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ON THE COMPARATIVE GROWTH ANALYSIS OF DIFFERENTIAL POLYNOMIALS DEPENDING UPON THEIR RELATIVE ORDERS, RELATIVE TYPE AND RELATIVE WEAK TYPE

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Abstract. In this paper the comparative growth properties of composition of entire and meromorphic functions on the basis of their relative orders (relative lower orders), relative types and relative weak types of differential polynomials generated by entire and meromorphic functions have been investigated.

Keywords: Entire function, meromorphic function, order (lower order), relative order (relative lower order), relative type, relative weak type, Property (A), growth, differential polynomial

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function relating to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. If f is non-constant then it has the following property:

Property (A) ([2]) : A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For exapmles of functions with or without the Property (A), one may see [2].

When f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic. In this situation one may define another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r) \ .$$

And given two meromorphic functions f and g the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's Characteristic function.

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When f is entire function, the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) \ .$$

We called the function $N_f(r,a)$ $\left(\bar{N}_f(r,a)\right)$ as counting function of a-points (distinct a-points) of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively. We put

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r \ ,$$

where we denote by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f and the quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows

$$\Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r,a;f)}{T(r,f)}.$$

Also we denote by $n_p(r,a;f)$ denotes the number of zeros of $f-a$ in $|z| \leq r$,where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geqslant p$ is counted exactly p times.

Accordingly, $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way and we set for any $a \in \mathbb{C} \cup \{\infty\}$

$$\delta_p(a;f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)} \quad \{\text{cf. } [14]\} ,$$

On the other hand, $m\left(r, \frac{1}{f-a}\right)$ is denoted by $m_f(r,a)$ and we mean $m_f(r,\infty)$ by $m_f(r)$, which is called the proximity function of f. We also put

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \ \text{ for all } x \geqslant 0 \ .$$

Further, for any non-constant meromorphic function f, $b \equiv b(z)$ $b = b(z)$ is called small with respect to f if $T_b(r) = S_f(r)$ where $S_f(r) = o\{T_f(r)\}$ i.e., $\frac{S_f(r)}{T_f(r)} \to 0$ as $r \to \infty$. Moreover for any non-constant meromorphic function f , we call $M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \ldots \ldots\ldots \left(f^{(k)}\right)^{n_{kj}}$ where $T_{A_j}(r) = S_f(r)$, to be a differential monomial generated by it where n_{0j}, n_{1j},......,n_{kj} $(k \geq 1)$ be non-negative integers such that for each $j, \sum_{i=0}^{k} n_{ij} \geq 1$. In this connection the numbers

$\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^{k} (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[6], [17]}. The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max_{1<j<s} \gamma_{M_j}$ and $\Gamma_P = \max_{1<j<s} \Gamma_{M_j}$ are called respectively the degree and weight of $P[f]$ {[6], [17]}. Also we call the numbers $\underline{\gamma}_P = \min_{1<j<s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma}_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e. for which $n_{0j} = 0$ for $j = 1, 2, ..., s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other. In this connection we denote $\gamma_{P_0[f]}$ as

$$\gamma_{P_0[f]} = \lim_{r\to\infty} \frac{T_{P_0[f]}(r)}{T_f(r)} .$$

Further, the following definition is also well known:

Definition 1.1. [3] $P[f]$ is said to be admissible if
(i) $P[f]$ is homogeneous, or
(ii) $P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.

If f is a non-constant entire function then $T_f(r)$ is rigorously increasing and continuous function of r and its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist where $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is known as growth of f with respect to g in terms of the Nevanlinna's Characteristic functions of the meromorphic functions f and g. Further, in the case of meromorphic functions, the growth markers such as order and lower order which are traditional in complex analysis are defined in terms of their growths with respect to the $\exp z$ function in the following way:

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)}\right),$$

and the growth of functions is said to be regular if their lower order coincides with their order.

