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ON A SEMI-SYMMETRIC NON-METRIC CONNECTION IN AN ALMOST KENMOTSU MANIFOLD WITH NULLITY DISTRIBUTION

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Abstract. We consider a semi-symmetric non-metric connection in an almost Kenmotsu manifold with its characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distribution. We first obtain the expressions of the curvature tensor and Ricci tensor with respect to the semi-symmetric non-metric connection in an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. Then we characterize an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution.
Keywords: semi-symmetric non-metric connection, almost Kenmotsu manifold, curvature tensor, Ricci tensor, nullity distribution.

1. Introduction

K. Yano [41] initiated systematic study of semi-symmetric connection in a Riemannian manifold. In 1924, Friedmann and Schouten [20] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\bar{\nabla}$ satisfies $T(X, Y) = \eta(Y)X - \eta(X)Y$, where η is a 1-form and ξ is a vector field defined by $\eta(X) = g(X, \xi)$, for all vector fields $X \in \chi(M)$, where $\chi(M)$ is the set off all differentiable vector fields on M.
In 1932, Hayden [22] introduced the idea of semi-symmetric metric connection on a Riemannian manifold (M, g). A semisymmetric connection $\bar{\nabla}$ is said to be a semi-symmetric metric connection if $\bar{\nabla}g = 0$. The study of a semi-symmetric metric connection was further developed by Amur and Pujar [1], T. Imai [23], U. C. De [16], Z. I. Szab̈o [30], T. Q. Binh [9], M. Pravanović and N. Pušić [27], N. Pušić [29], Lj. S. Velimirović et al [36, 37], Ajit Barman [10, 11, 12], Y. Liang [25] and many other geometers.
After a long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying $\bar{\nabla}g \neq 0$ was initiated by M. Prvanović [28] with the name pseudo-metric semi-symmetric connection. A semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection if $\bar{\nabla}g \neq 0$. Semi-symmetric non-metric connections have been studied

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by several authors such as N. S. Agashe and M. R. Chafle [2], O. C. Andonie [3], U. C. De et al [17, 18], D. Smaranda [31], B. Barua and S. Mukhopadhay [6], R. N. Singh et al [32, 33, 34] and many others.

The notion of k-nullity distribution was introduced by Gray [21] and Tanno [35] in the study of Riemannian manifolds (M, g), which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}, \tag{1.1}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Blair, Koufogiorgos and Papantoniou [4] introduced a generalized notion of the k-nullity distribution, named the (k,μ)-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu) \;=\; & \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ & +\mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{1.2}$$

where $h = \frac{1}{2}\pounds_{\xi}\phi$ and £ denotes the Lie derivative.

In [13], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' \;=\; & \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ & +\mu[g(Y,Z)h'X - g(X,Z)h'Y]\} \end{aligned} \tag{1.3}$$

where $h' = h \circ \phi$.

A Riemannian manifold is said to be Ricci semi-symmetric if $R(X,Y) \cdot S = 0$, where $R(X,Y)$ is considered as a field of linear operators, acting on S and S denotes the Ricci tensor of type $(0,2)$.
The present paper is organized in the following way. In section 2, we give a brief account on an almost Kenmotsu manifold, while section 3 contains some results on an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution. In section 4, we define a semi-symmetric non-metric connection. In section 5, we obtain the expressions of the curvature tensor and Ricci tensor with respect to the semi-symmetric non-metric connection. Section 6 is devoted to characterize Ricci semi-symmetric almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution. Finally, we prove that if an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution is locally ϕ-Ricci symmetric with respect to the semi-symmetric non-metric connection, then the manifold is Ricci symmetric under certain condition.

