JENSEN-TYPE INEQUALITIES FOR A CLASS OF CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new class of function to obtain some interesting inequalities of Jensen-type. Our results generalize and refine some existing results in literature. Some applications for Schur’s inequality are also included.

Key words: Jensen-type inequalities, Schur’s inequality, integral inequality, convex function.

1. Introduction

The study of Convex functions has been a widely known concept used by several researcher (see [5],[6], [7]). Some previous studies have employed a good number of classes of convex functions to derive new integral inequalities of practical interest (see [23], [24],[27]).

In this paper, $I$ and $J$ will be used to denote intervals of real numbers.

Definition 1.1. [24]. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be convex if

(1.1) $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y),$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 1.2. [8]. A function $f : I \rightarrow \mathbb{R}$ is a Godunova-Levin function or that $f$ belongs to the class $Q(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in (0, 1)$

(1.2) $f(tx + (1 - t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1 - t}.$

This class of convex functions was first introduced in [2] by Godunova and Levin and has gained attention in literature (see [7], [20], [21]).

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Definition 1.3. [2]. Let $s$ be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be $s$-convex (in the second sense), or that $f$ belongs to the class $K^2_s$, if

$$f(tx + (1-t)y) \leq t^sf(x) + (1-t)^sf(y),$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$. This was first introduced by Breckner in [2] and some of its properties given in [9].

Definition 1.4. [7]. We say that $f : I \to \mathbb{R}$ is a $P$–function, or that $f$ belongs to the class $P(I)$, if $f$ is a non-negative function and for all $x, y \in I, t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 1.5. [28]. Let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : I \to \mathbb{R}$ is an $h$-convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I, t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

Definition 1.6. [28]. A function $f : I \to \mathbb{R}$ is said to be super-multiplicative if

$$f(xy) \geq f(x)f(y),$$

for all $x, y \in I$. If the inequality (1.6) is reversed, then $f$ is sub-multiplicative.

Definition 1.7. [28]. If $f, g : I \to \mathbb{R}$ are functions such that for $x, y \in I$ the following inequality holds

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,$$

then, $f$ and $g$ are said to be similarly ordered.

By developing a new concept different from those in [3], [4], [11]-[18], [22], we prove interesting inequalities for a new class of convex functions.

2. Main Results

We now define a new class of convex functions.

Definition 2.1. Let $h : J \to (0, \infty)$, $s \in [0, 1]$, $t \in (0, 1)$ and $\phi$ be a given real-valued function. Then, $f : I \to [0, \infty)$ is a $\phi_{h,s}$ convex function if for all $x, y \in I$,

$$f(t\phi(x) + (1-t)\phi(y)) \leq (\frac{h(t)}{t})^{-s}f(\phi(x)) + (\frac{h(1-t)}{1-t})^{-s}f(\phi(y)).$$
Let us denote by \( Q_s(I) \) and \( SX(\phi_{h-s}, I) \) the class of \( s \)-Godunova-Levin functions and \( \phi_{h-s} \) convex functions respectively, then it is easy to see that

\[
P(I) = Q_0(I) = SX(\phi_{h-0}, I) \subseteq SX(\phi_{h-s_1}, I) \subseteq SX(\phi_{h-s_2}, I) \subseteq SX(\phi_{h-1}, I) = SX(\phi_h),
\]

for \( 0 \leq s_1 \leq s_2 \leq 1 \) whenever \( \phi \) is the identity function.

If the inequality sign in (2.1) is reversed, then \( f \) is \( \phi_{h-s} \) concave, i.e. \( f \in SV(\phi_{h-s}, I) \).

Next, we give some properties and examples of our new class of convex functions.

**Proposition 2.1.** If \( f, g \in SX(\phi_{h-s}, I) \) i.e they are \( \phi_{h-s} \) convex, and \( c \geq 0 \) is a positive real number, then \( f + g \) and \( cf \) are both \( \phi_{h-s} \) convex.