In this connection the following two definitions are also well known:

Definition 1.2. The *type* σ_f and *lower type* $\overline{\sigma}_f$ of a meromorphic function f are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty .$$

If f is entire then

$$\sigma_f = \limsup_{r\to\infty}\frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \liminf_{r\to\infty}\frac{\log M_f(r)}{r^{\rho_f}},\ 0<\rho_f<\infty\ .$$

Definition 1.3. [8] The *weak type* τ_f and the growth indicator τ_f of a meromorphic function f of finite positive lower order λ_f are defined by

$$\overline{\tau}_f = \limsup_{r\to\infty}\frac{T_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty}\frac{T_f(r)}{r^{\lambda_f}},\ 0<\lambda_f<\infty\ .$$

When f is entire then

$$\overline{\tau}_f = \limsup_{r\to\infty}\frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \liminf_{r\to\infty}\frac{\log M_f(r)}{r^{\lambda_f}},\ 0<\lambda_f<\infty\ .$$

However, extending the thought of relative order of entire functions as initiated by Bernal {[1], [2]} , Lahiri and Banerjee [15] introduced the definition of relative order of a meromorphic function f with respect to another entire function g, symbolized by $\rho_g(f)$ to avoid comparing growth just with $\exp z$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu>0 : T_f(r) < T_g(r^{\mu}) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one if $g(z)=\exp z$ {cf. [15] }.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log r}.$$

To compare the relative growth of two entire functions having same non zero finite *relative order* with respect to another entire function, Roy [16] introduced the notion of *relative type* of two entire functions in the following way:

Definition 1.4. [16] Let f and g be any two entire functions such that $0<\rho_g(f)<\infty$. Then the *relative type* $\sigma_g(f)$ of f with respect to g is defined as :

$$\begin{aligned}&\sigma_g(f) \\ =\ & \inf\left\{k>0 : M_f(r) < M_g\left(kr^{\rho_g(f)}\right) \text{ for all sufficiently large values of } r\right\} \\ =\ & \limsup_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}}\ .\end{aligned}$$

Likewise, one can define the *relative lower type* of an entire function f with respect to an entire function g denoted by $\overline{\sigma}_g(f)$ as follows :

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty}\frac{M_g^{-1}M_f(r)}{r^{\rho_g(f)}},\ 0<\rho_g(f)<\infty\ .$$

Analogously, to determine the relative growth of two entire functions having same non zero finite *relative lower order* with respect to another entire function, Datta and Biswas [9] introduced the definition of *relative weak type* of an entire function f with respect to another entire function g of finite positive *relative lower order* $\lambda_g(f)$ in the following way:

Definition 1.5. [9] The *relative weak type* $\tau_g(f)$ of an entire function f with respect to another entire function g having finite positive *relative lower order* $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} .$$

Also one may define the growth indicator $\overline{\tau}_g(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}},\ 0 < \lambda_g(f) < \infty .$$

In case of meromorphic functions, it therefore seems reasonable to define suitably the *relative type* and *relative weak type* of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite *relative order* or *relative lower order* with respect to an entire function. Datta and Biswas also [9] gave such definitions of *relative type* and *relative weak type* of a meromorphic function f with respect to an entire function g which are as follows:

Definition 1.6. [9] The *relative type* $\sigma_g(f)$ of a meromorphic function f with respect to an entire function g are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \text{where } 0 < \rho_g(f) < \infty.$$

Similarly, one can define the *lower relative type* $\overline{\sigma}_g(f)$ in the following way:

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 1.7. [9] The *relative weak type* $\tau_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 1.4 , Definition 1.5, Definition 1.6 and Definition 1.7 coincide with the classical definitions of type (lower type) and weak type of entire and meromorphic functions respectively.