2. Almost Kenmotsu manifold

A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ, ξ, η)-structure or an almost contact structure, if it admits a $(1,1)$-type tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([5],[7]),

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1) easily.
If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y of $T_p M^{2n+1}$, then M is said to have an almost contact structure (ϕ, ξ, η, g). The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \Phi Y)$ for any X, Y of $T_p M^{2n+1}$. The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [5]. Recently in ([13],[14],[15],[26]), almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y. It is well known [24] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution. Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations :

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$\begin{aligned} R(X, Y)\xi &= \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) \\ &\quad +(\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \end{aligned} \tag{2.5}$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that ([13], [40])

$$h = 0 \Leftrightarrow h' = 0, \ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.6}$$

Almost Kenmotsu manifold have been studied by several authors such as Dileo and Pastore ([13], [14], [15]), Wang and X. Liu ([39], [40]) and many others.

3. Almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution

This section is devoted to study almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue λ. Then $h'X = \lambda X$ implies $h'^2X = \lambda^2 X$. Therefore $\lambda^2 X = (k+1)\phi^2 X$, since in a $(k,\mu)'$-almost Kenmotsu manifold $h'^2 = (k+1)\phi^2$. Hence $\lambda^2 X = -(k+1)X$ which implies $\lambda^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigenvalue λ and $-\lambda$ of h', respectively. Before presenting our main theorems we recall some results:

Lemma 3.1. *(Prop. 4.1 and Prop. 4.3 of [13]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0,\lambda,-\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi]\oplus[\lambda]'$ and $[\xi]\oplus[-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as following:*

(a) $K(X,\xi) = k-2\lambda$ *if* $X \in [\lambda]'$ *and*
$K(X,\xi) = k+2\lambda$ *if* $X \in [-\lambda]'$,

(b) $K(X,Y) = k-2\lambda$ *if* $X,Y \in [\lambda]'$;
$K(X,Y) = k+2\lambda$ *if* $X,Y \in [-\lambda]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) M^{2n+1} *has constant negative scalar curvature* $r = 2n(k-2n)$.

Lemma 3.2. *(Lemma 3 of [38]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta\otimes\xi - 2nh'. \tag{3.1}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Lemma 3.3. *(Proposition 4.2 of [13]) Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k,-2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:*

$$\begin{aligned}
R(X_\lambda,Y_\lambda)Z_{-\lambda} &= 0,\\
R(X_{-\lambda},Y_{-\lambda})Z_\lambda &= 0,\\
R(X_\lambda,Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda,Z_\lambda)Y_{-\lambda},\\
R(X_\lambda,Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda},Z_{-\lambda})X_\lambda,\\
R(X_\lambda,Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda,Z_\lambda)X_\lambda - g(X_\lambda,Z_\lambda)Y_\lambda],\\
R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}].
\end{aligned}$$

Lemma 3.4. *(Lemma 4.1 of [13]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to the $(k, -2)'$-nullity distribution. Then, for any $X, Y \in \chi(M^{2n+1})$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X).$$

From (1.3) we have,

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (3.2)

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{3.3}$$

Contracting Y in (3.2) we have

$$S(X, \xi) = 2nk\eta(X). \tag{3.4}$$

Moreover in an almost Kenmotsu manifold with $(k, \mu)'$-nullity distribution,

$$\nabla_X \xi = X - \eta(X)\xi + h'X \tag{3.5}$$

and

$$(\nabla_X \eta)Y = g(Y,X) - \eta(X)\eta(Y) + g(Y, h'X) \tag{3.6}$$

4. Semi-symmetric non-metric connection

This section deals with a type of semi-symmetric non-metric connection on an almost Kenmotsu manifold.
A relation between semi-symmetric non-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ is given by [2],

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X,Y)\xi. \tag{4.1}$$

Using (4.1), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y. \tag{4.2}$$

Hence a relation satisfying (4.2) is called a semi-symmetric connection.
Further using (4.1), we have

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) &= \nabla_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z)\\ &= 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \neq 0.\end{aligned} \tag{4.3}$$

$\bar{\nabla}$ defined by (4.1) satisfying (4.2) and (4.3) is a type of semi-symmetric non-metric connection.