**Proof.** \( f, g \in SX(\phi_{h-s}, I) \) implies

\[
f(\lambda \phi(a) + (1 - \lambda) \phi(b)) = \left( \frac{h(\lambda)}{\lambda} \right)^{-s} f(\phi(a)) + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-s} f(\phi(b)).
\]

\[
g(\lambda \phi(a) + (1 - \lambda) \phi(b)) = \left( \frac{h(\lambda)}{\lambda} \right)^{-s} g(\phi(a)) + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-s} g(\phi(b)).
\]

Clearly,

\[
(f + g)(\lambda \phi(a) + (1 - \lambda) \phi(b)) \leq \left( \frac{h(\lambda)}{\lambda} \right)^{-s} [(f + g)(\phi(a))] + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-s} [(f + g)(\phi(b))]
\]

and,

\[
cf(\lambda \phi(a) + (1 - \lambda) \phi(b)) \leq \left( \frac{h(\lambda)}{\lambda} \right)^{-s} cf(\phi(a)) + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-s} cf(\phi(b)).
\]

We have just shown that addition and scalar multiplication holds for our newly defined class of convex functions so that \( SX(\phi_{h-s}, I) \) is a linear space for \( c \geq 0 \). Observe that, \( SX(\phi_{h-s}, I) \) satisfies the properties (and even more) enjoyed by many known classes of convex functions. We give examples in Proposition 2.2.
Proposition 2.2. Let \( f \) be a non-negative convex function on \( I \). If \( h \) is a non-negative function such that
\[
h(\lambda) \leq \lambda^{1-s}, \quad s \in (0, 1], \lambda \in (0, 1).
\]
Then, \( f \in SX(\phi_{h-s}, I) \).

Proof. Proposition 2.2 implies that all convex functions are examples of our newly defined class of convex function provided that the condition \( h(\lambda) \leq \lambda^{1-s} \) is satisfied. In particular, an example of such \( h(\lambda) \) is \( h(\lambda) = \lambda^k \) for \( k \geq 1 - \frac{1}{s}, s \in (0, 1] \). Now,
\[
f(\lambda \phi(a) + (1 - \lambda) \phi(b)) \leq \lambda f(\phi(a)) + (1 - \lambda) f(\phi(b))
\]
\[(2.6)\]
Hence, \( f \in SX(\phi_{h-s}, I) \).

Similarly, if \( h \) satisfies \( h(\lambda) \geq \lambda^{1-s} \) for any \( \lambda \in (0, 1) \), then any non-negative concave function \( f \) belongs to the class \( SV(\phi_{h-s}, I) \) i.e. is \( \phi_{h-s} \) concave. This new notion of \( \phi(x)_{h-s} \) convex functions generalizes quite a number of classes of convex functions which exist in literature. The immediate implication of this is that \( \phi(x)_{h-s} \) convex functions provide us with many inequalities which generalize and extend the Jensen-type inequalities for the classes of convex functions that already exist in literature. In this paper, we obtain some interesting results for our newly introduced class of convex functions.

Theorem 2.1. Let \( f \in SX(\phi_{h-s}, I) \).

(i) If \( g \) is a linear function, then \( f \circ g \) is \( \phi_{h-s}(I) \) convex.

(ii) If \( f \) is increasing and \( g \) is convex, then \( f \circ g \) is \( \phi_{h-s}(I) \) convex.

Proof. Since \( f \) is \( \phi_{h-s}(I) \) convex then,
\[
f(\lambda \phi(x) + (1 - \lambda) \phi(y)) \leq \left( \frac{\lambda}{h(t)} \right)^s f(\phi(x)) + \left( \frac{(1 - \lambda)}{h(1-\lambda)} \right)^s f(\phi(y)).
\]
\[(2.7)\]
Now,
\[
f \circ g (\lambda \phi(a) + (1 - \lambda) \phi(b)) = f (\lambda g(\phi(a)) + (1 - \lambda) g(\phi(b)))
\]
\[(2.8)\]
\[
\leq \left( \frac{\lambda}{h(t)} \right)^s f \circ g(\phi(a)) + \left( \frac{(1 - \lambda)}{h(1-\lambda)} \right)^s f \circ g(\phi(b)).
\]
and (i) is proved.

