5}$$

Next on multiplying both sides of (3.5) by $F_{q_2}(t,y)\, f^{\theta}(y)$ and integrating the resulting inequality with respect to y from 0 to t, we obtain

$$\begin{aligned} &I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\delta}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\delta}(t)\right] \\ \leq\ & I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\theta}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\theta}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right], \end{aligned} \tag{3.6}$$

which implies (3.1). □

Theorem 3.2. *Suppose that f is a positive, continuous and decreasing function on T_{t_0}. Then for all $t > 0, \delta \geq \theta > 0, \sigma > 0$ and $0 < q_1, q_2 < 1$, we have*

$$\begin{aligned} &I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\delta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] \\ \leq\ & I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\theta}(t)\right]. \end{aligned} \tag{3.7}$$

where $\alpha > \max(0,-\beta), \omega > \max(0,-\lambda), \beta, \lambda < 1,\ \eta-\beta, \gamma-\lambda > -1.$

Proof. Multiplying both sides of (3.4) by $G_{q_2}(t,y)\, f^{\theta}(y)$, where

$$\begin{aligned} G_q(t,y) \ =\ & \frac{t^{-\lambda-1} q^{-\gamma(\omega+\lambda)}}{\Gamma_q(\omega)} \left(\frac{qy}{t};q\right)_{\omega-1} \\ & \times \Pi_{q,\frac{q^{\omega+1}y}{t}}\left({}_2\Omega_1\left[q^{\omega+\lambda}, q^{-\gamma}; q^{\omega}; q, q\right]\right), \end{aligned} \tag{3.8}$$

for $y \in (0,t), t > 0$. We can see that the function $G_q(t,y)$ remains positive under the conditions stated with Theorem 3.2. Integrating the resulting inequality obtained with respect to y from 0 to t, we have

$$\begin{aligned} &x^{\sigma} f^{\delta-\theta}(x)\, I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f^{\delta}(t)\right] \\ \leq\ & f^{\delta-\theta}(x)\, I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f^{\theta}(t)\right] + x^{\sigma} I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right]. \end{aligned} \tag{3.9}$$

Now, multiplying both sides of (3.9) by $F_{q_1}(t,x)\, f^{\theta}(x)$ and integrating the resulting inequality with respect to x from 0 to t, we have

$$\begin{aligned} &I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\delta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] \\ \leq\ & I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f^{\theta}(t)\right]. \end{aligned} \tag{3.10}$$

which implies (3.7). □

Remark 3.1. For $\alpha = \omega$, $\beta = \lambda$ and $\eta = \gamma$, Theorem 3.2 immediately reduce to Theorem 3.1.

Theorem 3.3. *Let f and h be two positive and continuous functions on T_{t_0}, such that f is decreasing and h is increasing on T_{t_0}. Then for all $t > 0, 0 < q_1, q_2 < 1$ and $\delta \geq \theta > 0, \sigma > 0$, we have*

$$\begin{aligned} (3.11) \qquad & I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right]+I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] \\ \leq\; & I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right]+I_{q_2}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right], \end{aligned}$$

where $\alpha > \max(0, -\beta), \omega > \max(0, -\lambda), \beta, \lambda < 1, \eta - \beta, \gamma - \lambda > -1.$

Proof. Since f and h are two positive and continuous functions on T_{t_0} such that f is decreasing and h is increasing on T_{t_0}, then we have

$$(3.12) \qquad \left(f^{\delta-\theta}(x)-f^{\delta-\theta}(y)\right)\left(h^{\sigma}(y)-h^{\sigma}(x)\right) \geq 0,$$

for all $\sigma > 0, \delta \geq \theta > 0, x, y \in (0, t), t > 0$,
which implies

$$(3.13) \qquad h^{\sigma}(y) f^{\delta-\theta}(y)+h^{\sigma}(x) f^{\delta-\theta}(x) \leq h^{\sigma}(y) f^{\delta-\theta}(x)+f^{\delta-\theta}(y) h^{\sigma}(x).$$

for $x \in (0, t), t > 0$.

Now, on multiplying both sides of (3.13) by $F_{q_1}(t, x) f^{\theta}(x)$ and integrating the resulting inequality with respect to x from 0 to t, we get

$$\begin{aligned} (3.14) \qquad & h^{\sigma}(y) f^{\delta-\theta}(y) I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right]+I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] \\ \leq\; & h^{\sigma}(y) I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right]+f^{\delta-\theta}(y) I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right]. \end{aligned}$$

Next, on multiplying both sides of (3.14) by $F_{q_2}(t, y) f^{\theta}(y)$ and integrating the resulting inequality with respect to y from 0 to t, we obtain

$$\begin{aligned} (3.15) \qquad & I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right]+I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] \\ \leq\; & I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right]+I_{q_2}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right]. \end{aligned}$$