For entire and meromorphic functions, the notion of their growth indicators such as *order*, *type* and *weak type* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* and consequently *relative type* as well as *relative weak type* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysts and they are not aware of the technical advantages of using the relative growth indicators of the functions. In this paper we wish to prove some newly developed results based on the growth properties of *relative order, relative type* and *relative weak type* of differential polynomials generated by entire and meromorphic functions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [12] and [18].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[4] Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)) \ .$$

Lemma 2.2. *[5] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \geq T_f(\exp(r^\mu)) \ .$$

Lemma 2.3. *[13] Let f be meromorphic and g be entire such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) > T_g(\exp(r^\mu)) ,$$

where $0 < \mu < \rho_g$.

Lemma 2.4. *[7] Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) < T_f(\exp(r^\mu)) \ .$$

Lemma 2.5. *[7] Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f \circ g}(r) < T_g \left(\exp\left(r^{\mu}\right)\right) .$$

Lemma 2.6. *[10] Let f be an entire function which satisfy the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f\left(r\right) < T_f\left(\alpha r^{\delta}\right).$$

Lemma 2.7. *[11] If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta\left(\infty; f\right) = \sum\limits_{a \neq \infty} \delta_p\left(a; f\right) = 1$ or $\delta\left(\infty; f\right) = \sum\limits_{a \neq \infty} \delta\left(a; f\right) = 1$ and g be an entire function of regular growth having non zero finite order and $\Theta\left(\infty; g\right) = \sum\limits_{a \neq \infty} \delta_p\left(a; g\right) = 1$ or $\delta\left(\infty; g\right) = \sum\limits_{a \neq \infty} \delta\left(a; g\right) = 1$. Then the relative order of $P_0\left[f\right]$ with respect to $P_0\left[g\right]$ are same as those of f with respect to g where $P_0\left[f\right]$ and $P_0\left[g\right]$ are homogeneous.*

Lemma 2.8. *[11] If f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta\left(\infty; f\right) = \sum\limits_{a \neq \infty} \delta_p\left(a; f\right) = 1$ or $\delta\left(\infty; f\right) = \sum\limits_{a \neq \infty} \delta\left(a; f\right) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta\left(\infty; g\right) = \sum\limits_{a \neq \infty} \delta_p\left(a; g\right) = 1$ or $\delta\left(\infty; g\right) = \sum\limits_{a \neq \infty} \delta\left(a; g\right) = 1$. Then the relative type and relative lower type of $P_0\left[f\right]$ with respect to $P_0\left[g\right]$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_g\left(f\right)$ is positive finite and $P_0\left[f\right]$ and $P_0\left[g\right]$ are homogeneous.*

Lemma 2.9. [11] *Let f be a meromorphic function either of finite order or of non-zero lower order with $\Theta\left(\infty; f\right) = \sum\limits_{a \neq \infty} \delta_p\left(a; f\right) = 1$ or $\delta\left(\infty; f\right) = \sum\limits_{a \neq \infty} \delta\left(a; f\right) = 1$ and g be an entire function of regular growth having non zero finite type and $\Theta\left(\infty; g\right) = \sum\limits_{a \neq \infty} \delta_p\left(a; g\right) = 1$ or $\delta\left(\infty; g\right) = \sum\limits_{a \neq \infty} \delta\left(a; g\right) = 1$. Then $\tau_{P_0[g]}\left(P_0\left[f\right]\right)$ and $\overline{\tau}_{P_0[g]}\left(P_0\left[f\right]\right)$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g i.e., $\tau_{P_0[g]}\left(P_0\left[f\right]\right) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \cdot \tau_g\left(f\right)$ and $\overline{\tau}_{P_0[g]}\left(P_0\left[f\right]\right) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_g\left(f\right)$ when $\lambda_g\left(f\right)$ is positive finite and $P_0\left[f\right]$ and $P_0\left[g\right]$ are homogeneous.*

3. Main Results

In this section we present the main results of the paper.

In the paper, it is needless to mention that the admissibility and homogeneity of $P_0[f]$ will be needed as per the requirements of the theorems.

