FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 31, No 2 (2016), 543–557

SCREEN SLANT RADICAL TRANSVERSAL NULL SUBMANIFOLDS OF PARA-SASAKIAN MANIFOLDS

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Abstract. In our paper we introduce totally paracontact umbilical radical transversal null submanifolds and screen slant radical transversal null submanifolds of para-Sasakian manifolds. On a screen slant radical transversal null submanifold of a para-Sasakian manifold, we find integrability conditions of distributions.
Keywords: Null submanifold, Para-Sasakian manifold

1. Introduction

Differential geometry of null (lightlike) submanifolds is different from non-degenerate submanifolds because of the fact that the normal vector bundle has non-trivial intersection with the tangent vector bundle. So, one cannot use the classical submanifold theory for null submanifolds. For this problem K. L. Duggal and A. Bejancu introduced a new method and presented a book about null submanifolds [13] (see also [14]). The term of totally contact umbilical null submanifolds was considered by several geometers ([9, 12, 17]). In 2009, B. Şahin studied screen slant null submanifolds [5]. Radical transversal null submanifolds were defined and studied by C. Yıldırım and B. Şahin in 2010. Since then many authors have studied null submanifolds ([3, 6, 7, 8, 15, 16, 20]).

In 1985, on a semi-Riemannian manifold M^{2n+1}, S. Kaneyuki and M. Konzai [19] introduced a structure which is called almost paracontact structure and then they characterized the almost paracomplex structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [21] studied paracontact metric manifolds and some subclasses which are known para-Sasakian manifolds. The study of paracontact geometry was continued by several papers ([10, 11, 18, 22, 23]) which include role of paracontact geometry about semi-Riemannian geometry, mathematical physics and relationships with the para-Kähler manifolds.

The goal of the present article is to examine some null submanifolds of a para-Sasakian manifold. There are some basic definitions for almost paracontact metric

Received January 25, 2015; accepted March 25, 2016
2010 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25

manifolds and null submanifolds in section 2. Totally paracontact umbilical radical transversal null submanifolds of a para-Sasakian manifold are introduced in Section 3. Finally, Section 4 is devoted to screen slant radical transversal null submanifolds of a para-Sasakian manifold and integrability conditions of distributions on screen slant radical transversal null submanifolds.

2. Preliminaries

2.1. Null Submanifolds

Let $(\bar{M}^{n+m}, \bar{g})$ be a semi-Riemannian manifold with index q, such that $m, n \geqslant 1, 1 \leq q \leq m+n-1$ and (M^m, g) be a submanifold of $\bar{M}$, where g induced metric from $\bar{g}$ on M. In this case M is called a *null* (*lightlike) submanifold* of $\bar{M}$ if g is degenerate on M. Now consider a degenerate metric g on M. Thus $TM^{\perp}$ is a degenerate $n-$dimensional subspace of $T_x\bar{M}$ and orthogonal subspaces T_xM and $T_xM^{\perp}$ are degenerate but no longer complementary. So, there exists a subspace $Rad\,T_xM = T_xM \cap T_xM^{\perp}$ which is called radical space. If the mapping $Rad\,TM : x \in M \to RadT_xM$, defines a smooth distribution, named *Radical distribution*, on M of rank $r > 0$ then the submanifold M is called an $r-$*null submanifold* [13].

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,TM$ in TM. So we can state

$$TM = S(TM) \perp Rad\,TM, \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad\,TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and $Rad\,TM$ in $S(TM^{\perp})^{\perp}$, respectively. In this case, we arrive at

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = \{Rad\,TM \oplus ltr(TM)\} \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Theorem 2.1. *[13] Let $(M, g, S(TM), S(TM^{\perp}))$ be a null submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exist a complementary vector subbundle $ltr(TM)$ of $Rad\,TM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM))\mid_U$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_U$, where U is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, E_i) = 1, \qquad \bar{g}(N_i, N_j) = 0, \tag{2.4}$$

where $\{E_1, E_2, ..., E_n\}$ is a null basis of $\Gamma(Rad\,TM)$.

For a null submanifold $(M, g, S(TM), S(TM^{\perp}))$,

* If $r < min\{m, n\}$ then M is a $r-$*null* submanifold,

* If $r = n < m$, $S(TM^{\perp}) = \{0\}$ then M is a *coisotropic* null submanifold,
* If $r = m < n$, $S(TM) = \{0\}$ then M is a *isotropic* null submanifold,
* If $r = m = n$, $S(TM) = \{0\} = S(TM^{\perp})$ then M is a *totally* null submanifold.