We now prove (ii). Since \( g \) is convex then
\[
g(\lambda \phi(a) + (1 - \lambda) \phi(b)) \leq \lambda g(\phi(a)) + (1 - \lambda) g(\phi(b)).
\]
Now,
\[ f \circ g(\lambda \phi(a) + (1 - \lambda)\phi(b)) \leq f(\lambda g(\phi(a)) + (1 - \lambda)g(\phi(b))) \leq \left( \frac{\lambda}{h(\lambda)} \right)^s f(\phi(a)) + \left( \frac{1 - \lambda}{h(1 - \lambda)} \right)^s f(\phi(b)). \] \hspace{1cm} (2.9)

**Remark 2.1.** This result generalizes Theorems 1 and 3 recently obtained by Ardic and Özdemir in [1] with appropriate choices of \( h(\lambda), \phi \) and \( s \). In particular, with \( h(\lambda) = \lambda^2 \) and \( s = 1 \), we obtain Theorem 3 in [1] and with \( h(\lambda) = 1 \), we obtain Theorem 1 in [1].

**Theorem 2.2.** Let \( 0 \in I \) and \( \phi(x) \) be an identity function. Then the following holds.

(i). If \( f \in SX(\phi_h-s, I) \), \( f(0) = 0 \) and \( h \) is super-multiplicative, then the inequality
\[ f(\alpha x + \beta y) \leq \left( \frac{h(\alpha)}{\alpha} \right)^s f(x) + \left( \frac{h(\beta)}{\beta} \right)^s f(y), \] \hspace{1cm} (2.10)
holds for all \( x, y \in I \) and \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \).

(ii). Let \( h \) be non-negative function with \( \left( \frac{h(\alpha)}{\alpha} \right)^s < \frac{1}{2} \) for some \( \alpha \in (0, \frac{1}{2}) \), \( s \in [0, 1] \). If \( f \) is a non-negative function satisfying (2.10) for all \( x, y \in I \), and \( \alpha, \beta > 0 \) with \( \alpha + \beta \leq 1 \), then \( f(0) = 0 \).

(iii). If \( f \in SV(\phi_h-s, I) \), \( f(0) = 0 \) and \( h \) is sub-multiplicative then the inequality
\[ f(\alpha x + \beta y) \geq \left( \frac{h(\alpha)}{\alpha} \right)^s f(x) + \left( \frac{h(\beta)}{\beta} \right)^s f(y), \] \hspace{1cm} (2.11)
holds for all \( x, y \in I \) and \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \).

(iv). Let \( h \) be a non-negative function with \( \left( \frac{h(\alpha)}{\alpha} \right)^s > \frac{1}{2} \) for some \( \alpha \in (0, \frac{1}{2}) \), \( s \in [0, 1] \). If \( f \) is a non-negative function satisfying (2.11) for all \( x, y \in I \) and \( \alpha, \beta > 0 \) with \( \alpha + \beta \leq 1 \), then \( f(0) = 0 \).

**Proof.** We first prove (i). The proof is trivial for \( \alpha, \beta > 0 \), \( \alpha + \beta = 1 \) since (2.10) reduces to (2.1) in definition 2.1. Let \( \alpha, \beta > 0 \), \( \alpha + \beta = r < 1 \) and let \( a \) and \( b \) be
numbers such that \( a = \frac{2}{r} \) and \( b = \frac{2}{r} \), then \( a + b = 1 \) and

