SOLVABILITY AND STABILITY FOR NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL SYSTEMS OF HIGH FRACTIONAL ORDER

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Abstract. In this paper, using Riemann-Liouville integral and Caputo derivative, we study an \( n \)-dimensional coupled system of nonlinear fractional integro-differential equations of high arbitrary order. The contraction mapping principle and Schaefer fixed point theorem are applied to prove the existence and the uniqueness of solutions in Banach spaces. Furthermore, we derive the Ulam-Hyers and the generalized Ulam-Hyers stabilities of solutions. Some illustrative examples are also presented.

Key words: Riemann-Liouville integral, Caputo derivative, fractional integro-differential equations, Banach space.

1. Introduction and Preliminaries

The fractional calculus has attracted interest of researchers in several areas of mathematics, physics, chemistry and engineering sciences. For more details, we refer the reader to the monographs by Hilfer [19], Lakshmikantham [26], Podlubny [33] and the papers of [3, 5, 18, 20, 24, 30, 31, 32]. In addition, many authors paid much attention to the existence and uniqueness of solutions for some fractional differential equations and systems, see for example [1, 2, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 21, 27, 28, 35, 38] and the references therein. Moreover, the study of Ulam type stability problems has grown to be one of the most important subjects in the mathematical analysis area. For instance, J. Wang et al. [17] obtained some results on Ulam type stability in the case of impulsive ordinary differential equations. Then, J. Wang, L. Lv and Y. Zhou [37] presented some interesting results on Ulam stability for fractional differential equations with Caputo derivative. Recently, R. Ibrahim [22, 23] obtained some results about Ulam-Hyers stability of solutions for fractional order dynamic equations. Very recently, in [12, 36], the Ulam-Hyers stability of solutions has been done for that problems of singular fractional differential equations.

Motivated by the above works, in this paper, we discuss the existence, the uniqueness and the Ulam-Hyers stability as well as the generalized Ulam-Hyers stability for the following fractional problem:
where $c (1.5)$

For Lemma 1.1.

Finally, the functions $g_k \in C$ for $k = 1, \ldots, n$, and $i = 1, \ldots, m$, will be specified later.

Now, let us recall some lemmas that we need to prove our main results [25, 29, 34].

**Lemma 1.1.** For $l \in \mathbb{N}^* \setminus \{1\}$, and $l - 1 < \alpha < l$, the general solution of the fractional differential equation $D^\alpha x(t) = 0$, is given by

$$
(1.4) \quad x(t) = \sum_{j=0}^{l-1} c_j t^j,
$$

where $c_j \in \mathbb{R}$, $j = 0, \ldots, l - 1$.

**Lemma 1.2.** Given $l \in \mathbb{N}^* \setminus \{1\}$, and $l - 1 < \alpha < l$. Then

$$
(1.5) \quad J^\alpha D^\alpha x(t) = x(t) + \sum_{j=0}^{l-1} c_j t^j,
$$

where $c_j \in \mathbb{R}$, $j = 0, 1, \ldots, l - 1$. 

\[
\left\{ \begin{array}{l}
D^\alpha x_1 (t) = \frac{m}{i=1} g_i^1 (t, x_1 (t), \ldots, x_n (t)) + \frac{m}{i=1} J^\delta y_i^1 (t, x_1 (t), \ldots, x_n (t)), \\
D^\alpha x_2 (t) = \frac{m}{i=1} g_i^2 (t, x_1 (t), \ldots, x_n (t)) + \frac{m}{i=1} J^\delta y_i^2 (t, x_1 (t), \ldots, x_n (t)), \\
\vspace{1em}
\vdots \\
D^\alpha x_n (t) = \frac{m}{i=1} g_i^n (t, x_1 (t), \ldots, x_n (t)) + \frac{m}{i=1} J^\delta y_i^n (t, x_1 (t), \ldots, x_n (t)), \\
\end{array} \right.
\]
Lemma 1.3. Let $q > p > 0$, $g \in L^1([a,b])$. Then,

$$D^p J^q g(t) = J^{q-p} g(t), \quad t \in [a,b].$$

Lemma 1.4. Let $E$ be a Banach space and $T : E \to E$ be a completely continuous operator. If the set $V := \{ x \in E : x = \nu Tx, 0 < \nu < 1 \}$ is bounded, then $T$ has a fixed point in $E$.

