FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 2 (2016), 503–512

SOME CHARACTERIZATIONS OF CURVES IN GALILEAN 3-SPACE $\mathbb{G}_3$

Sezgin Büyükkütük, İlim Kişi, Vishnu Narayan Mishra & Günay Öztürk

Abstract. In this paper, we consider a unit speed curve in Galilean 3-space $\mathbb{G}_3$ as a curve whose position vector can be written as a linear combination of its Serret-Frenet vectors. We show that there is no T-constant curve in Galilean 3-space $\mathbb{G}_3$, and we obtain some results of N-constant type of curves in Galilean 3-space $\mathbb{G}_3$.
Keywords Galilean 3-space, curve, Serret-Frenet vectors, Galilean geometry

1. Introduction

The basic concepts of Euclidean plane geometry are points and straight lines, and the best known theorem is Pisagor theorem. In nature, however, every surface is not a plane and every line is not a straight line like in Euclidean geometry. Ömer Hayyam and Tusi were the first scholars to study Euclid's postulate. However, in the 19th century, non-Euclidean geometries were set forth by C.F. Gauss, N.I. Lobachevsky and J. Bolyai, with the discovery of hyperbolic geometry, which accepts a new postulate (infinite number of parallels can be drawn to a line from a point outside the given line) instead of the parallel postulate. G.F.B. Riemann laid the foundations of a new geometry called the elliptic geometry afterwards. Those geometries were generalized by F. Klein, and Euclid presented the existence of the nine geometries including the hyperbolic and elliptic ones [18]. Galilean geometry is a non-Euclidean geometry and associated with Galileo's principle of relativity. This principle can be explained briefly as "in all inertial frames, all laws of physics are the same."

(Except for the Euclidean geometry in some cases), Galilean geometry is the easiest of all Klein geometries, and it is relevant to the theory of relativity of Galileo and Einstein. For a comprehensive study of Galilean geometry, one can have a look at the studies by Yaglom [19] and Röschel [17]. Furthermore, many works related to Galilean geometry have been done by several authors. In [13], the authors studied helices in the Galilean space $\mathbb{G}_3$ and in [5] the authors studied some curves in the

Received November 05, 2015; Accepted February 06, 2016
2010 *Mathematics Subject Classification.* Primary 53A35; Secondary 53B30

Galilean space. Similar studies about Galilean geometry may be found in [1, 2, 14]. In [13], the authors obtained characterizations of helix for a curve with respect to the Frenet frame in a 3-dimensional Galilean space $\mathbb{G}_3$. In [9], the authors give a short and understandable exposition on differential operators over modules and rings as a path to the generalized differential geometry. Also, in [16], the authors gave projective flatness of a new class of metrics.

For a regular curve $\alpha(x)$, the position vector α can be decomposed into its tangential and normal components at each point:

$$\alpha = \alpha^T + \alpha^N \tag{1.1}$$

A curve α in $\mathbb{E}^n$ is said to be of *constant ratio* if the ratio $\|\alpha^T\| : \|\alpha^N\|$ is constant on $\alpha(I)$ where $\|\alpha^T\|$ and $\|\alpha^N\|$ denote the length of α^T and α^N, respectively [6].

Moreover, a curve in $\mathbb{E}^n$ is called T-*constant* (resp. N-*constant*) if the tangential component α^T (resp. the normal component α^N) of its position vector α is of constant length [7, 8]. Recently, the authors give the necessary and sufficient conditions for curves in Euclidean and Minkowski spaces to become T-constant or N-constant [3, 4, 10, 12].

In the present study, we consider a unit speed curve in Galilean space $\mathbb{G}_3$ whose position vector satisfies the parametric equation

$$\alpha(x) = m_0(x)T(x) + m_1(x)N(x) + m_2(x)B(x), \tag{1.2}$$

for some differentiable functions, $m_i(x)$, $0 \le i \le 2$. We characterize the twisted curves in terms of their curvature functions $m_i(x)$ and give the necessary and sufficient conditions for these curves to become T-constant or N-constant.

2. Basic Notations

Galilean space is a three dimensional complex projective space $\mathbf{P}_3$, in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points, $I_1, I_2 \in f$ (the absolute points) [11].

We shall take, as a real model of the space $\mathbb{G}_3$, a real projective space $\mathbf{P}_3$, with the absolute $\{w, f\}$ consisting of a real plane $w \subset \mathbb{G}_3$, and a real line $f \subset w$, on which an elliptic involution ε has been defined.