The proof is done. □

Theorem 3.4. *Let f and h are two positive and continuous functions on T_{t_0}, such that f is decreasing and h is increasing on T_{t_0}. Then for all $t > 0, 0 < q_1, q_2 < 1$ and $\delta \geq \theta > 0, \sigma > 0$, we have*

$$\begin{aligned} (3.16) \qquad & I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right]+I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\delta}(t)\right] \\ \leq\; & I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right]+I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f^{\theta}(t)\right], \end{aligned}$$

where $\alpha > \max(0, -\beta), \omega > \max(0, -\lambda), \beta, \lambda < 1,\ \eta - \beta, \gamma - \lambda > -1.$

Proof. Multiplying both sides of (3.13) by $G_{q_2}(t, y) f^{\theta}(y)$ and integrating with respect to y from 0 to t, we have

$$\begin{aligned} (3.17) \qquad & I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\delta}(t)\right]+h^{\sigma}(x) f^{\delta}(x) I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] \\ \leq\; & f^{\delta-\theta}(x) I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t) f^{\theta}(t)\right]+h^{\sigma}(x) I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right]. \end{aligned}$$

Multiplying both sides of (3.17) by $F_{q_1}(t,x)\, f^{\theta}(x)$ and integrating the resulting inequality with respect to x from 0 to t, we obtain

$$\begin{aligned}(3.18)\qquad & I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\, f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\theta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\, f^{\delta}(t)\right] \\ \leq\ & I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\, f^{\theta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f^{\delta}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[f^{\delta}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\, f^{\theta}(t)\right].\end{aligned}$$

This ends proof of Theorem 3.4. □

Remark 3.2. For $\alpha = \omega$, $\beta = \lambda$ and $\eta = \gamma$, Theorem 3.4 immediately reduces to Theorem 3.3.

Now, by using Saigo fractional $q-$integral, we generate new class of Saigo fractional $q-$integral inequalities involving a family of n positive functions defined on T_{t_0}.

Theorem 3.5. *Suppose that $(f_i)\,,\ _{i=1,\dots,n}$ are n positive and continuous functions on T_{t_0}. Then, for all $t>0, 0<q_1,q_2<1$ and $\sigma>0, \delta\geq\theta_k>0,\ k\in\{1,...,n\}\,,$ the following fractional inequality*

$$\begin{aligned}(3.19)\qquad & I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & + I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\ \leq\ & I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\ & + I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right].\end{aligned}$$

is valid for any $\alpha>\max(0,-\beta)\,,\beta<1,\ \beta-1<\eta<0.$

Proof. Suppose $(f_i)\,,\ _{i=1,\dots,n}$ are n positive continuous functions on T_{t_0}, then we can write

$$(3.20)\qquad \left(f_k^{\delta-\theta_k}(x) - f_k^{\delta-\theta_k}(y)\right)(y^{\sigma}-x^{\sigma}) \geq 0,$$

for any fixed $k\in\{1,...,n\}$ and for any $\delta\geq\theta_k>0, \sigma>0, x,y\in(0,t)\,,t>0.$

From (3.20), we obtain

$$(3.21)\qquad y^{\sigma} f_k^{\delta-\theta_k}(y) + x^{\sigma} f_k^{\delta-\theta_k}(x) \leq y^{\sigma} f_k^{\delta-\theta_k}(x) + x^{\sigma} f_k^{\delta-\theta_k}(y)\,.$$

Now, multiplying both sides of (3.21) by $F_{q_1}(t,x)\prod_{i=1}^{n} f_i^{\theta_i}(x)$ and integrating with respect to x from a to t, we obtain

$$\begin{aligned}(3.22)\qquad & y^{\sigma} f_k^{\delta-\theta_k}(y)\, I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ \leq\ & y^{\sigma} I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] + f_k^{\delta-\theta_k}(y)\, I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].\end{aligned}$$

Next, multiplying both sides of (3.22) by $F_{q_2}(t,y)\prod_{i=1}^{n} f_i^{\theta_i}(y)$ and integrating the resulting inequality with respect to y from a to t, we get

$$\begin{aligned}(3.23)\qquad & I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & + I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\ \leq\ & I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\ & + I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right].\end{aligned}$$

which implies (3.19). □

Theorem 3.6. *Suppose that $(f_i)_{,i=1,\dots,n}$ are n positive and continuous functions on T_{t_0}. Then, for all $t>0, 0<q_1,q_2<1$ and $\sigma>0, \delta\geq\theta_k>0,\ k\in\{1,\dots,n\}$, the following fractional inequality*

$$\begin{aligned}(3.24)\qquad & I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\ & + I_{q_2}^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ \leq\ & I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \\ & + I_{q_2}^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].\end{aligned}$$