Also, the following auxiliary result is important to give the integral solution of (1.1):

Lemma 1.5. Let $l \in \mathbb{N}^* \setminus \{1\}$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, and a family $(G^k_i) \in C(J, \mathbb{R})$ for $i = 1, \ldots, m$, $k = 1, \ldots, n$, and consider the problem

$$
\begin{align*}
D^a x_1(t) &= \sum_{i=1}^{m} G^1_i(t) + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_1-1}}{\Gamma(a_1)} G^1_i(s) ds, \; t \in J,
D^a x_2(t) &= \sum_{i=1}^{m} G^2_i(t) + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_2-1}}{\Gamma(a_2)} G^2_i(s) ds, \; t \in J,
& \quad \vdots
D^a x_n(t) &= \sum_{i=1}^{m} G^n_i(t) + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_n-1}}{\Gamma(a_n)} G^n_i(s) ds, \; t \in J,
\end{align*}
$$

(1.6)

where $\delta_k \in \mathbb{R}^+$, $l-1 < \alpha_k < l$,

with the conditions:

$$
\begin{align*}
\sum_{k=1}^{n} |x_k(0)| &= \sum_{k=1}^{n} |x_k'(0)| = \cdots = \sum_{k=1}^{n} |x_k^{(l-2)}(0)| = 0,
\end{align*}

(1.7)

$$
\begin{align*}
x_k^{(l-1)}(1) = 0, \; k = 1, 2, \ldots, n.
\end{align*}

Then, the solution $(x_1,x_2,\ldots,x_n)$ of (1.6) – (1.7) is given by

$$
\begin{align*}
x_k(t) &= \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_k-1}}{\Gamma(a_k)} G^k_i(s) ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} G^k_i(s) ds \\
& \quad - \frac{t^{\alpha_k-1}}{(l-1)! \Gamma(\alpha_k+\delta_k-\alpha_k)} \sum_{i=1}^{m} \int_0^1 (1-s)^{\alpha_k-\delta_k} G^k_i(s) ds \\
& \quad - \frac{t^{\alpha_k-1}}{(l-1)! \Gamma(\alpha_k+\delta_k+\alpha_k)} \sum_{i=1}^{m} \int_0^1 (1-s)^{\alpha_k+\delta_k-\alpha_k} G^k_i(s) ds, \; k = 1, 2, \ldots, n.
\end{align*}
$$

(1.8)

Proof. Using Lemma 1.1 and Lemma 1.2 and (1.6), we get

$$
\begin{align*}
x_k(t) &= \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_k-1}}{\Gamma(a_k)} G^k_i(s) ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{a_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} G^k_i(s) ds - \sum_{j=0}^{l-1} c_j^k t^j,
\end{align*}
$$

(1.9)
where \( c_j^k \in \mathbb{R}, j = 0, 1, 2, \ldots, l - 1 \) and \( l - 1 < \alpha_k < l, \ k = 1, 2, \ldots, n \).

For all \( k = 1, 2, \ldots, n, j = 0, 1, \ldots, l - 2, \) we can write
\[
x_k^{(j)}(0) = -j! c_j^k.
\]

Using Lemma 1.3 and applying the boundary conditions (1.7), we obtain:
\[
(1.10) \quad c_j^k = 0, \quad j = 0, 1, \ldots, l - 2,
\]
and
\[
(1.11) \quad c_{l-1}^k = \sum_{i=1}^{m} \int_{0}^{1} \frac{(1-s)^{\alpha_k-l}}{(l-1)!\Gamma(\alpha_k-l+1)} \ G_i^k(s) \ ds
+ \sum_{i=1}^{m} \int_{0}^{1} \frac{(1-s)^{\alpha_k+\delta_k-l}}{(l-1)!\Gamma(\alpha_k+\delta_k-l+1)} \ G_i^k(s) \ ds.
\]

Substituting the values of \( c_j^k \) into (1.9), we end the proof of Lemma 1.5.

Now, to study the problem (1.1), we introduce the Banach space
\[
S := \{(x_1, x_2, \ldots, x_n) : x_k \in C([0, 1], \mathbb{R}), k = 1, 2, \ldots, n\},
\]
equipped with the norm
\[
(1.12) \quad \|(x_1, x_2, \ldots, x_n)\|_S = \max(\|x_1\|_\infty, \|x_2\|_\infty, \ldots, \|x_n\|_\infty).
\]

2. Main Results

In this section, we will formulate and prove sufficient conditions for the existence and uniqueness of solutions for the system (1.1). Then, we study its Ulam-Hyers stability as well as its generalized Ulam-Hyers stability.

We begin by listing the following hypotheses:

\((H_1)\) : There exist nonnegative constants \( \left( \omega_{i, j, k, i=1,2,\ldots,m} \right) \), \( j = 1, 2, \ldots, n \), such that for all \( t \in [0, 1] \) and all \( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \), we have
\[
|g_i^k(t, x_1, x_2, \ldots, x_n) - g_i^k(t, y_1, y_2, \ldots, y_n)| \leq \sum_{j=1}^{n} (\omega_{i, j, k}) |x_j - y_j|, \ k = 1, \ldots, n, i = 1, \ldots, m.
\]

\((H_2)\) : The functions \( g_i^k : [0, 1] \times \mathbb{R}^n \to \mathbb{R} \) are continuous for each \( i = 1, 2, \ldots, m \) and \( k = 1, 2, \ldots, n, m, n \in \mathbb{N}^* \).