Let ε be in homogeneous coordinates

$$w...x_0 = 0, \quad f...x_0 = x_1 = 0$$

$$\varepsilon : (0:0:x_2:x_3) \to (0:0:x_3:-x_2)\,.$$

In $\mathbb{G}_3$ there are four classes of lines:

a) (proper) nonisotropic lines - they do not meet the absolute line f.

b) (proper) isotropic lines - lines that do not belong to the plane w but meet the absolute line f.

c) unproper nonisotropic lines - all lines of w but f.

d) the absolute line f.

Planes $x = const$ are Euclidean and so is the plane w. Other planes are isotropic [13].

The scalar product and cross product of two vectors $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ in $\mathbb{G}_3$ are respectively defined by:

$$\langle v_1, v_2 \rangle = \begin{cases} x_1 x_2, & \text{if } x_1 \neq 0 \vee x_2 \neq 0 \\ y_1 y_2 + z_1 z_2 & \text{if } x_1 = 0 \wedge x_2 = 0, \end{cases} \tag{2.1}$$

$$v_1 \times v_2 = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}, & \text{if } x_1 \neq 0 \vee x_2 \neq 0 \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}, & \text{if } x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.2}$$

Also the length of the vector $v = (x, y, z)$ is given by

$$\|v\| = \begin{cases} |x|\,, & \text{if } x \neq 0 \\ \sqrt{y^2 + z^2}, & \text{if } x = 0 \end{cases} \tag{2.3}$$

[15].

A curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ parameterized by the Galilean invariant parameter (the arc-length on α) is given in the coordinate form

$$\alpha(x) = (x, y(x), z(x))\,, \tag{2.4}$$

the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by

$$\kappa(x) = \sqrt{y''^2(x) + z''^2(x)} \tag{2.5}$$

and

$$\tau(x) = \det \frac{(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)}. \tag{2.6}$$

The associated moving trihedron is given by

$$\begin{aligned} T(x) &= \alpha'(x) = (1, y'(x), z'(x)) \\ N(x) &= \frac{\alpha''(x)}{\kappa(x)} = \frac{1}{\kappa(x)} (0, y''(x), z''(x)) \\ B(x) &= (T \times N)(x) = \frac{1}{\kappa(x)} (0, -z''(x), y''(x))\,. \end{aligned} \tag{2.7}$$

The vectors T, N and B are called the vectors of the tangent, principal normal and the binormal line, respectively. For their derivatives the following Frenet's formulas hold [15]

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= -\tau N. \end{aligned} \tag{2.8}$$

3. Characterization of Curves in $\mathbb{G}_3$

In the present section, we characterize the unit speed curves given with the invariant parameter x in $\mathbb{G}_3$ in terms of their curvatures. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed regular curve with curvatures $\kappa(x) \geq 0$ and $\tau(x)$. The position vector of the curve (also defined by α) satisfies the vectorial equation (1.2), for some differential functions $m_i(x)$, $0 \leq i \leq 2$. Differentiating (1.2) with respect to the arc length parameter x and using the Serret-Frenet equations (2.8), we obtain

$$\begin{aligned} \alpha'(x) &= m_0'(x)T(x) \\ &\quad +(m_1'(x) + \kappa(x)m_0(x) - \tau(x)m_2(x))N(x) \\ &\quad +(m_2'(x) + \tau(x)m_1(x))B(x). \end{aligned}$$

It follows that

$$\begin{aligned} m_0'(x) &= 1 \\ m_1'(x) + \kappa(x)m_0(x) - \tau(x)m_2(x) &= 0 \\ m_2'(x) + \tau(x)m_1(x) &= 0. \end{aligned} \tag{3.1}$$

The general solution of the equation system (3.1) is obtained in Theorem 3.1 in [1].

Theorem 3.1. *[1] The position vector $\alpha(x)$ of an arbitrary curve with curvature $\kappa(x)$ and torsion $\tau(x)$ in the Galilean space $\mathbb{G}_3$ is computed from the natural representation form*

$$\alpha(x) = (x, \int \left[\int \kappa(x) \cos\left[\tau(x)dx\right] dx \right] dx, \int \left[\int \kappa(x) \sin\left[\tau(x)dx\right] dx \right] dx).$$

Definition 3.1. [1] Let α be a regular curve in Galilean space $\mathbb{G}_3$ with the Frenet frame $\{T, N, B\}$ and κ be its curvature. If $\kappa = 0$, then α is called a straight line with respect to the Frenet frame.