Theorem 3.1. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\sigma_g < \infty$ and also h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \sigma_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Proof. Let us suppose that $\alpha > 2$.
Since $T_h^{-1}(r)$ is an increasing function r, it follows from Lemma 2.1, Lemma 2.6 and the inequality $T_g(r) \leq \log M_g(r)$ $\{cf.$ [12]$\}$ for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1} T_{f \circ g}(r) &\leqslant T_h^{-1}\left[\{1 + o(1)\} T_f(M_g(r))\right] \\ i.e.,\ T_h^{-1} T_{f \circ g}(r) &\leqslant \alpha \left[T_h^{-1} T_f(M_g(r))\right]^{\delta} \\ i.e.,\ \log T_h^{-1} T_{f \circ g}(r) &\leqslant \delta \log T_h^{-1} T_f(M_g(r)) + O(1) \end{aligned} \tag{3.1}$$

$$i.e.,\ \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})}$$
$$\leq \frac{\delta \log T_h^{-1} T_f(M_g(r)) + O(1)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} = \frac{\delta \log T_h^{-1} T_f(M_g(r)) + O(1)}{\log M_g(r)} \cdot \frac{\log M_g(r)}{r^{\rho_g}} \cdot \frac{\log \exp r^{\rho_g}}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})}$$

$$\begin{aligned} & i.e.,\ \limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} \\ & \leq \limsup_{r \to \infty} \frac{\delta \log T_h^{-1} T_f(M_g(r)) + O(1)}{\log M_g(r)} \cdot \limsup_{r \to \infty} \frac{\log M_g(r)}{r^{\rho_g}} \cdot \\ & \qquad \limsup_{r \to \infty} \frac{\log \exp r^{\rho_g}}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} \end{aligned} \tag{3.2}$$

$$i.e.,\ \limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} \leq \delta \cdot \rho_h(f) \cdot \sigma_g \cdot \frac{1}{\lambda_{P_0[h]}(P_0[f])} .$$

Therefore in view of Lemma 2.7, we obtain from the above that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \sigma_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Thus the theorem is established. □

In the line of Theorem 3.1, the following theorem can be proved :

Theorem 3.2. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum\limits_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum\limits_{a \neq \infty} \delta(a; g) = 1$ and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum\limits_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum\limits_{a \neq \infty} \delta(a; h) = 1$ with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\sigma_g < \infty$ and also h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \sigma_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Using the notion of l*ower type*, we may state the following two theorems without their proofs because those can be carried out in the line of Theorem 3.1 and Theorem 3.2 respectively.

Theorem 3.3. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum\limits_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a \neq \infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum\limits_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum\limits_{a \neq \infty} \delta(a; h) = 1$ such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\overline{\sigma}_g < \infty$ and also h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \overline{\sigma}_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Theorem 3.4. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum\limits_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum\limits_{a \neq \infty} \delta(a; g) = 1$ and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum\limits_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum\limits_{a \neq \infty} \delta(a; h) = 1$ with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\overline{\sigma}_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^{\rho_g})} \leq \frac{\delta \cdot \overline{\sigma}_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Using the concept of the growth indicators τ_g and $\overline{\tau}_g$ of an entire function g, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 respectively.

Theorem 3.5. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non-zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\overline{\tau}_g < \infty$ and also h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \overline{\tau}_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Theorem 3.6. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\overline{\tau}_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \overline{\tau}_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Theorem 3.7. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $\tau_g < \infty$ and also h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \tau_g \cdot \rho_h(f)}{\lambda_h(f)} .$$

Theorem 3.8. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$, $\tau_g < \infty$ and h satisfy the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^{\lambda_g})} \leq \frac{\delta \cdot \tau_g \cdot \rho_h(f)}{\lambda_h(g)} .$$