\((H_3)\) : There exist nonnegative constants \( \left( L_{i, k}^k \right)_{i=1, \ldots, m} \), such that for each \( t \in J \) and all \( (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), we have
\[
|g_i^k(t, x_1, x_2, \ldots, x_n)| \leq L_{i, k}^k, \ i = 1, \ldots, m, \ k = 1, 2, \ldots, n.
\]
We also need to define the following quantities:

\[ \Delta_k = \frac{1}{\Gamma(\alpha_k + 1)} + \frac{1}{\Gamma(\alpha_k + \delta_k + 1)}, \quad k = 1, 2, ..., n, \]  

(2.1)

\[ \Sigma_k = \sum_{i=1}^{m} \left( (\omega_i^k)_1 + (\omega_i^k)_2 + ... + (\omega_i^k)_n \right), \quad k = 1, 2, ..., n, \]  

(2.2)

and

\[ \theta_k = \frac{1}{(l-1)!\Gamma(\alpha_k + 2 - l)} + \frac{1}{(l-1)!\Gamma(\alpha_k + \delta_k + 2 - l)}, \quad k = 1, 2, ..., n. \]  

(2.3)

2.1. Existence and Uniqueness of Solutions

The first result is based on Banach contraction principle. We have:

**Theorem 2.1.** Assume that \((H_1)\) holds and

\[ \max \left( \Sigma_1 \Delta_1 + \theta_1, \Sigma_2 \Delta_2 + \theta_2, ..., \Sigma_n \Delta_n + \theta_n \right) < 1. \]  

(2.4)

Then, the system (1.1) has a unique solution on \(J\).

**Proof.** We define the following nonlinear operator \(T : S \to S\), by

\[ T(x_1, ..., x_n)(t) := (T_1(x_1, ..., x_n)(t), T_2(x_1, ..., x_n)(t), ..., T_n(x_1, ..., x_n)(t)), \]  

(2.5)

such that,

\[ T_k(x_1, ..., x_n)(t) = \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} G_k^i(s) \, ds + \sum_{i=1}^{m} \int_0^t \frac{(t-s)^{\alpha_k+\delta_k-1}}{\Gamma(\alpha_k+\delta_k)} G_k^i(s) \, ds \]

\[ - \frac{t^{l-1}}{(l-1)!\Gamma(\alpha_k - l + 1)} \sum_{i=1}^{m} \int_0^1 (1 - s)^{\alpha_k-1} G_k^i(s) \, ds \]

\[ - \frac{t^{l-1}}{(l-1)!\Gamma(\alpha_k + \delta_k - l + 1)} \sum_{i=1}^{m} \int_0^1 (1 - s)^{\alpha_k+\delta_k-1} G_k^i(s) \, ds, \]

(2.6)

where,

\[ G_k^i(s) = g_i^k(s, x_1(s), x_2(s), ..., x_n(s)), \quad k = 1, 2, ..., n, i = 1, 2, ..., m. \]  

(2.7)

We show that \(T\) is a contractive operator.

Let \((x_1, ..., x_n), (y_1, ..., y_n) \in S\). Then, for each \(k = 1, 2, ..., n\) and \(t \in J\), we have:
\[ T_k(x_1, \ldots, x_n)(t) - T_k(y_1, \ldots, y_n)(t) \]

\[ \leq \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)} \sup_{s \in J} \sum_{i=1}^{m} \left| g_k^i(s, x_1(s), \ldots, x_n(s)) - g_k^i(s, y_1(s), \ldots, y_n(s)) \right| \]

\[ + \frac{t^{\alpha_k + \delta_k}}{\Gamma(\alpha_k + \delta_k + 1)} \sup_{s \in J} \sum_{i=1}^{m} \left| g_k^i(s, x_1(s), \ldots, x_n(s)) - g_k^i(s, y_1(s), \ldots, y_n(s)) \right| \]

\[ + \frac{t^{l-1}}{(l-1)! \Gamma(\alpha_k + \delta_k + 2 - l)} \sup_{s \in J} \sum_{i=1}^{m} \left| g_k^i(s, x_1(s), \ldots, x_n(s)) - g_k^i(s, y_1(s), \ldots, y_n(s)) \right| . \]

By the hypothesis \((H_1)\), for each \(k = 1, 2, \ldots, n\), we can write

\[ \| T_k(x_1, \ldots, x_n) - T_k(y_1, \ldots, y_n) \| \]

\[ \leq \left( \frac{1}{\Gamma(\alpha_k + 1)} + \frac{1}{\Gamma(\alpha_k + \delta_k + 1)} + \frac{1}{(l-1)! \Gamma(\alpha_k + \delta_k - l + 2)} \right) \times \max \left( \| x_1 - y_1 \|, \ldots, \| x_n - y_n \| \right) \sum_{i=1}^{m} \sum_{j=1}^{n} (\omega_k^i)_j . \]