Similar to [6], we give the following definition;

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve given with the invariant parameter x. Then the position vector α can be decomposed into its tangential and normal components at each point as in (1.1). If the ratio $\left\|\alpha^T\right\| : \left\|\alpha^N\right\|$ is constant on $\alpha(I)$ then α is said to be of constant ratio.

Clearly, for a constant ratio curve in Galilean space $\mathbb{G}_3$, we have

$$\frac{m_0^2}{m_1^2+m_2^2}=c_1 \tag{3.2}$$

for some constant c_1.

Theorem 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. Then α is of constant ratio if and only if*

$$\left(\frac{\kappa'+\kappa^3 c_1(x+c)}{c_1\kappa^2\tau}\right)'=\frac{\tau}{c_1\kappa}$$

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve given with the invariant parameter x. Then from (3.2), the curvature functions satisfy

$$m_1(x)m_1'(x)+m_2(x)m_2'(x)=\frac{x+c}{c_1} \tag{3.3}$$

Also, by the use of the equations (3.1) with (3.3) , we have

$$m_1=-\frac{1}{c_1\kappa}$$

Then

$$\begin{aligned} m_2 &= \frac{\kappa'+\kappa^3 c_1(x+c)}{c_1\kappa^2\tau} \\ m_2' &= \frac{\tau}{c_1\kappa} \end{aligned}$$

So, we get the result. □

Example 3.1. Let us consider the following curve

$$\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$$

$$\alpha(x)=(x,\frac{x}{2}\left[\sin(\ln x)-\cos(\ln x)\right],\frac{-x}{2}\left[\cos(\ln x)+\sin(\ln x)\right]). \tag{3.4}$$

Differentiating (3.4), we have

$$\alpha'(x)=(1,\sin(\ln x),-\cos(\ln x)). \tag{3.5}$$

Galilean inner product follows that $<\alpha',\alpha'>=1$. So the curve is parameterized by the arc length and the tangent vector is (3.5). In order to calculate the first curvature let us express

$$T'=(0,\frac{1}{x}\cos(\ln x),\frac{1}{x}\sin(\ln x))$$

Taking the norm of both sides, we have $\kappa(x) = \frac{1}{x}$. Thereafter, we arrive at

$$N = (0, \cos(\ln x), \sin(\ln x))$$

and binormal vector

$$B = (0, -\sin(\ln x), \cos(\ln x))$$

By the use of the parametric equation (2.2), we have the curvature functions:

$$\begin{aligned} m_0 &= x \\ m_1 &= -\frac{x}{2} \\ m_2 &= \frac{x}{2} \end{aligned}$$

So, from (3.2), we get,

$$\frac{m_0^2}{m_1^2 + m_2^2} = 2$$

As a result of this, α is of constant ratio and the ratio is equal to 2.

3.1. T-constant Curves in $\mathbb{G}_3$

Similar to [7, 8], we give the following definition.

Definition 3.3. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. If $\left\|\alpha^T\right\|$ is constant, then α is called a *T-constant curve*. Further, a T-constant curve α is called first kind if $\left\|\alpha^T\right\| = 0$, otherwise second kind.

As a consequence of (1.2) with (3.1) we get the following result.

Proposition 3.1. *There is no T-constant unit speed curve in Galilean space $\mathbb{G}_3$.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. Then $\left\|\alpha^T\right\| = m_0$ is zero or a nonzero constant. However, $m_0 = x + c$ from the equations (3.1). This is a contradiction. Thus, we get the result. □

3.2. N-constant Curves in $\mathbb{G}_3$

Similar to [7, 8], we give the following definition.

Definition 3.4. Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. If $\left\|\alpha^N\right\|$ is constant then α is called a N-constant curve. For a N-constant curve α, either $\left\|\alpha^N\right\| = 0$ or $\left\|\alpha^N\right\| = \mu$ for some non-zero smooth function μ. Further, a N-constant curve α is called first kind if $\left\|\alpha^N\right\| = 0$, otherwise second kind.

Note that, for a N-constant curve α in $\mathbb{G}_3$, we can write;

$$\left\|\alpha^N(x)\right\|^2 = m_1^2(x) + m_2^2(x) = c_1, \tag{3.6}$$

where c_1 is a real constant.