5. Curvature tensor of an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution with respect to the semi-symmetric non-metric connection

In this section we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature of M^{2n+1} with respect to the semi-symmetric non-metric connection defined by (4.1).
Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ given by,

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, \tag{5.1}$$

where X, Y, Z, are the vector fields on M^{2n+1}.
Using (4.1) in (5.1) we get,

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - (\nabla_X\eta)(Y)Z + (\nabla_Y\eta)(X)Z \\ &\quad -2\eta(Y)g(X,Z)\xi + 2\eta(X)g(Y,Z)\xi + g(Y,Z)\nabla_X\xi \\ &\quad -\eta(Y)g(X,Z)\xi.\end{aligned} \tag{5.2}$$

Using (3.5) and (3.6) we get from (5.2),

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)X - g(X,Z)Y \\ &\quad +g(Y,Z)h'X - g(X,Z)h'Y + \eta(X)g(Y,Z)\xi \\ &\quad -\eta(Y)g(X,Z)\xi.\end{aligned} \tag{5.3}$$

From (5.3) it follows that

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$$

and

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0,$$

the first Bianchi identity with respect to the semi-symmetric non-metric connection $\bar{\nabla}$. Putting $X = \xi$ in (5.3) and by the help of (3.3), we get

$$\begin{aligned}\bar{R}(\xi,Y)Z &= (k+2)g(Y,Z)\xi - (k+1)\eta(Z)Y + \eta(Z)h'Y - \\ &\quad \eta(Y)\eta(Z)\xi.\end{aligned} \tag{5.4}$$

Again putting $Z = \xi$ in (5.4) we get,

$$\bar{R}(\xi,Y)\xi = (k+1)\eta(Y)\xi - (k+1)Y + h'Y. \tag{5.5}$$

Let $\{e_1, e_2, ...e_{2n+1}\}$ be a local orthonormal basis of fields in M. Then by putting $X = Z = e_i$ in (5.3) and taking summation over i, $1 \leq i \leq 2n+1$

$$\bar{S}(Y, Z) = S(Y, Z) + (4n+1)g(Y, Z) - g(h'Y, Z) - \eta(Y)\eta(Z), \tag{5.6}$$

where S and $\bar{S}$ are the Ricci tensor of M with respect to ∇ and $\bar{\nabla}$ respectively. From (5.6) it is clear that,

$$\bar{S}(Y, Z) = \bar{S}(Z, Y). \tag{5.7}$$

Let $\bar{r}$ and r denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇ respectively. Again let $\{e_1, e_2, ...e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $Y = Z = \xi$ in (5.6) and taking summation over i, $1 \leq i \leq 2n+1$,

$$\bar{r} = r + 2n(4n+3). \tag{5.8}$$

Therefore we have the following:

Theorem 5.1. *For an almost Kenmotsu manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$*

- *(i) The curvature tensor is given by* (5.3)*,*
- *(ii) The Ricci tensor is given by* (5.6)*,*
- *(iii) The scalar curvature is given by* $\bar{r} = r + 2n(4n+3)$
- *(iv)* $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$*,*
- *(v)* $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0$*,*
- *(vi) The Ricci tensor* $\bar{S}$ *is symmetric.*

Now using (3.1), we have

$$S(Y, Z) = -2ng(Y, Z) + 2n(k+1)\eta(Y)\eta(Z) - 2ng(h'Y, Z). \tag{5.9}$$

6. Ricci semi-symmetric almost Kenmotsu manifold such that ξ belongs to$(k, \mu)'$-nullity distribution with respect to the semi-symmetric non-metric connection

In this section we characterize Ricci semi-symmetric almost Kenmotsu manifolds with respect to the semi-symmetric non-metric connection.
Now we prove the following:

Theorem 6.1. *Let M^{2n+1} be an almost Kenmotsu manifold with characteristic vector ξ belonging to $(k, \mu)'$-nullity distribution and $h' \neq 0$. If the manifold is Ricci semi-symmetric with respect to the semi-symmetric non-metric connection then the manifold M^{2n+1} is an Einstein manifold with respect to the semi-symmetric non-metric connection or locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold with constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Suppose $(\bar{R}(X,Y)\cdot\bar{S})(Z,W)=0$ for all vector fields X, Y, Z, W on M^{2n+1}. Then

$$\bar{S}(\bar{R}(X,Y)Z,W)+\bar{S}(Z,\bar{R}(X,Y)W)=0. \tag{6.1}$$

Putting $X=Z=\xi$ in (6.1) yields

$$\bar{S}(\bar{R}(\xi,Y)\xi,W)+\bar{S}(\xi,\bar{R}(\xi,Y)W)=0. \tag{6.2}$$

Using (5.4) and (5.5) in (6.2) we get,

$$\begin{aligned}&(k+1)\eta(Y)\bar{S}(\xi,W)-(k+1)\bar{S}(Y,W)+\bar{S}(h'Y,W)\\&+(k+2)g(Y,W)\bar{S}(\xi,\xi)-(k+1)\eta(W)\bar{S}(\xi,Y)-\eta)(Y)\eta(W)\bar{S}(\xi,\xi)=0,\end{aligned} \tag{6.3}$$

for any vector fields Y, W on M^{2n+1}.
By the help of (5.6) and (5.9) we obtain

$$\begin{aligned}&-(k+1)\bar{S}(Y,W)+\bar{S}(h'Y,W)+2n(k+2)^2g(Y,W)\\&-2n(k+2)\eta(Y)\eta(W)=0.\end{aligned} \tag{6.4}$$

Putting $Y=h'Y$ in (6.4) yields

$$-(k+1)\bar{S}(h'Y,W)+\bar{S}(h'^2Y,W)+2n(k+2)^2g(h'Y,W)=0, \tag{6.5}$$

Again substituting $h'^2=(k+1)\phi^2$ in (6.5), yields

$$-(k+1)\bar{S}(h'Y,W)+\bar{S}((k+1)\phi^2Y,W)+2n(k+2)^2g(h'Y,W)=0. \tag{6.6}$$

Using (2.1) we get from (5.4),

$$\begin{aligned}&(k+1)\bar{S}(h'Y,W)-(k+1)\bar{S}(Y,W)+(k+1)\eta(Y)\bar{S}(\xi,W)\\&+2n(k+2)^2g(h'Y,W),\end{aligned} \tag{6.7}$$

By the help of (5.6) it follows that

$$\begin{aligned}&-(k+1)\bar{S}(h'Y,W)-(k+1)\bar{S}(Y,W)+2n(k+1)(k+2)\eta(Y)\eta(W)\\&+2n(k+2)^2g(h'Y,W)=0,\end{aligned} \tag{6.8}$$

Multiplying (6.5) by $(k+1)$ and then adding with (6.8) we obtain

$$\begin{aligned}&(k+2)[(k+1)\bar{S}(Y,W)-2n(k+1)(k+2)g(Y,W)\\&-2n(k+2)g(h'Y,W)]=0,\end{aligned} \tag{6.9}$$

Let $Y,Z\in[\lambda]'$, then (6.9) takes the form

$$(k+2)[\bar{S}(Y,W)-2n(k+2)\left(1-\frac{\lambda}{(k+1)}\right)g(Y,W)]=0, \tag{6.10}$$

since $(k+1) \neq 0$. In [13], Dileo and Pastore prove that in almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution if $k=-1$, then $h'=0$ and the manifold M^{2n+1} is locally a wrapped product of an almost Kähler manifold and an open interval. Thus $k+1=0$, contradicts our hypothesis $h' \neq 0$. Then the following two cases occur:
Case 1:

$$\bar{S}(Y,W) = 2n(k+2)\left(1 - \frac{\lambda}{(k+1)}\right) g(Y,W),$$

which implies that the manifold is an Einstein manifold with respect to the semi-symmetric non-metric connection.
Case 2: $(k+2)=0$, that is, $k=-2$.
Without loss of generality we may choose $\lambda = 1$. Then we have from Lemma (3.3),

$$\text{R}(X_\lambda, Y\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \tag{6.11}$$

$$\text{R}(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0, \tag{6.12}$$

for any vector field $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also noticing $\mu = -2$ it follows from lemma (3.1) that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from lemma (3.1), we see that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$;$K(X,Y) = 0$ for any $X,Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [13] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $H^{2n+1}(-4) \times \mathbb{R}^n$. This completes the proof. □