Then, we obtain

\[ \| T_k(x_1, \ldots, x_n) - T_k(y_1, \ldots, y_n) \| \leq \max_{1 \leq k \leq n} (\Delta_k \Sigma_k + \theta_k) \left( \| x_1 - y_1, \ldots, x_n - y_n \| \right) S . \]

\[ (2.10) \]

Therefore,

\[ \| T(x_1, \ldots, x_n) - T(y_1, \ldots, y_n) \|_S \leq \]

\[ \max (\Delta_1 \Sigma_1 + \theta_1, \ldots, \Delta_n \Sigma_n + \theta_n) \left( \| x_1 - y_1, \ldots, x_n - y_n \| \right) S . \]

\[ (2.11) \]

Using (2.4), we can see that \(T\) is a contractive operator. Consequently, by Banach fixed point theorem, \(T\) has a fixed point which is a solution of the system (1.1). Theorem 2.1 is thus proved. \( \square \)

Our second main result is based on Lemma 1.4. We have:

**Theorem 2.2.** Assume that the hypotheses \((H_2)\) and \((H_3)\) are satisfied. Then, the system (1.1) has at least a solution on \(J\).

**Proof.** (i) : We prove that \(T\) is completely continuous.
Let us take for \( \mu > 0 \), the set \( B_\mu := \{(x_1, ..., x_n) \in S, \| (x_1, ..., x_n) \|_S \leq \mu \} \). Then for any \((x_1, ..., x_n) \in B_\mu\) and for each \( k = 1, 2, ..., n \), we can write

\[
(2.12) \quad \| T_k (x_1, ..., x_n) \| \leq \left( \frac{t^{\alpha_k}}{\Gamma (\alpha_k + 1)} + \frac{t^{\alpha_k + \delta_k}}{\Gamma (\alpha_k + \delta_k + 1)} \right) \sup_{s \in J} \sum_{i=1}^{m} | g^k_i (s, x_1(s), ..., x_n(s)) | \\
+ \theta_k \sup_{s \in J} \sum_{i=1}^{m} | g^k_i (s, x_1(s), ..., x_n(s)) | \\
\leq (\Delta_k + \theta_k) \sum_{i=1}^{m} L^k_i.
\]

Thus,

\[
(2.13) \quad \| T (x_1, x_2, ..., x_n) \|_S \leq \max \left( \sum_{i=1}^{m} L^1_i (\Delta_1 + \theta_1), ..., \sum_{i=1}^{m} L^n_i (\Delta_n + \theta_n) \right) < \infty.
\]

Using the above inequality (2.13), we deduce that \( T \) maps bounded sets into bounded sets in \( S \).

The operator \( T \) is continuous on \( S \), in view of the continuity of \( g^k_i \) given in the hypothesis (H2).

For any \( 0 \leq t_1 < t_2 \leq 1 \), \((x_1, x_2, ..., x_n) \in B_\mu\) and \( k = 1, ..., n \), we have:

\[
| T_k (x_1, ..., x_n) (t_2) - T_k (x_1, ..., x_n) (t_1) | \leq M_k \sup_{s \in J} \sum_{i=1}^{m} | g^k_i (s, x_1(s), ..., x_n(s)) |,
\]

where

\[
M_k = \frac{1}{\Gamma (\alpha_k + 1)} (2 (t_2 - t_1)^{\alpha_k} + (t_2^{\alpha_k} - t_1^{\alpha_k})) \\
+ \frac{1}{\Gamma (\alpha_k + \delta_k + 1)} \left( 2 (t_2 - t_1)^{\alpha_k + \delta_k} + \left( t_2^{\alpha_k + \delta_k} - t_1^{\alpha_k + \delta_k} \right) \right) \\
+ \frac{1}{(l - 1)! \Gamma (\alpha_k + 2 - l)} + \frac{1}{(l - 1)! \Gamma (\alpha_k + \delta_k + 2 - l)} (t_2^{l-1} - t_1^{l-1}).
\]

Therefore,

\[
(2.14) \quad | T_k (x_1, ..., x_n) (t_2) - T_k (x_1, ..., x_n) (t_1) | \leq M_k \sum_{i=1}^{m} L^k_i,
\]

such that,

\[
\| T (x_1, ..., x_n) (t_2) - T (x_1, ..., x_n) (t_1) \|_S = \max_{1 \leq k \leq n} | T_k (x_1, ..., x_n) (t_2) - T_k (x_1, ..., x_n) (t_1) |.
\]
The right-hand side of (26) is independent of \((x_1, x_2, ..., x_n) \in B_{\mu}\) and tends to zero as \(t_2 - t_1 \to 0\). Therefore, \(T\) is an equi-continuous operator. We conclude that \(T\) is a completely continuous operator.