As a consequence of (1.2), (3.1) and (3.6), we get the following result.

Lemma 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. Then α is a N-constant curve if and only if*

$$\begin{aligned} m_0'(x) &= 1 \\ m_1'(x) + \kappa(x)m_0(x) - \tau(x)m_2(x) &= 0 \\ m_2'(x) + \tau(x)m_1(x) &= 0 \\ m_1(x)m_1'(x) + m_2(x)m_2'(x) &= 0 \end{aligned}$$

hold, where $m_i(x)$, $0 \leq i \leq 2$ are differentiable functions.

Proposition 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. Then α is a $N-$constant curve of first kind if and only if, α is a straight line in $\mathbb{G}_3$.*

Proof. Suppose that α is N-constant curve of first kind in $\mathbb{G}_3$. Then $m_1^2 + m_2^2 = 0$, which means $m_1 = m_2 = 0$. Using the second equation of (3.1), we get $\kappa = 0$. From definition, α is a straight line in $\mathbb{G}_3$. □

Example 3.2. [1] The position vector $\alpha(x)$ of a straight line in Galilean space $\mathbb{G}_3$ is given by

$$\alpha(x) = (x, c_1x + c_3, c_2x + c_4)$$

where c_i, $(i = 1, 2, 3, 4)$ are arbitrary constants.

Proposition 3.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ be a unit speed curve in $\mathbb{G}_3$. If α is a $N-$constant curve of second kind, then the position vector of the curve has the parametrization of the form*

$$\begin{aligned} \alpha(x) &= (x + c)T(x) \\ &\pm \left[\sec^2(u(x)) \sqrt{\frac{c_1}{\tan(u(x))^2 + 1}} \right. \\ &\left. + \tan(u(x)) \left(\frac{c_1}{\tan(u(x))^2 + 1} \right)^{\frac{-1}{2}} \left(\frac{-c_1 u(x) \sec^2(u(x))^2}{\left(\tan(u(x))^2 + 1\right)^2} \right) \right] N(x) \\ &\mp \tan(u(x)) \sqrt{\frac{c_1}{\tan(u(x))^2 + 1}} B(x), \end{aligned}$$

where $u(x) = \int \tau(x)dx + c_2$ and c_2 is integral constant.

Proof. Using the equations (3.1) with (3.6), we get $m_0(x) = (x + c)$ and the differential equation

$$\left(m_2'(x)\right)^2 + \left(\tau(x)\right)^2 \left(m_2(x)\right)^2 - c_1 \left(\tau(x)\right)^2 = 0$$

where $c_1 \neq 0$ is a real constant. Then the solution of this differential equation is

$$m_2(x) = \mp \tan u(x) \sqrt{\frac{c_1}{\tan u(x)^2 + 1}}. \tag{3.7}$$

Substituting the equation (3.7) in the third equation of (3.1), one can find

$$\begin{aligned} m_1(x) &= \pm \sec^2(u(x))\sqrt{\frac{c_1}{\tan(u(x))^2+1}} \\ &+ \tan(u(x))\left(\frac{c_1}{\tan(u(x))^2+1}\right)^{\frac{-1}{2}}\left(\frac{-c_1 u(x)\sec^2(u(x))^2}{\left(\tan(u(x))^2+1\right)^2}\right), \end{aligned}$$

which completes the proof of Proposition 3.3. □

Acknowledgements

The authors would like to express their deep gratitude to the anonymous referee(s) and the editor for their valuable suggestions and constructive comments, which resulted in the subsequent improvement of this research article. The authors declare that there is no conflict of interests regarding the publication of this research article. The third author Vishnu Narayan Mishra acknowledges that this project was supported by the Cumulative Professional Development Allowance (CPDA), S.V.N.I.T., Surat (Gujarat), India. All the authors contributed equally and significantly in writing this manuscript. All the authors drafted the manuscript, read and approved the final version of manuscript in FU Math Inform. The authors are thankful to all the editors and referees of this esteemed journal, i.e. Facta Universitatis, Series: Mathematics and Informatics (FU Math Inform).