\[
f(\alpha x + \beta y) = f(ax + by) 
\leq \left( \frac{h(a)}{a} \right)^{-s} f(\phi(rx)) + \left( \frac{h(b)}{b} \right)^{-s} f(\phi(ry)) 
\leq \left( \frac{h(a)}{a} \right)^{-s} \left[ \left( \frac{h(r)}{r} \right)^{-s} f(x) + \left( \frac{h(1-r)}{1-r} \right)^{-s} f(0) \right] 
+ \left( \frac{h(b)}{b} \right)^{-s} \left[ \left( \frac{h(r)}{r} \right)^{-s} f(y) + \left( \frac{h(1-r)}{1-r} \right)^{-s} f(0) \right] 
\leq \left( \frac{h(ar)}{ar} \right)^{-s} f(x) + \left( \frac{h(br)}{br} \right)^{-s} f(y) 
= \left( \frac{h(\alpha)}{\alpha} \right)^{-s} f(x) + \left( \frac{h(\beta)}{\beta} \right)^{-s} f(y).
\]

We now prove (\(ii\)). Suppose \( f(0) \neq 0 \), then \( f(0) > 0 \). By setting \( x = y = 0 \) in (2.10) we get

\[
(2.12) 
f(0) \leq \left( \frac{h(\alpha)}{\alpha} \right)^{-s} f(0) + \left( \frac{h(\beta)}{\beta} \right)^{-s} f(0).
\]

Again, we set \( \alpha = \beta \), where \( \alpha \in (0, \frac{1}{2}) \) and dividing both sides of the inequality (2.12) by \( f(0) \) to obtain

\[
2 \left( \frac{h(\alpha)}{\alpha} \right)^{-s} \geq 1, \quad \text{for all } \alpha \in (0, \frac{1}{2}).
\]

This contradicts the assumption that \( \left( \frac{h(\alpha)}{\alpha} \right)^{-s} < \frac{1}{2} \), so \( f(0) \) must be 0. Hence the proof of (\(ii\)).

The proofs of (\(iii\)) and (\(iv\)) follow by similar arguments and are thus omitted. \(\Box\)

**Theorem 2.3.** Let \( f \) and \( g \) be similarly ordered on \( I \) for all \( a, b \in I \). If \( f \in SX(\phi_{h_1-s}, I) \) and \( g \in SX(\phi_{h_2-s}, I) \) such that

\[
\left( \frac{h(\lambda)}{\lambda} \right)^{-s} + \left( \frac{h(1-\lambda)}{1-\lambda} \right)^{-s} \leq c^{-s}, \quad \forall \lambda \in (0,1) \text{ with } h(\lambda) = \max_{\lambda \in (0,1)} \{h_1(\lambda), h_2(\lambda)\} \text{ and } c \text{ is a fixed positive number. Then, } f, g \in SX(\phi_{h-s}, I).
\]

**Proof.** Since \( f \) and \( g \) are similarly ordered, then

\[
(f(\phi(a)) - f(\phi(b))(g(\phi(a)) - g(\phi(b))) \geq 0, \quad \forall \phi(a), \phi(b) \in I.
\]

ImPLYING,

\[
f(\phi(a))g(\phi(a)) + f(\phi(b))g(\phi(b)) \geq f(\phi(a))g(\phi(b)) + f(\phi(b))g(\phi(a)).
\]

Now,
Proof. Since \( x \to \phi(x) \) is an increasing function, we have \( g(\lambda \phi(a) + (1 - \lambda) \phi(b)) = f(\lambda \phi(a) + (1 - \lambda) \phi(b)) g((\lambda \phi(a) + (1 - \lambda) \phi(b))) \)

\[
\leq \left[ \left( \frac{h_1(\lambda)}{\lambda} \right)^{-s} f(\phi(a)) + \left( \frac{h_1(1 - \lambda)}{1 - \lambda} \right)^{-s} f(\phi(b)) \right] \\
\times \left[ \left( \frac{h_2(\lambda)}{\lambda} \right)^{-s} g(\phi(a)) + \left( \frac{h_2(1 - \lambda)}{1 - \lambda} \right)^{-s} g(\phi(b)) \right]
\]