\((ii)\) : We shall show that the set \(\Omega\) defined by
\[
\Omega := \{(x_1, ..., x_n) \in S, (x_1, ..., x_n) = \lambda T (x_1, ..., x_n), 0 < \lambda < 1 \},
\]
is bounded. Let \((x_1, x_2, ..., x_n) \in \Omega\), then \((x_1, ..., x_n) = \lambda T (x_1, ..., x_n)\), for some \(0 < \lambda < 1\). We have:
\[
x_k (t) = \lambda T_k (x_1, ..., x_n) (t), \ k = 1, 2, ..., n.
\]
Corresponding to (24), we get:
\[
\|x_k\| \leq \lambda (\sum_{i=1}^{m} L_i^k (\Delta_i + \theta_i)) \sum_{i=1}^{L_i} L_i^k, \ k = 1, ..., n.
\]
Thus,
\[
\|(x_1, ..., x_n)\|_S \leq \lambda \max \left( \sum_{i=1}^{m} L_i^1 (\Delta_1 + \theta_1), ..., \sum_{i=1}^{m} L_i^n (\Delta_n + \theta_n) \right) < \infty.
\]
Consequently, the set \(\Omega\) is bounded. So by Lemma 1.4, we deduce that the operator \(T\) has at least one fixed point, which is a solution of the system (1.1). Theorem 2.2 is thus proved.

2.2. Ulam-Hyers and Generalized Ulam-Hyers Stabilities

In this section, we prove some results on the Ulam-Hyers and the generalized Ulam-Hyers stabilities for the solutions of (1.1).

**Definition 2.1.** The fractional system (1.1) is Ulam-Hyers stable if there exists a real number \(C > 0\), such that for each \(\epsilon_k > 0, k = 1, 2, ..., n\), and for each solution \((x_1, x_2, ..., x_n) \in S\) of

\[
D^\alpha x_k (t) - \sum_{i=1}^{m} g_i^k (t, x_1 (t), ..., x_n (t)) - \sum_{i=1}^{m} j^i g_i^k (t, x_1 (t), ..., x_n (t)) \leq \epsilon_k,
\]

there exists \((y_1, y_2, ..., y_n) \in S\) of (1.1) with
\[
\sum_{k=1}^{m} |y_k (0)| = \sum_{k=1}^{m} |y_k' (0)| = ... = \sum_{k=1}^{m} |y_k^{(l-2)} (0)| = 0, y_k^{(l-1)} (1) = 0, k = 1, 2, ..., n,
\]

satisfying
\[
\|(x_1 - y_1, x_2 - y_2, ..., x_n - y_n)\|_S \leq C \epsilon, \ \epsilon > 0.
\]
Definition 2.2. The fractional system (1.1) is generalized Ulam-Hyers stable if there exists \( \Upsilon \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+) \), such that for each \( \epsilon > 0 \) and for each solution \((x_1, x_2, \ldots, x_n) \) \( \in S \) of (2.19), there exists \((y_1, y_2, \ldots, y_n) \) \( \in S \) of (1.1) with
\[
\| (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \|_S \leq \Upsilon (\epsilon).
\]

We prove the following result:

Theorem 2.3. Suppose that:

(a) : The assumptions of Theorem 2.1 are satisfied.
(b) : The hypotheses \((H_2)\) and \((H_3)\) are valid.
(c) : The quantity (2.2) satisfies: \( 0 < \Sigma_k < 1 \).
(d) : For each \( k = 1, 2, \ldots, n \),
\[
(2.21) \quad \sup_{t \in J} |D^{\alpha_k} x_k (t)| \geq (\Delta_k + \theta_k) \sum_{i=1}^{m} L^k_i.
\]

Then, the problem (1.1) has the generalized Ulam-Hyers stability in \( S \).

Proof. Let
\[
(2.22) \quad \left| D^{\alpha_k} x_k (t) - \sum_{i=1}^{m} g^k_i (t, x_1 (t), x_2 (t), \ldots, x_n (t)) - \sum_{i=1}^{m} J^{\beta_i} g^k_i (t, x_1 (t), x_2 (t), \ldots, x_n (t)) \right| < \epsilon_k,
\]
where, \( \epsilon_k > 0 \) and \( k = 1, 2, \ldots, n \).