In view of (2.3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^t_X U, \qquad \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ belong to $\Gamma(TM)$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(tr(TM))$. $\bar{\nabla}$ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. In view of (2.2), we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$. Therefore (2.5) and (2.6) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \quad X \in \Gamma(TM), N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \quad X \in \Gamma(TM), W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$, $\nabla^l_X N$, $D^l(X,W) \in \Gamma(ltr(TM))$, $\nabla^s_X W$, $D^s(X,N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y, A_N X$, $A_W X \in \Gamma(TM)$.

Let P be a projection of TM on $S(TM)$, from (2.1), we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY), \quad X, Y \in \Gamma(TM), \tag{2.10}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E, \quad X \in \Gamma(TM),\ E \in \Gamma(RadTM), \tag{2.11}$$

where $\{\nabla^*_X PY, A^*_E X\}$ belong to $\Gamma(S(TM))$ and $\{h^*(X,PY), \nabla^{*t}_X E\}$ belong to $\Gamma(RadTM)$.

Using (2.10) and (2.11), we have

$$\bar{g}(h^*(X,PY),N) = \bar{g}(A_N X, PY), \tag{2.12}$$

$$\bar{g}(h^l(X,PY),E) = \bar{g}(A^*_E X, PY), \tag{2.13}$$

$$A^*_E E = 0, \quad \bar{g}(h^l(X,E),E) = 0. \tag{2.14}$$

In genereal, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.7), we have

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.15}$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

2.2. Almost paracontact metric manifolds

A paracontact manifold $\bar{M}^{2n+1}$ is a smooth manifold equipped with a 1−form η, a characteristic vector field ξ and a tensor field $\bar{\phi}$ of type $(1,1)$ such that [19]:

$$\eta(\xi) = 1, \tag{2.16}$$

$$\bar{\phi}^2 = I - \eta \otimes \xi, \tag{2.17}$$

$$\bar{\phi}\xi = 0, \tag{2.18}$$

$$\eta \circ \bar{\phi} = 0, \tag{2.19}$$

If we set $D = \ker \eta = \{X \in \Gamma(T\bar{M}) : \ \eta(X) = 0\}$, then $\bar{\phi}$ induces an almost paracomplex structure on the codimension 1 distribution defined by D [19].

Moreover, if the manifold $\bar{M}$ is equipped with a semi-Riemannian metric $\bar{g}$ of signature $(n+1,n)$ which is called *compatible metric* satisfying [21]

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = -\bar{g}(X,Y) + \eta(X)\eta(Y), \quad X,\, Y \in \Gamma(T\bar{M}), \tag{2.20}$$

then we say that $\bar{M}$ is an *almost paracontact metric manifold* with an *almost paracontact metric structure* $(\bar{\phi}, \xi, \eta, \bar{g})$.

From the definition, one can see that [21],

$$\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y), \tag{2.21}$$

$$\bar{g}(X, \xi) = \eta(X). \tag{2.22}$$

If $\bar{g}(X, \bar{\phi}Y) = d\eta(X,Y)$ the almost paracontact metric manifold is said to be a *paracontact metric manifold*.

For an almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$, one can always find a local orthonormal basis which is called $\bar{\phi}-$*basis* $(X_i, \bar{\phi}X_i, \xi)$ $(i = 1, 2, ..., n)$ [21].

An almost paracontact metric manifold $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is a *para-Sasakian manifold* if and only if [21]

$$(\bar{\nabla}_X \bar{\phi})Y = -\bar{g}(X,Y)\xi + \eta(Y)X, \qquad X,\, Y \in \Gamma(T\bar{M}), \tag{2.23}$$

where $\bar{\nabla}$ is Levi-Civita connection of $\bar{M}$.

From (2.23), we also have

$$\bar{\nabla}_X \xi = -\bar{\phi}X. \tag{2.24}$$

Example 2.1. [1] Let $\bar{M} = \mathbb{R}^{2n+1}$ be $(2n+1)-$dimensional real number space with $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$ standard coordinate system. Defining

$$\begin{aligned} \overline{\phi}\frac{\partial}{\partial x_\alpha} &= \frac{\partial}{\partial y_\alpha}, \qquad \overline{\phi}\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \qquad \overline{\phi}\frac{\partial}{\partial z} = 0, \\ \xi &= \frac{\partial}{\partial z}, \qquad \overline{\eta} = dz, \\ \overline{g} &= \eta \otimes \eta + \sum_{\alpha=1}^{n}(dx_\alpha \otimes dx_\alpha - dy_\alpha \otimes dy_\alpha), \end{aligned} \tag{2.25}$$

where $\alpha = 1, 2, ..., n$, then the set $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ is an almost paracontact metric manifold.