Theorem 3.9. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum\limits_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum\limits_{a\neq\infty} \delta(a; h) = 1$ such that (i) $0 < \rho_h(f) < \infty$, (ii) $\rho_h(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \sigma_h(f) < \infty$ and also h satisfy Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\delta\cdot\rho_h(f)\cdot\sigma_g}{\sigma_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Proof. From (3.1), we get for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)\log M_g(r) + O(1). \tag{3.3}$$

Using Definition 1.2, we obtain from (3.3) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon)\cdot r^{\rho_g} + O(1). \tag{3.4}$$

Now in view of the condition (ii), we obtain from (3.4) for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leqslant \delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon)\cdot r^{\rho_h(f)} + O(1). \tag{3.5}$$

Again in view of Definition 1.6, we get for a sequence of values of r tending to infinity that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \geq \left(\sigma_{P_0[h]}(P_0[f]) - \varepsilon\right) r^{\rho_{P_0[h]}(P_0[f])}. \tag{3.6}$$

Now in view of Lemma 2.7 and Lemma 2.8, we get from the above for a sequence of values of r tending to infinity that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \geq \left(\sigma_h(f)\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}. \tag{3.7}$$

Therefore from (3.5) and (3.7), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \frac{\delta(\rho_h(f) + \varepsilon)(\sigma_g + \varepsilon)\cdot r^{\rho_h(f)} + O(1)}{\left(\sigma_h(f)\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from the above that

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\delta\cdot\rho_h(f)\cdot\sigma_g}{\sigma_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Hence the theorem follows. □

Using the notion of *lower type* and *relative lower type*, we may state the following theorem without its proof as it can be carried out in the line of Theorem 3.9:

Theorem 3.10. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ with (i) $0 < \rho_h(f) < \infty$, (ii) $\rho_h(f) = \rho_g$, (iii) $\overline{\sigma}_g < \infty$, (iv) $0 < \overline{\sigma}_h(f) < \infty$ and also h satisfies Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\delta \cdot \rho_h(f) \cdot \overline{\sigma}_g}{\overline{\sigma}_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Similarly using the notion of *type* and *relative lower type,* one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 3.9 :

Theorem 3.11. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that (i) $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, (ii) $\rho_h(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \overline{\sigma}_h(f) < \infty$ and also h satisfies the Property (A). Then for any $\delta > 1$,*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\delta \cdot \lambda_h(f) \cdot \sigma_g}{\overline{\sigma}_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Theorem 3.12. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ with (i) $0 < \rho_h(f) < \infty$, (ii) $\rho_h(f) = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $0 < \overline{\sigma}_h(f) < \infty$ and also h satisfies Property (A). Then for any $\delta > 1$,*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\delta \cdot \rho_h(f) \cdot \sigma_g}{\overline{\sigma}_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Similarly, using the concept of *weak type* and *relative weak type*, we may state next four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.9, Theorem 3.10, Theorem 3.11 and Theorem 3.12 respectively.