\[
\leq \left( \frac{h(\lambda)}{\lambda} \right)^{-2s} f(g(\lambda \phi(a) + (1 - \lambda) \phi(b))) \\
+ \frac{h(1 - \lambda)}{1 - \lambda} \left( \frac{h(\lambda)}{\lambda} \right)^{-s} f(\phi(b)) g(\phi(a)) + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-2s} f(\phi(b)) g(\phi(a)) \\
\leq \left( \frac{h(\lambda)}{\lambda} \right)^{-2s} f(g(\lambda \phi(a) + (1 - \lambda) \phi(b))) \\
+ \frac{h(1 - \lambda)}{1 - \lambda} \left( \frac{h(\lambda)}{\lambda} \right)^{-s} f(\phi(b)) g(\phi(a)) + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-2s} f(\phi(b)) g(\phi(a)) \\
\leq \left( \frac{h(\lambda)}{\lambda} \right)^{-2s} f(g(\lambda \phi(a) + (1 - \lambda) \phi(b))) \\
+ \frac{h(1 - \lambda)}{1 - \lambda} \left( \frac{h(\lambda)}{\lambda} \right)^{-s} f(\phi(b)) g(\phi(a)) + \left( \frac{h(1 - \lambda)}{1 - \lambda} \right)^{-2s} f(\phi(b)) g(\phi(a)).
\]

Hence, \( fg ) \in SX(\phi_{c, h}, I) \). □

Remark 2.2. Theorem 2.4 extends the following result of Varošanec on \( SX(h, I) \) i.e the class of \( h \)-convex functions (see [28]). For arbitrary \( \lambda \in (0, 1) \) and \( c \geq 0 \), define \( \phi(x) = x \) and \( \lambda \mapsto \phi(x) = \chi(h) \) in Theorem 2.3, then Corollary 2.1 follows immediately.

Corollary 2.1. Let \( f \) and \( g \) be similarly ordered functions on \( I \), \( \forall \ x, y \in I \). If \( f \in SX(h_1, I) \) and \( g \in SX(h_2, I) \) such that \( h(\lambda) + h(1 - \lambda) \leq c \) for all \( \lambda \in (0, 1) \), with \( h(\lambda) = \max\{h_1, h_2\} \) and \( c \) a fixed positive number. Then the product \( fg \) belongs to \( SX(ch, I) \).

Theorem 2.4. Let \( h : J \to R \) be a non-negative super-multiplicative and let \( f : I \to R \) be a function such that \( f \in SX(\phi_{c, h}, I) \) where \( \phi(x) = x \). Then, for all \( x_1, x_2, x_3 \in I \) such that \( x_1 < x_2 < x_3 \) and \( x_3 - x_1, x_3 - x_2, x_2 - x_1 \in J \), the following holds:

\[
[(x_3 - x_1)(x_2 - x_1)h(x_3 - x_2)]^{-s} f(x_1) + [(x_3 - x_2)(x_2 - x_1)h(x_3 - x_1)]^{-s} f(x_2) \\
+ [(x_3 - x_1)(x_3 - x_2)h(x_2 - x_1)]^{-s} f(x_3) \geq 0.
\]

(2.13)

Proof. Since \( f \in SX(\phi_{c, h}, I) \), then it is easy to see that

\[
\frac{x_3 - x_2}{x_3 - x_1}, \frac{x_2 - x_1}{x_3 - x_1} \in J \quad \text{and} \quad \frac{x_3 - x_2}{x_3 - x_1} + \frac{x_2 - x_1}{x_3 - x_1} = 1.
\]
Also,
\[(h(x_3 - x_2))^{-s} = \left( h\left(\frac{x_3 - x_2}{x_3 - x_1}(x_3 - x_1)\right) \right)^{-s} \geq \left( h\left(\frac{x_3 - x_2}{x_3 - x_1}h(x_3 - x_1)\right) \right)^{-s}.
\]

Similarly,
\[(h(x_2 - x_1))^{-s} \geq \left( h\left(\frac{x_2 - x_1}{x_3 - x_1}h(x_3 - x_1)\right) \right)^{-s}.
\]