3. Totally paracontact umbilical radical transversal null submanifold

The aim of this section is to examine totally paracontact umbilical radical transversal null submanifolds of a para-Sasakian manifold. We state the following definition given [9] for a radical transversal null submanifold of a para-Sasakian manifold.

Definition 3.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a null submanifold of a para-Sasakian manifold $(\bar{M}, \bar{g})$ such that $\xi \in \Gamma(TM)$. If the following conditions given by

$$\bar{\phi}(RadTM) = ltr(TM), \tag{3.1}$$

$$\bar{\phi}(D) = D, \tag{3.2}$$

are provided on M then M is called radical transversal null submanifold,where $S(TM) = D \perp \{\xi\}$ and D is complementary non-degenerate distribution to $\{\xi\}$ in $S(TM)$.

Example 3.1. Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be a $9-$dimensional almost paracontact metric manifold given in Example 2.1. Assume that M is a submanifold defined by

$$x_1 = -y_3,\ x_2 = y_4,$$
$$x_3 = -y_1,\ x_4 = y_2.$$

In this case TM of M is spanned by

$$\left\{ \begin{array}{ll} \Psi_1 = -\frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_1}, \quad \Psi_2 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_2}, \quad \Psi_3 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_3}, \\ \Psi_4 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_4}, \qquad \Psi_5 = \frac{\partial}{\partial z} \end{array} \right\}.$$

Hence the radical distribution $RadTM = Sp\{\Psi_1, \Psi_3\}$ and $ltr(TM)$ is spanned by

$$\Omega_1 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_1}, \quad \Omega_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_3}.$$

It follows that $\bar{\phi}\Psi_1 = -\Omega_1$, $\bar{\phi}\Psi_3 = -\Omega_2$, $\bar{\phi}\Psi_2 = \Psi_4$, $\bar{\phi}\Psi_4 = \Psi_2$. Thus

$$\bar{\phi}(RadTM) = ltr(TM)$$

and

$$\bar{\phi}(D) = D,$$

which implies that M is a radical transversal 2-null submanifold.

Proposition 3.1. [2] *There does not exist an isotropic or totally null radical transversal null submanifold of a para-Sasakian manifold.*

Proposition 3.2. [2] *There exists no 1-null radical transversal null submanifold of a para-Sasakian manifold.*

For a radical transversal null submanifold M of a para-Sasakian manifold $\bar{M}$, assume that ω_1 and ω_2 are the projection morphisms on $S(TM)$ and $RadTM$, respectively. Then, for $X \in \Gamma(TM)$, one can write

$$X = \omega_1 X + \omega_2 X, \tag{3.3}$$

where $\omega_1 X \in \Gamma(S(TM))$ and $\omega_2 X \in \Gamma(RadTM)$.

If we apply $\bar{\phi}$ to (3.3), we get

$$\bar{\phi}X = \bar{\phi}\omega_1 X + \bar{\phi}\omega_2 X. \tag{3.4}$$

Taking $\bar{\phi}\omega_1 X = TX$ and $\bar{\phi}\omega_2 X = QX$ in (3.4), we have

$$\bar{\phi}X = TX + QX, \tag{3.5}$$

where $TX \in \Gamma(S(TM))$ and $QX \in \Gamma(ltr(TM))$.

From (2.23), for a radical transversal null submanifold M, we get

$$\begin{aligned}(\bar{\nabla}_X\bar{\phi})Y &= \bar{\nabla}_X\bar{\phi}Y - \bar{\phi}\bar{\nabla}_X Y \\ &= -\bar{g}(X,Y)\xi + \eta(Y)X.\end{aligned}$$

By use of (2.7), (2.8) with (3.5), we obtain

$$\begin{aligned}-g(X,Y)\xi + \eta(Y)X &= \nabla_X TY + h^l(X,TY) + h^s(X,TY) \\ &\quad -A_{QY}X + \nabla^l_X QY + D^s(X,QY) \\ &\quad -T\nabla_X Y - Q\nabla_X Y - \bar{\phi}h^l(X,Y) \\ &\quad -\bar{\phi}h^s(X,Y).\end{aligned}$$

Considering the tangential, lightlike transversal and screen transversal components of above equation, we get

$$(\nabla_X T)Y = \bar{\phi}h^l(X,Y) + A_{QY}X - g(X,Y)\xi + \eta(Y)X, \tag{3.6}$$

$$h^l(X,TY) + \nabla^l_X QY - Q\nabla_X Y = 0, \tag{3.7}$$

$$h^s(X,TY) + D^s(X,QY) - \bar{\phi}h^s(X,Y) = 0. \tag{3.8}$$

It is well known that the induced connection of a null submanifold is not a metric connection. The following theorem shows the necessary and sufficient condition for the induced connection to be a metric connection.