Theorem 3.13. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ such that (i) $0<\lambda_h(f)\leq\rho_h(f)<\infty$, (ii) $\lambda_h(f)=\lambda_g$, (iii) $\overline{\tau}_g<\infty$, (iv) $0<\overline{\tau}_h(f)<\infty$ and also h satisfies Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\leq\left(\frac{\delta\cdot\rho_h(f)\cdot\overline{\tau}_g}{\overline{\tau}_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.14. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ with (i) $0<\lambda_h(f)\leq\rho_h(f)<\infty$, (ii) $\lambda_h(f)=\lambda_g$, (iii) $\tau_g<\infty$, (iv) $0<\tau_h(f)<\infty$ and also h satisfies Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\leq\left(\frac{\delta\cdot\rho_h(f)\cdot\tau_g}{\tau_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.15. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ such that (i) $0<\lambda_h(f)<\infty$, (ii) $\lambda_h(f)=\lambda_g$, (iii) $\overline{\tau}_g<\infty$, (iv) $0<\tau_h(f)<\infty$ and also h satisfies Property (A). Then for any $\delta>1$,*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\leq\left(\frac{\delta\cdot\lambda_h(f)\cdot\overline{\tau}_g}{\tau_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.16. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ with (i) $0<\lambda_h(f)\leq\rho_h(f)<\infty$, (ii) $\lambda_h(f)=\lambda_g$, (iii) $\overline{\tau}_g<\infty$, (iv) $0<\tau_h(f)<\infty$ and also h satisfies Property (A). Then for any $\delta>1$,*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}\leq\left(\frac{\delta\cdot\rho_h(f)\cdot\overline{\tau}_g}{\tau_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.17. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $0 < \lambda_h(f) \le \rho_h(f) < \rho_g \le \infty$ and $\sigma_h(f) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \ge \left(\frac{\lambda_h(f)}{\sigma_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Proof. Since $\rho_h(f) < \rho_g$ and $T_h^{-1}(r)$ is a increasing function of r, we get from Lemma 2.2 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &\ge \log T_h^{-1} T_f(\exp(r^{\mu})) \\ i.e.,\ \log T_h^{-1} T_{f\circ g}(r &\ge (\lambda_h(f) - \varepsilon) \cdot r^{\mu} \end{aligned}$$

$$i.e.,\ \log T_h^{-1} T_{f\circ g}(r) \ge (\lambda_h(f) - \varepsilon) \cdot r^{\rho_h(f)} . \tag{3.8}$$

Again in view of Definition 1.6, we get for all sufficiently large values of r that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \le \left(\sigma_{P_0[h]}(P_0[f]) + \varepsilon\right) r^{\rho_{P_0[h]}(P_0[f])} .$$

Therefore in view of Lemma 2.7 and Lemma 2.8, we obtain from the above for all sufficiently large values of r that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \le \left(\sigma_h(f) \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} + \varepsilon\right) r^{\rho_h(f)} . \tag{3.9}$$

Now from (3.8) and (3.9), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \ge \frac{(\lambda_h(f) - \varepsilon) r^{\rho_h(f)}}{\left(\sigma_h(f) \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} + \varepsilon\right) r^{\rho_h(f)}} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \ge \left(\frac{\lambda_h(f)}{\sigma_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Thus the theorem follows. □

In the line of Theorem 3.17, the following theorem can be proved and therefore its proof is omitted:

Theorem 3.18. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or*

$\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$*and* h *be an entire function of regular growth having non zero finite order and* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ *such that* $0 < \lambda_h(f)$, $0 < \rho_h(g) < \rho_g \leq \infty$ *and* $\sigma_h(g) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \geq \left(\frac{\lambda_h(f)}{\sigma_h(g)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}} .$$

The following two theorems can also be proved in the line of Theorem 3.17 and Theorem 3.18 respectively and with help of Lemma 2.3. Hence their proofs are omitted.

Theorem 3.19. *Let* f *be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$, g *be entire function and* h *be an entire function of regular growth having non zero finite order and* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ *such that* $0 < \lambda_h(g)$, $0 < \lambda_f$, $0 < \rho_h(f) < \rho_g < \infty$ *and* $\sigma_h(f) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \left(\frac{\lambda_h(g)}{\sigma_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}} .$$

Theorem 3.20. *Let* f *be a meromorphic function,* g *be an entire function either of finite order or of non-zero lower order with* $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ *or* $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$*and* h *be an entire function of regular growth having non zero finite order and* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ *such that* $0 < \lambda_h(g)$, $0 < \lambda_f$, $0 < \rho_h(g) < \rho_g < \infty$ *and* $\sigma_h(g) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \geq \left(\frac{\lambda_h(g)}{\sigma_h(g)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}} .$$

Now we state the following four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.17, Theorem 3.18, Theorem 3.19 and Theorem 3.20 and with the help of Definition 1.7:

Theorem 3.21. *Let* f *be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$, g *be entire function and* h *be an entire function of regular growth having non zero finite order and* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$ *such that* $0 < \lambda_h(f) < \rho_g \leq \infty$ *and* $\overline{\tau}_h(f) < \infty$. *Then*

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \left(\frac{\lambda_h(f)}{\overline{\tau}_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Theorem 3.22. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with* $\Theta(\infty;g) = \sum_{a\neq\infty}\delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a\neq\infty}\delta(a;g) = 1$ *and h be an entire function of regular growth having non zero finite order and* $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ *such that* $0 < \lambda_h(f)$, $0 < \lambda_h(g) < \rho_g \leq \infty$ *and* $\overline{\tau}_h(g) < \infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \geq \left(\frac{\lambda_h(f)}{\overline{\tau}_h(g)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.23. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f) = \sum_{a\neq\infty}\delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum_{a\neq\infty}\delta(a;f) = 1$, *$g$ be entire function and h be an entire function of regular growth having non zero finite order and* $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ *such that* $0 < \lambda_h(g) < \rho_g < \infty$, $0 < \lambda_f$ *and* $\overline{\tau}_h(f) < \infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \geq \left(\frac{\lambda_h(g)}{\overline{\tau}_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.24. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order with* $\Theta(\infty;g) = \sum_{a\neq\infty}\delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a\neq\infty}\delta(a;g) = 1$ *and h be an entire function of regular growth having non zero finite order and* $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ *such that* $0 < \lambda_h(g) < \rho_g < \infty$, $0 < \lambda_f$ *and* $\overline{\tau}_h(g) < \infty$. *Then*

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[g]}(r)} \geq \left(\frac{\lambda_h(g)}{\overline{\tau}_h(g)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.25. *Let f be a meromorphic function either of non-zero finite order or non zero lower order such that* $\Theta(\infty;f) = \sum_{a\neq\infty}\delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum_{a\neq\infty}\delta(a;f) = 1$, *$g$ be entire function and h be an entire function of regular growth having non zero finite order and* $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ *such that* $0 < \lambda_g < \rho_h(f) < \infty$ *and* $\overline{\sigma}_h(f) > 0$. *Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{P_0[h]}^{-1}T_{P_0[f]}(r)} \leq \left(\frac{\rho_h(f)}{\overline{\sigma}_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Proof. As $\lambda_g < \rho_h(f)$ and $T_h^{-1}(r)$ is a increasing function of r, it follows from Lemma 2.4 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f\circ g}(r) &< \log T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right) \\ i.e.,\ \log T_h^{-1} T_{f\circ g}(r &< \left(\rho_h(f)+\varepsilon\right)\cdot r^{\mu} \\ i.e.,\ \log T_h^{-1} T_{f\circ g}(r) &< \left(\rho_h(f)+\varepsilon\right)\cdot r^{\rho_h(f)}\ . \qquad (3.10) \end{aligned}$$

Further in view of Definition 1.6, we obtain for all sufficiently large values of r that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \geq \left(\overline{\sigma}_{P_0[h]}\left(P_0[f]\right)-\varepsilon\right) r^{\rho_{P_0[h]}(P_0[f])}\ .$$

Now in view of Lemma 2.7 and Lemma 2.8, we obtain from the above for all sufficiently large values of r that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \geq \left(\overline{\sigma}_h(f)\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}\ . \qquad (3.11)$$

Since $\varepsilon\,(>0)$ is arbitrary, therefore from (3.10) and (3.11) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} &\leq \frac{\left(\rho_h(f)+\varepsilon\right)\cdot r^{\rho_h(f)}}{\left(\overline{\sigma}_h(f)\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} - \varepsilon\right) r^{\rho_h(f)}} \\ i.e.,\ \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} &\leq \left(\frac{\rho_h(f)}{\overline{\sigma}_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}\ . \end{aligned}$$

Hence the theorem is established. □

In the line of Theorem 3.25, the following theorem can be proved and therefore its proof is omitted:

Theorem 3.26. *Let f be a meromorphic function with non-zero finite order and lower order, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty;g) = \sum\limits_{a\neq\infty} \delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum\limits_{a\neq\infty} \delta(a;g) = 1$and h be an entire function of regular growth having non zero finite order and $\Theta(\infty;h) = \sum\limits_{a\neq\infty} \delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum\limits_{a\neq\infty} \delta(a;h) = 1$ such that $\rho_h(f) < \infty$, $0 < \lambda_g < \rho_h(g) < \infty$ and $\overline{\sigma}_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \left(\frac{\rho_h(f)}{\overline{\sigma}_h(g)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}}\ .$$

Moreover, the following two theorems can also be deduced in the line of Theorem 3.17 and Theorem 3.18 respectively and with help of Lemma 2.5 and therefore their proofs are omitted.

Theorem 3.27. *Let f be a meromorphic function of finite order with $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $\rho_h(g) < \infty$, $0 < \lambda_g < \rho_h(f) < \infty$ and $\overline{\sigma}_h(f) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\rho_h(g)}{\overline{\sigma}_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.28. *Let f be a meromorphic function with finite order, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$and h be an entire function of regular growth having non-zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $0 < \lambda_g < \rho_h(g) < \infty$ and $\overline{\sigma}_h(g) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \left(\frac{\rho_h(g)}{\overline{\sigma}_h(g)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}}.$$

Finally we state the following four theorems without their proofs as those can be carried out with the help of Lemma 2.9 and in the line of Theorem 3.25, Theorem 3.26, Theorem 3.27 and Theorem 3.28 using the concept of *relative weak type:*

Theorem 3.29. *Let f be a meromorphic function either of non zero finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$, g be entire function and h be an entire function of regular growth having non-zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $0 < \lambda_g < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\tau_h(f) > 0$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\rho_h(f)}{\tau_h(f)}\right) \left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}}.$$

Theorem 3.30. *Let f be a meromorphic function with non zero finite order and lower order, g be an entire function either of finite order or of non-zero lower order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$and h be an entire function of regular growth having non-zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ such that $\rho_h(f) < \infty$, $0 < \lambda_g <$*

$\lambda_h(g) < \infty$ *and* $\tau_h(g) > 0$. *Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \left(\frac{\rho_h(f)}{\tau_h(g)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}} .$$

Theorem 3.31. *Let f be a meromorphic function of finite order with* $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$, *$g$ be entire function and h be an entire function of regular growth having non zero finite order and* $\Theta(\infty; h) = \sum\limits_{a\neq\infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum\limits_{a\neq\infty} \delta(a; h) = 1$ *such that* $\rho_h(g) < \infty$, $0 < \lambda_g < \lambda_h(f) < \infty$ *and* $\tau_h(f) > 0$. *Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \leq \left(\frac{\rho_h(g)}{\tau_h(f)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[f]}}\right)^{\frac{1}{\rho_h}} .$$

Theorem 3.32. *Let f be a meromorphic function with finite order, g be an entire function either of finite order or of non-zero lower order with* $\Theta(\infty; g) = \sum\limits_{a\neq\infty} \delta_p(a; g) = 1$ *or* $\delta(\infty; g) = \sum\limits_{a\neq\infty} \delta(a; g) = 1$*and h be an entire function of regular growth having non-zero finite order and* $\Theta(\infty; h) = \sum\limits_{a\neq\infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum\limits_{a\neq\infty} \delta(a; h) = 1$ *such that* $0 < \lambda_g < \lambda_h(f) \leq \rho_h(g) < \infty$ *and* $\tau_h(g) > 0$. *Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} \leq \left(\frac{\rho_h(g)}{\tau_h(g)}\right)\left(\frac{\gamma_{P_0[h]}}{\gamma_{P_0[g]}}\right)^{\frac{1}{\rho_h}} .$$

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