Setting \(\alpha = \frac{x_1 - x_2}{x_3 - x_1}, \beta = \frac{x_2 - x_1}{x_3 - x_1}\) in (2.1), we have \(x_2 = \alpha x + \beta y\) and

\[
(2.14) \quad f(x_2) \leq \left( h\left(\frac{x_2 - x_1}{x_3 - x_1}\right) \right)^{-s} f(x_1) + \left( h\left(\frac{x_2 - x_1}{x_3 - x_1}\right) \right)^{-s} f(x_3)
\]

\[
(2.15) \quad \leq \left( h\left(\frac{x_3 - x_2}{x_3 - x_1}\right) \right)^{-s} f(x_1) \left( \frac{h(x_2 - x_1)}{h(x_3 - x_1)} \right)^{-s} f(x_3).
\]

Multiplying both sides of (2.15) by \(\left(\frac{x_3 - x_2}{x_3 - x_1}\right)^{-s} (h(x_3 - x_1))^{-s}\) and rearranging, gives (2.13). This result has several implications for the Schurs inequality (interested reader can see [19] and references therein). \(\square\)

**Theorem 2.5.** Let \(w_1, w_2, \cdots, w_n\) be positive real numbers and let \((m, M)\) be an interval in \(I\). If \(h : (0, \infty) \to R\) is a non-negative super-multiplicative function and \(f \in SX(\phi_n, s, I)\) where \(\phi\) is an identity function, then for all \(x_1, x_2, \cdots, x_n \in (m, M)\) the following inequality holds,

\[
(2.16) \quad \sum_{i=1}^{n} p_i f(x_i) \leq f(m) \sum_{i=1}^{n} p_i \left( h\left(\frac{M-x_i}{M-m}\right) \right)^{-s} + f(M) \sum_{i=1}^{n} p_i \left( h\left(\frac{x_i-m}{M-m}\right) \right)^{-s},
\]

where

\[p_i = \left( h\left(\frac{W_i}{W_n}\right) \right)^{-s}, \quad W_n = \sum_{i=1}^{n} w_i.
\]

**Proof.** Setting \(x_1 = m, x_2 = x_i, x_3 = M\) in (2.14) we obtain

\[
(2.17) \quad f(x_1) \leq \left( h\left(\frac{M-x_1}{M-m}\right) \right)^{-s} f(m) + \left( h\left(\frac{x_i-m}{M-m}\right) \right)^{-s} f(M).
\]

Multiplying both sides of (2.17) by \(p_i\) and adding both sides of the resulting inequality for \((i = 1, \cdots, n)\), we obtain (2.16). \(\square\)
Theorem 2.6. Let \( f \in SX(\phi_{h-s}, I) \) and \( \sum_{i=1}^{n} t_i = T_n = 1, \ t_i \in (0,1), \ i = 1,2, \cdots , n \). Then
\[
f \left( \sum_{i=1}^{n} t_i \phi(x_i) \right) \leq \left[ \frac{h(t_i)}{t_i} \right]^{-s} \sum_{i=1}^{n} f(\phi(x_i)).
\]

Proof. Observe that
\[
f \left( \sum_{i=1}^{n} t_i \phi(x_i) \right) = f \left( T_{n-1} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} \phi(x_i) + t_n \phi(x_n) \right)
\leq \left[ \frac{h(T_{n-1})}{T_{n-1}} \right]^{-s} f \left( \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \phi(x_i) + t_{n-1} \phi(x_{n-1}) \right)
+ \left[ \frac{h(t_n)}{t_n} \right]^{-s} f(\phi(x_n))
\leq \left[ \frac{h(T_{n-1})}{T_{n-1}} \right]^{-s} \left[ \frac{h(T_{n-2})}{T_{n-2}} \right]^{-s} \left[ \frac{T_{n-1}}{T_{n-2}} \right]^{-s} f \left( \sum_{i=1}^{n-2} \frac{t_i}{T_{n-2}} \phi(x_i) \right)
+ \left[ \frac{h(t_n)}{t_n} \right]^{-s} f(\phi(x_n))
\leq \left[ \frac{h(t)}{t_i} \right]^{-s} \sum_{i=1}^{n} f(\phi(x_i)).
\]

\[\square\]

Remark 2.3. Set \( h(t) = 1 \) and \( s = 1 \) in Theorem 2.6, then we obtain what is known in literature as the discrete version of the Jensen’s inequality (see [10]).