Using Theorem 2.1, we conclude that (1.1) has a unique solution \((y_1, y_2, \ldots, y_n) \) \( \in S \) satisfying:
\[
(2.23) \quad \sum_{i=1}^{m} g^k_i (t, y_1 (t), y_2 (t), \ldots, y_n (t)) + \sum_{i=1}^{m} J^{\beta_i} g^k_i (t, y_1 (t), y_2 (t), \ldots, y_n (t)) ,
\]
with, \( \sum_{k=1}^{n} |y_k (0)| = \sum_{k=1}^{n} |y_k' (0)| = \ldots = \sum_{k=1}^{n} |y_k^{(l-2)} (0)| = 0, y_k^{(l-1)} (1) = 0, k = 1, 2, \ldots, n. \)

Thanks to \((H_2)\) and \((H_3)\), we get
\[
(2.24) \quad |x_k (t)| \leq (\Delta_k + \theta_k) \sum_{i=1}^{m} L^k_i, \quad k = 1, 2, \ldots, n.
\]

Combining (2.21) and (2.24), we obtain:
\[
(2.25) \quad \sup_{t \in J} |x_k (t)| \leq \sup_{t \in J} |D^{\alpha_k} x_k (t)|.
\]
Then,
\begin{equation}
\left| D^{\alpha_k} x_k (t) - y_k (t) \right| \leq \sup_{t \in J} \left| D^{\alpha_k} (x_k (t) - y_k (t)) \right| \tag{2.26}
\end{equation}
\begin{equation}
\left( D^{\alpha_k} x_k (t) - \sum_{i=1}^{m} g^k_i (t, x_1 (t), x_2 (t), \ldots, x_n (t)) \right) \\
- \sum_{i=1}^{m} J^{\delta_k} g^k_i (t, x_1 (t), x_2 (t), \ldots, x_n (t)) \\
+ \left( -D^{\alpha_k} y_k (t) + \sum_{i=1}^{m} g^k_i (t, y_1 (t), y_2 (t), \ldots, y_n (t)) \right) \\
+ \sum_{i=1}^{m} J^{\delta_k} g^k_i (t, y_1 (t), y_2 (t), \ldots, y_n (t)) \\
- \sum_{i=1}^{m} g^k_i (t, y_1 (t), y_2 (t), \ldots, y_n (t)) \\
+ \sum_{i=1}^{m} J^{\delta_k} g^k_i (t, y_1 (t), y_2 (t), \ldots, y_n (t)) \right) \\
\leq \sup_{t \in J} \left| x_k (t) - y_k (t) \right| \tag{2.27}
\end{equation}

Using (34) and (35), we get
\begin{equation}
\sup_{t \in J} \left| x_k (t) - y_k (t) \right| \leq \epsilon_k + \sum_{i=1}^{m} \left( (\omega^k_{i1})_1 + (\omega^k_{i2})_2 + \ldots + (\omega^k_{in})_n \right) \max_{1 \leq k \leq n} \left| x_k (t) - y_k (t) \right| . \tag{2.28}
\end{equation}

By (2.28), we obtain
\begin{equation}
\max_{1 \leq k \leq n} \left| x_k (t) - y_k (t) \right| \leq \max_{1 \leq k \leq n} \frac{\epsilon_k}{1 - \sum_{k}^n} = C \epsilon , \tag{2.29}
\end{equation}

\begin{equation}
\epsilon = \max_{1 \leq k \leq n} \epsilon_k , \quad C = \max_{1 \leq k \leq n} \frac{1}{1 - \sum_{k}^n} , \tag{2.30}
\end{equation}

which implies that
\begin{equation}
\| (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \|_{S} = \max_{1 \leq k \leq n} \| (x_k - y_k) \| \leq C \epsilon . \tag{2.31}
\end{equation}

Thus, the system (1.1) is stable in the sense of Ulam-Hyers.
Taking \( \Upsilon (\epsilon) = C \epsilon \), we see that the system (1.1) is generalized Ulam-Hyers stable.
This completes the proof of Theorem 2.3. \( \square \)

3. Examples

In this section, we present two examples to illustrate the application of our main result.
Example 3.1. Consider the following system:

\[
\begin{align*}
D^\frac{2}{\pi^2} x_1(t) &= \frac{|x_1(t)+x_2(t)+x_3(t)|}{8\pi^2 (1+|x_1(t)+x_2(t)+x_3(t)|)} \\
&+ \frac{1}{2\pi^2} \left( \frac{\sin(x_1(t))+\sin(x_2(t))}{2\pi t} + \sin(x_3(t)) \right) \\
&+ \int_0^t \left( \frac{|x_1(t)+x_2(t)+x_3(t)|}{2\pi (1+|x_1(t)+x_2(t)+x_3(t)|)} \right) \\
&+ \frac{1}{2\pi^2} \left( \frac{\sin(x_1(t))+\sin(x_2(t))}{2\pi t} + \sin(x_3(t)) \right) dx, \quad t \in [0, 1],
\end{align*}
\]