REFERENCES

1. T. A. Ahmad: *Position Vectors of Curves in Galilean Space* $\mathbb{G}_3$. Matematicki Vesnik. **3**(2012),200-210.
2. A. Z. Azak, M. Akyiğit and S. Ersoy: *Involute-Evolute Curves in Galilean Space* $\mathbb{G}_3$.Scientia Magna,**6**(2010), 75-80.
3. S. Büyükkütük and G. Öztürk: *Constant Ratio Curves According to Bishop Frame in Euclidean* $3-space$ $\mathbb{E}^3$. Gen. Math. Notes, **28** (2015), 81–91.
4. S. Büyükkütük and G. Öztürk: *Constant Ratio Curves According to Parallel Transport Frame in Euclidean* $4-space$ $\mathbb{E}^4$. NTMSCI,**3** (2015), 171–178.
5. H. Balgetir Öztekin and S. Tatlipinar: *On Some Curves in Galilean Plane and 3-Dimensional Galilean Space*.Journal of Dynamical Systems and Geometric Theories,**10**(2012),189-196.
6. B. Y. Chen: *Constant ratio Hypersurfaces*. Soochow J. Math. **28** (2001), 353–362.
7. B. Y. Chen: *Geometry of Warped Products as Riemannian Submanifolds and Related Problems*, Soochow J. Math. **28** (2002), 125–156.
8. B. Y. Chen: *Geometry of position functions of Riemannian submanifolds in pseudo-Euclidean space*, Journal of Geo. **74** (2002), 61–77.
9. Deepmala, L.N. Mishra: *Differential operators over modules and rings as a path to the generalized differential geometry,* Facta Universitatis (NIŠ) Ser. Math. Inform., **30** (2015), 753–764.

10. S. Gürpinar, K. Arslan, G. Öztürk: *A Characterization of Constant-ratio Curves in Euclidean 3-space* $\mathbb{E}^3$,. Acta Universitatis Apulensis, **44** (2015), 39–51.
11. I. Kamenarovic: *Existence Theorems for Ruled Surfaces In The Galilean Space* $\mathbb{G}_3$. Rad Hazu Math, **10** (1991), 183–196.
12. İ. Kışı and G.Öztürk: *Constant Ratio Curves According to Bishop Frame in Minkowski* $3-$*space* $\mathbb{E}_1^3$. Facta Universitatis Ser. Math. Inform.,**30** (2015), 527–538.
13. A. O. Öğrenmiş, M. Ergüt and M. Bektaş: *On the Helices in the Galilean Space* $\mathbb{G}_3$. Iranian Journal of Science & Technology, Transaction A. **31** (2007), 177–181.
14. A. O. Öğrenmiş, H. Öztekin and M. Ergüt: *Bertrand Curves in Galilean Space and Their Characterizations*. Kragujevac J. Math. **32** (2009), 139–147.
15. B. J. Pavkovic and I. Kamenarovic: *The equiform differential geometry of curves in the Galilean space* $\mathbb{G}_3$. Glasnik Matematicki. **42** (1987), 449–457.
16. L. Piscoran and V. N. Mishra: *Projectively flatness of a new class of metrics.* Georgian Math. Journal, accepted on Dec. 14, 2015, in press.
17. O. Röschel: *"Die Geometrie Des Galileischen Raumes"*.Forschungszentrum Graz Research Centre, Austria, 1986.
18. S. Sertöz: *Matematiğin Aydınlık Dünyası*.Tübitak Popüler Bilim Kitapları, 2009.
19. I. M. Yaglom: *A Simple Non-Euclidean Geometry and Its Physical Basis.* Springer-Verlag Inc. New York, 1979.

Sezgin Büyükkütük
Department of Mathematics
Kocaeli University
41380 Kocaeli, Turkey
sezgin.buyukkutuk@kocaeli.edu.tr

İlim Kişi
Department of Mathematics
Kocaeli University
41380 Kocaeli, Turkey
ilim.ayvaz@kocaeli.edu.tr

Vishnu Narayan Mishra (Corresponding author)
Applied Mathematics and Humanities Department,
Sardar Vallabhbhai National Institute of Technology,
Ichchhanath Mahadev Dumas Road, Surat 395 007, Gujarat, India
L. 1627 Awadh Puri Colony Phase III, Beniganj,
Opposite - Industrial Training Institute (I.T.I.),
Ayodhya Main Road, Faizabad 224 001, Uttar Pradesh, India

vishnu_narayanmishra@yahoo.co.in, vishnunarayanmishra@gmail.com

Günay Öztürk
Department of Mathematics
Kocaeli University
41380 Kocaeli, Turkey
ogunay@kocaeli.edu.tr