7. Locally ϕ-Ricci symmetric almost Kenmotsu manifolds with respect to the semi-symmetric non-metric connection

E. Boeckx, P. Buecken and L. Vanhecke [8] introduced the notion of ϕ symmetry. Recently ϕ-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [19]. An almost Kenmotsu manifold M^{2n+1} is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)Y) = 0,$$

for all vector fields $X, Y \in M^{2n+1}$ and $S(X,Y) = g(QX,Y)$. If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

Now from (5.9) we can write,

$$\bar{Q}Y = QY + (4n+1)Y - h'Y - \eta(Y)\xi. \tag{7.1}$$

Again we have,

$$(\bar{\nabla}_X \bar{Q})Y = \bar{\nabla}_X \bar{Q}Y - \bar{Q}(\bar{\nabla}_X Y). \tag{7.2}$$

Using (6.1) in (6.2) we get,

$$(\bar{\nabla}_X \bar{Q})Y = (\bar{\nabla}_X Q)Y - (\bar{\nabla}_X h')Y - ((\bar{\nabla}_X \eta)Y)\xi, \tag{7.3}$$

Now we prove the following:

Theorem 7.1. *If an almost Kenmotsu manifold with ξ belonging to $(k,\mu)'$-nullity distribution is locally ϕ-Ricci symmetric with respect to the semi-symmetric non-metric connection, then the manifold is Ricci symmetric.*

Proof. We suppose that the manifold under consideration is locally ϕ-Ricci symmetric. Then $\phi^2(\bar{\nabla}_X \bar{Q})Y = 0$. Now using (2.1) we get from (7.3),

$$\phi^2(\bar{\nabla}_X Q)Y - \phi^2(\bar{\nabla}_X h')Y = 0. \tag{7.4}$$

Hence from Lemma (3.2) we can easily obtain,

$$\phi^2(\bar{\nabla}_X h')Y = -\eta(Y)h'X - \eta(Y)h'^2 X. \tag{7.5}$$

Setting $h'^2 = (k+1)\phi^2$ in (6.5) we get,

$$\phi^2(\bar{\nabla}_X h')Y = -\eta(Y)h'X - (k+1)\eta(Y)X - (k+1)\eta(X)\eta(Y)\xi. \tag{7.6}$$

Using (2.1) and (7.6) in (7.4) we have,

$$\begin{aligned} &-(\bar{\nabla}_X Q)Y + \eta(\bar{\nabla}_X Q)Y + \eta(Y)h'X - (k+1)\eta(Y)X \\ &+(k+1)\eta(X)\eta(Y)\xi. \end{aligned} \tag{7.7}$$

From (4.1) it follows that,

$$\bar{\nabla}_X QY = \nabla_X QY - \eta(X)QY + g(X, QY)\xi. \tag{7.8}$$

and

$$Q(\bar{\nabla}_X Y) = Q(\nabla_X Y) - \eta(X)QY - 2nkg(X, QY)\xi. \tag{7.9}$$

Also from (6.10) we obtain,

$$\eta((\bar{\nabla}_X Q)Y) = \eta((\nabla_X Q)Y) + (1+2nk)g(X, QY). \tag{7.10}$$

Using (7.2), (7.8), (7.9) and (7.10) we get from (7.7),

$$\begin{aligned}&-(\nabla_X Q)Y - (1+2nk)g(X,QY)\xi + (\nabla_X S)(Y,\xi)\xi\\&+(1+2nk)g(X,QY)\xi + \eta(Y)h'X - (k+1)\eta(Y)X\\&+\eta(X)\eta(Y)\xi = 0.\end{aligned} \tag{7.11}$$

Taking inner product with Z of (7.11) and considering X, Y, Z orthogonal to ξ, we get

$$(\nabla_X S)(Y, W) = 0,$$

which implies the manifold is Ricci symmetric with respect to the Levi-Civita connection provided X, Y, Z are orthogonal to ξ. This completes the proof. □

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