Theorem 2.7. Let \( t_1, \cdots , t_n \) be positive real numbers \((n \geq 2)\). If \( h \) is a non-negative super-multiplicative function and if \( f \in SX(\phi_{h-s}, I) \) where \( \phi \) is an identity function, for \( x_1, \cdots , x_n \in I \), then
\[
f \left( \frac{1}{T_n} \sum_{i=1}^{n} t_i x_i \right) \leq \sum_{i=1}^{n} \left[ \frac{h(t_i)}{t_i} \right]^{-s} f(x_i),
\]
where \( T_n = \sum_{i=1}^{n} t_i \).

If \( h \) is sub-multiplicative and \( f \in SV(\phi_{h-s}, I) \) then the inequality (2.18) is reversed.

Proof. Let us suppose that \( f \in SX(\phi_{h-s}, I) \). If \( n=2 \), then the inequality (2.18) is equivalent to the definition of \( \phi_{h-s} \) convex functions with \( \lambda = \frac{t_1}{t_2} \) and \( 1 - \lambda = \frac{t_2}{t_1} \). Suppose the inequality holds for \( n = 1 \), then for \( n \)-tuples \((x_1, \cdots , x_n)\) and \((t_1, \cdots , t_n)\), we have,
\[ f \left( \frac{1}{T_n} \sum_{i=1}^{n} t_i x_i \right) = f \left( \frac{T_n}{T_n} x_n + \sum_{i=1}^{n-1} \frac{t_i}{T_n} x_i \right) \]

\[ = f \left( \frac{T_n}{T_n} x_n + \frac{T_{n-1}}{T_n} \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} x_i \right) \]

\[ \leq \left[ h \left( \frac{t_n}{T_n} \right) \right]^{-s} f(x_n) + \left[ h \left( \frac{T_{n-1}}{T_n} \right) \right]^{-s} f \left( \sum_{i=1}^{n-1} \frac{t_i}{T_{n-1}} x_i \right) \]

\[ \leq \left[ h \left( \frac{t_n}{T_n} \right) \right]^{-s} f(x_n) + \left[ h \left( \frac{T_{n-1}}{T_n} \right) \right]^{-s} \sum_{i=1}^{n-1} \left[ h \left( \frac{t_i}{T_{n-1}} \right) \right]^{-s} f(x_i) \]

\[ \leq \left[ h \left( \frac{t_n}{T_n} \right) \right]^{-s} f(x_n) + \sum_{i=1}^{n-1} \left[ h \left( \frac{t_i}{T_n} \right) \right]^{-s} f(x_i) = \sum_{i=1}^{n} \left[ h \left( \frac{t_i}{T_n} \right) \right]^{-s} f(x_i). \]

**Remark 2.4.** If we choose \( h(\lambda) = 1 \) and \( s = 1 \), then Theorem 2.11 becomes the well known discrete version of the classical Jensen inequality for convex functions (see [10], [26]). If we choose \( h(\lambda) = 1 \), then our result is the Jensen-type inequality for s-convex functions (see [26]). If \( h(\lambda) = \lambda^2 \), then we obtain the very recent result known as the Jensen-type inequality for s-Godunova-Levin functions (see [25]).

3. Open Problems

It is known in literature that several direct and converse results can be obtained for Jensen’s inequality by using convex functions. Is it possible to obtain similar results for functions belonging to the class \( \text{SX}(\phi_{h-s}, I) \)?

Also, recent developments has shown that time scaled inequalities are more general and widely applicable, can new inequalities on time scales be obtained for functions belonging to the class \( \text{SX}(\phi_{h-s}, I) \)?

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