Proof. Multiplying both sides of (3.21) by $G_{q_2}(t,y)\prod_{i=1}^{n} f_i^{\theta_i}(y)$ and integrating the resulting inequality with respect to y over $(0,t)$, we obtain

$$I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] + x^{\sigma} f_k^{\delta-\theta_k}(x)\, I_{q_2}^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \tag{3.25}$$
$$\leq f_k^{\delta-\theta_k}(x)\, I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] + x^{\sigma} I_{q_2}^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right].$$

Multiplying both sides of (3.25) by $F_{q_1}(t,x)\prod_{i=1}^{n} f_i^{\theta_i}(y)$ and integrating the resulting inequality with respect to y over $(0,t)$, we obtain

$$I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \tag{3.26}$$
$$+I_{q_2}^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma} f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]$$
$$\leq I_{q_2}^{\omega,\lambda,\gamma}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]$$
$$+I_{q_2}^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_1}^{\alpha,\beta,\eta}\left[t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].$$

The result is proved. □

Remark 3.3. Applying Theorem 3.6 for $\alpha=\omega$, $\beta=\lambda$ and $\eta=\gamma$, we obtain Theorem 3.5 immediately.

Theorem 3.7. *Let $(f_i)_{,i=1,\ldots,n}$ and h be positive continuous functions on T_{t_0}, such that h is increasing and $(f_i)_{,i=1,\ldots,n}$ are decreasing on T_{t_0}. Then for all $t>0$,*

$0 < q_1, q_2 < 1,$ *and* $\sigma > 0, \delta \geq \theta_k > 0, k \in \{1, ..., n\},$ *we have*

$$I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \leq I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]. \tag{3.27}$$

where $\alpha > \max(0, -\beta), \beta < 1, \eta - \beta > -1.$

Proof. Let $x, y \in (0, t), t > 0$, we have

$$h^{\sigma}(y) f_k^{\delta-\theta_k}(y) + h^{\sigma}(x) f_k^{\delta-\theta_k}(x) \leq h^{\sigma}(y) f_k^{\delta-\theta_k}(x) + f_k^{\delta-\theta_k}(y) h^{\sigma}(x), \tag{3.28}$$

for any $\sigma > 0,\ \delta \geq \theta_k > 0, k \in \{1, 2, ..., n\}.$

Multiplying both sides of (3.28) by $F_{q_1}(t, x)\prod_{i=1}^{n} f_i^{\theta_i}(x)$ and integrating with respect to x over $(0, t)$, we obtain

$$h^{\sigma}(y) f_k^{\delta-\theta_k}(y) I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] \leq h^{\sigma}(y) I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] + f_k^{\delta-\theta_k}(y) I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]. \tag{3.29}$$

Now, multiplying both sides of (3.29) by $F_{q_2}(t, y)\prod_{i=1}^{n} f_i^{\theta_i}(y)$ and integrating with respect to y from 0 to t, we have

$$I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t) f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \leq I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] + I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_{q_2}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n} f_i^{\theta_i}(t)\right]. \tag{3.30}$$

This completes proof of Theorem 3.7. □

Theorem 3.8. *Let* $(f_i)\,,\ _{i=1,\ldots,n}$ *and* h *be positive continuous functions on* T_{t_0}, *such that* h *is increasing and* $(f_i)\,,\ _{i=1,\ldots,n}$ *are decreasing on* T_{t_0}. *Then for all* $t>0, 0<q<1$ *and* $\sigma>0, \delta\geq\theta_k>0,\ k\in\{1,...,n\}$, *we have*

$$I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]$$
$$\leq I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right], \tag{3.31}$$

where $\alpha>\max(0,-\beta)\,,\omega>\max(0,-\lambda)\,,\beta,\lambda<1,\ \eta-\beta,\gamma-\lambda>-1.$

Proof. Multiplying both sides of (3.28) by $G_{q_2}(t,y)\prod_{i=1}^{n}f_i^{\theta_i}(y)$ and integrating with respect to y over $(0,t)$, we obtain

$$I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right] + h^{\sigma}(x)\,f_k^{\delta-\theta_k}(x)\,I_{q_2}^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]$$
$$\leq f_k^{\delta-\theta_k}(x)\,I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] + h^{\sigma}(x)\,I_{q_2}^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]. \tag{3.32}$$

Now, multiplying both sides of (3.32) by $F_{q_1}(t,x)\prod_{i=1}^{n}f_i^{\theta_i}(x)$ and integrating with respect to x over $(0,t)$, we have

$$I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\,f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]$$
$$\leq I_{q_2}^{\omega,\lambda,\gamma}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right] + I_{q_2}^{\omega,\lambda,\gamma}\left[f_k^{\delta}(t)\prod_{i\neq k}^{n}f_i^{\theta_i}(t)\right]I_{q_1}^{\alpha,\beta,\eta}\left[h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]. \tag{3.33}$$

Theorem 3.8 is thus proved. □

Remark 3.4. Applying Theorem 3.8 for $\alpha = \omega$, $\beta = \lambda$ and $\eta = \gamma$, we obtain Theorem 3.7.

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