(3.1)

\[
\begin{align*}
D^\frac{3}{\pi^2} x_2(t) &= \frac{|x_1(t)+x_2(t)+x_3(t)|}{8\pi^2 (1+|x_1(t)+x_2(t)+x_3(t)|)} \\
&+ \frac{1}{16\pi^2 t^2} \left( \frac{\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))}{1+\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))} \right) \\
&+ \frac{1}{16\pi^2 t^4} \left( \frac{\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))}{1+\sin(x_1(t))+\cos(x_2(t))+\cos(x_3(t))} \right) dx, \quad t \in [0, 1],
\end{align*}
\]

\[
\begin{align*}
D^\frac{2}{\pi^2} x_3(t) &= \cos(x_1(t))+\cos(x_2(t))+\cos(x_3(t)) \\
&+ \frac{1}{16\pi^2 t^2} \left( \sin(x_1(t)) + \frac{|x_2(t)+x_3(t)|}{8\pi^2 (1+|x_1(t)+x_2(t)+x_3(t)|)} \right) \\
&+ \frac{1}{16\pi^2 t^4} \left( \sin(x_1(t)) + \frac{|x_2(t)+x_3(t)|}{8\pi^2 (1+|x_1(t)+x_2(t)+x_3(t)|)} \right) dx, \quad t \in [0, 1],
\end{align*}
\]

(3.1)

\[
\begin{align*}
|x_1(0)| + |x_2(0)| + |x_3(0)| &= |x_1(0)| + |x_2(0)| + |x_3(0)| = 0, \\
x_1(1) = x_2(1) = x_3(1) = 1.
\end{align*}
\]

In this example, we have:

- \( n = 3, m = 2, l = 3, \alpha_1 = \frac{5}{2}, \alpha_2 = \frac{7}{3}, \alpha_3 = \frac{9}{4}, \delta_1 = \frac{7}{2}, \delta_2 = \frac{5}{4}, \delta_3 = \frac{3}{4}, J = [0, 1], \)

\[
\begin{align*}
g_1^1(t, x_1, x_2, x_3) &= \frac{|x_1 + x_2 + x_3|}{8\pi^2 \left( 1 + |x_1 + x_2 + x_3| \right)}, \\
g_2^1(t, x_1, x_2, x_3) &= \frac{1}{32\pi^2} \left( \frac{\sin x_1 + \sin x_2}{2\pi t + 1} + \sin x_3 \right),
\end{align*}
\]

(3.2)

\[
\begin{align*}
g_1^2(t, x_1, x_2, x_3) &= \frac{|x_1 + x_2 + x_3|}{8\pi^4 \left( 1 + |x_1 + x_2 + x_3| \right)}, \\
g_2^2(t, x_1, x_2, x_3) &= \frac{t^2}{16\pi^2 \left( 1 + |\sin x_1 + \cos x_2 + \cos x_3| \right)}.
\end{align*}
\]

(3.3)
We can take
\[ g_1^3(t, x_1, x_2, x_3) = \frac{\cos x_1 + \cos x_2 + \cos x_3}{4\pi e^2}, \]
\[ g_2^3(t, x_1, x_2, x_3) = \frac{1}{16\pi (t^2 + 1)} \left( \sin x_1 + \frac{|x_2 + x_3|}{3\pi^3 (1 + |x_2 + x_3|)} \right). \]

So, for \( t \in [0, 1] \) and \( (x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3 \), we have:
\[ |g_1^3(t, x_1, x_2, x_3) - g_1^3(t, y_1, y_2, y_3)| \leq \frac{1}{8\pi^2} |x_1 - y_1| + \frac{1}{8\pi^2} |x_2 - y_2| + \frac{1}{8\pi^2} |x_3 - y_3|, \]
\[ |g_2^3(t, x_1, x_2, x_3) - g_2^3(t, y_1, y_2, y_3)| \leq \frac{1}{64\pi^2} |x_1 - y_1| + \frac{1}{64\pi^2} |x_2 - y_2| + \frac{1}{32\pi^2} |x_3 - y_3|, \]
\[ |g_1^3(t, x_1, x_2, x_3) - g_2^3(t, y_1, y_2, y_3)| \leq \frac{1}{8\pi^2} |x_1 - y_1| + \frac{1}{8\pi^2} |x_2 - y_2| + \frac{1}{8\pi^2} |x_3 - y_3|, \]
\[ |g_2^3(t, x_1, x_2, x_3) - g_2^3(t, y_1, y_2, y_3)| \leq \frac{1}{16\pi^2} |x_1 - y_1| + \frac{1}{16\pi^2} |x_2 - y_2| + \frac{1}{16\pi^2} |x_3 - y_3|, \]
\[ |g_1^3(t, x_1, x_2, x_3) - g_3^3(t, y_1, y_2, y_3)| \leq \frac{1}{4\pi^2} |x_1 - y_1| + \frac{1}{4\pi^2} |x_2 - y_2| + \frac{1}{4\pi^2} |x_3 - y_3|, \]
\[ |g_2^3(t, x_1, x_2, x_3) - g_3^3(t, y_1, y_2, y_3)| \leq \frac{1}{16\pi} |x_1 - y_1| + \frac{1}{48\pi^2} |x_2 - y_2| + \frac{1}{48\pi^2} |x_3 - y_3|, \]