Theorem 3.1. [2] *Let M be a radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then ∇ is a metric connection on M if and only if $A_{\bar{\phi}Y}X$ has no component in $S(TM)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$.*

Lemma 3.1. *Let M be a radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then for all $X, Y \in (\Gamma(TM) - \{\xi\})$, we have*

$$g(\nabla_X Y, \xi) = \bar{g}(Y, \bar{\phi}X). \tag{3.9}$$

Proof. From (2.7), we obtain

$$g(\nabla_X Y, \xi) = g(\bar{\nabla}_X Y, \xi). \tag{3.10}$$

Since $\bar{\nabla}$ is a metric connection, by use of (3.10), we get

$$g(\nabla_X Y, \xi) = Xg(Y, \xi) - \bar{g}(Y, \bar{\nabla}_X \xi),$$

which implies

$$g(\nabla_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}_X \xi). \tag{3.11}$$

In view of (2.24) and (3.11), we complete the proof. □

Definition 3.2. For a null submanifold M of a para-Sasakian manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$, if the second fundamental form h of M satisfies;

$$h^l(X,Y) = (g(X,Y) - \eta(X)\eta(Y))\alpha_Q + \eta(X)h^l(Y,\xi) + \eta(Y)h^l(X,\xi), \tag{3.12}$$

$$h^s(X,Y) = (g(X,Y) - \eta(X)\eta(Y))\alpha_T + \eta(X)h^s(Y,\xi) + \eta(Y)h^s(X,\xi), \tag{3.13}$$

then M is said to be totally paracontact umbilical radical transversal null submanifold, where $\alpha_Q \in \Gamma(ltr(TM))$, $\alpha_T \in \Gamma(S(TM^{\perp}))$ and $X, Y \in \Gamma(TM)$.

Theorem 3.2. *Let M be a totally paracontact umbilical radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then $\alpha_Q = 0$ if and only if*

$$h^*(X, \bar{\phi}Y) = 0, \tag{3.14}$$

for any $X, Y \in \Gamma(D)$.

Proof. Assume that M is totally paracontact umbilical radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. From (3.6), we get

$$\begin{aligned} -\bar{g}(X,Y)\xi+\eta(Y)X &= \bar{\nabla}_X\bar{\phi}Y-\bar{\phi}\bar{\nabla}_XY \\ &= \nabla_X\bar{\phi}Y+h^l(X,\bar{\phi}Y)+h^s(X,\bar{\phi}Y) \\ &\quad -T\nabla_XY-Q\nabla_XY-\bar{\phi}h^l(X,Y)-\bar{\phi}h^s(X,Y), \end{aligned} \tag{3.15}$$

for any $X,Y\in\Gamma(D)$.
Now, using (3.15) for $X,Y\in\Gamma(D)$ and $Z\in\Gamma(RadTM)$, we have

$$\bar{g}(\nabla_X\bar{\phi}Y,\bar{\phi}Z)-\bar{g}(\bar{\phi}h^l(X,Y),\bar{\phi}Z)=0. \tag{3.16}$$

In view of (2.20), (2.10) in (3.16), we obtain

$$\bar{g}(h^*(X,\bar{\phi}Y),\bar{\phi}Z)+\bar{g}(h^l(X,Y),Z)=0. \tag{3.17}$$

From the definition of totally paracontact umbilical radical transversal null submanifold, we find

$$\bar{g}(h^*(X,\bar{\phi}Y),\bar{\phi}Z)+\bar{g}(g(X,Y)\alpha_Q,Z)=0, \tag{3.18}$$

completes the proof. □

Theorem 3.3. *Let M be a totally paracontact umbilical radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then ∇ is a metric connection if and only if*

$$A_{\bar{\phi}Y}X=\eta(X)Y,$$

for any $X\in\Gamma(TM)$ and $Y\in\Gamma(RadTM)$.