We can take
\[ (\omega_1^1)_1 = (\omega_1^1)_2 = (\omega_1^1)_3 = \frac{1}{8\pi^2}, \quad (\omega_2^1)_1 = (\omega_2^1)_2 = \frac{1}{64\pi^2 e^2}, \quad (\omega_3^1)_3 = \frac{1}{32\pi^2 e}, \]
\[ (\omega_2^2)_1 = (\omega_2^2)_2 = (\omega_2^2)_3 = \frac{1}{32\pi^2 e}, \quad (\omega_3^2)_1 = (\omega_3^2)_2 = (\omega_3^2)_3 = \frac{1}{16\pi^2 e}. \]
(3.13) \( (\omega^3_1)_1 = (\omega^3_2)_2 = (\omega^3_3)_3 = \frac{1}{4\pi e^2}, \) \( (\omega^2_1)_1 = \frac{1}{10\pi}, \) \( (\omega^2_2)_2 = (\omega^2_3)_3 = \frac{1}{48\pi^4} \).

It follows that

(3.14) \( \Sigma_1 = 0.039589, \Sigma_2 = 0.008626, \Sigma_3 = 0.052631, \)

(3.15) \( \Delta_1 = 0.302290, \Delta_2 = 0.370998, \Delta_3 = 0.468782, \)

(3.16) \( \theta_1 = 0.585, \theta_2 = 0.41996, \theta_3 = 0.79544. \)

Which implies that the condition (2.4) holds

(3.17) \max (\Sigma_1 \Delta_1 + \theta_1, \Sigma_2 \Delta_2 + \theta_2, \Sigma_3 \Delta_3 + \theta_3) < 1.

Then by Theorem 2.1, we deduce that the fractional coupled system (3.1) has a unique solution on \([0, 1] \).

**Example 3.2.** To illustrate the second main result, we consider the following system:

\[
D^\alpha x_1 (t) = \left(\frac{\pi e^t}{\pi e^t + \sin(x_1(t) + x_2(t) + x_3(t))}\right) + \frac{\pi e^t}{\pi e^t + \sin(x_1(t) + x_3(t))} + \frac{e^t \cos(x_2(t))}{\pi e^t + \sin(x_2(t) + x_3(t))} ds, \quad t \in [0, 1],
\]

\[
D^\beta x_2 (t) = \frac{\cos(x_1(t) + x_3(t))}{\pi e^t + \cos(x_2(t) + x_3(t))} + \frac{e^t \cos(x_2(t))}{\pi e^t + \cos(x_2(t) + x_3(t))} ds, \quad t \in [0, 1],
\]

\[
D^\gamma x_3 (t) = \frac{\cos(x_1(t))}{\pi e^t + \sin(x_1(t) + x_2(t))} + \sin(x_1(t)) \sin(x_2(t) + x_3(t)) ds, \quad t \in [0, 1],
\]

\[
\left| x_1(0) \right| + \left| x_2(0) \right| + \left| x_3(0) \right| = x_1'(1) = x_2'(1) = x_3'(1) = 0.
\]

We have:

\( n = 3, m = 2, l = 2, \alpha_1 = \frac{3}{2}, \alpha_2 = \frac{2}{3}, \alpha_3 = \frac{2}{9}, \delta_1 = \frac{3}{5}, \delta_2 = \frac{2}{7}, \delta_3 = \frac{4}{3}, \) \( \) \( J = [0, 1]. \)

And for \( i = 1, 2, 3, \) the functions \( g_i^k \) are continuous.

It is clear that:

(3.19) \( \left| g_1^1 (t, x_1, x_2, x_3) \right| \leq \pi e, \) \( \left| g_2^1 (t, x_1, x_2, x_3) \right| \leq \frac{e}{2\pi - 1}, \)

(3.20) \( \left| g_1^2 (t, x_1, x_2, x_3) \right| \leq \frac{1}{2\pi e - 1}, \) \( \left| g_2^2 (t, x_1, x_2, x_3) \right| \leq \frac{1}{e - 1}, \)

(3.21) \( \left| g_1^3 (t, x_1, x_2, x_3) \right| \leq \frac{1}{\pi - 1}, \) \( \left| g_2^2 (t, x_1, x_2, x_3) \right| \leq 1. \)

Also, the functions \( g_i^k \) are also bounded on \([0, 1] \times \mathbb{R}^3 \). So, by Theorem 2.2, the system (3.18) has at least one solution on \([0, 1] \).
REFERENCES


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