Proof. It is known that the induced connection is a metric connection if and only if [2]

$$\nabla_XY\in\Gamma(RadTM),\ X\in\Gamma(TM),\ Y\in\Gamma(RadTM). \tag{3.19}$$

By use of (3.6), (3.7) and (3.8) we have

$$\begin{aligned} &T\nabla_XY+Q\nabla_XY \\ &+\bar{\phi}h^l(X,Y)+\bar{\phi}h^s(X,Y) = \nabla^l_XQY-A_{QY}X+D^s(X,QY). \end{aligned} \tag{3.20}$$

Now, using (3.12) and (3.13) in (3.20), we obtain

$$\begin{aligned} &T\nabla_XY+Q\nabla_XY \\ &+\eta(X)\bar{\phi}h^l(Y,\xi)+\eta(X)\bar{\phi}h^s(Y,\xi) = -A_{LY}X+\nabla^l_XQY+D^s(X,QY). \end{aligned} \tag{3.21}$$

Taking the tangential part of (3.21), we get

$$T\nabla_XY+\eta(X)\bar{\phi}h^l(Y,\xi)=-A_{\bar{\phi}Y}X. \tag{3.22}$$

Also, from (2.7) and (2.24), we have

$$-\bar{\phi}Y = h^l(Y,\xi), \qquad Y \in \Gamma(RadTM). \tag{3.23}$$

Using (3.23) in (3.22), we find

$$T\nabla_X Y = -A_{\bar{\phi}Y}X + \eta(X).$$

The proof follows from the previous equation. □

4. Screen slant radical transversal null submanifold

Now, we investigate screen slant radical transversal null submanifolds of a para-Sasakian manifold.

Definition 4.1. Let M be a $2q-$lightlike submanifold of a para-Sasakian manifold $\bar{M}$ of index $2q$ such that $2q \leq \dim(M)$. We say that M is a screen slant radical transversal null submanifold of $\bar{M}$ if the following conditions are provided:

i) $\bar{\phi}(RadTM) = ltr(TM)$,

ii) For each non-zero vector field X tangent to D at $x \in U \subset M$, the angle $\theta(X)$ between, $\bar{\phi}X$ and the vector space D_x is constant, i.e. it is independent of the choice of $x \in U \subset M$ and $X \in D_x$, where D is complementary non-degenerate distribution to $\{\xi\}$ in $S(TM)$ such that $S(TM) = D\perp\{\xi\}$.

This constant angle $\theta(X)$ is called *slant angle* of distribution D. From the definition, we have the following decomposition:

$$TM = RadTM \perp D \perp \{\xi\}. \tag{4.1}$$

Now, we denote the projection on $RadTM$ and D in TM by f_1 and f_2, respectively. Thus, we get

$$X = f_1X + f_2X + \eta(X)\xi, \qquad \forall X \in \Gamma(TM). \tag{4.2}$$

Applying $\bar{\phi}$ to the both sides of equation (4.2), we get

$$\bar{\phi}X = \bar{\phi}f_1X + \bar{\phi}f_2X, \tag{4.3}$$

which implies

$$\bar{\phi}X = \bar{\phi}f_1X + TX + LX, \qquad \forall X \in \Gamma(TM), \tag{4.4}$$

where TX and LX denote the tangential and transversal component of $\bar{\phi}f_2X$,, respectively. Therefore we have $\bar{\phi}f_1X \in \Gamma(ltr(TM))$, $TX \in \Gamma(D)$ and $LX \in \Gamma(S(TM^\perp))$. Similarly, we denote the projections on $ltr(TM)$ and $S(TM^\perp)$ in $tr(TM)$ by t_1 and t_2, respectively. Then, we get

$$U = t_1U + t_2U, \qquad \forall U \in \Gamma(tr(TM)). \tag{4.5}$$

On applying $\bar{\phi}$ to (4.5), we have

$$\bar{\phi}U = \bar{\phi}t_1U + \bar{\phi}t_2U, \tag{4.6}$$

which gives

$$\bar{\phi}U = \bar{\phi}t_1U + P_1U + P_2U, \tag{4.7}$$

where P_1U and P_2U denote tangential and transversal component of $\bar{\phi}t_2U$, respectively. Hence, we have $\bar{\phi}t_1U \in \Gamma(RadTM)$, $P_1U \in \Gamma(D)$ and $P_2U \in \Gamma(S(TM^{\perp}))$.

By use of (2.7), (2.8), (2.9), (2.23), (4.4) and (4.7), and comparing tangential, lightlike transversal and screen transversal components, we have

$$\begin{aligned} -\bar{g}(X,Y)\xi + \eta(Y)X &= \nabla_X TY - A_{LY}X - A_{\bar{\phi}f_1Y}X \\ &\quad -T\nabla_X Y + P_1h^s(X,Y) + \bar{\phi}h^l(X,Y), \end{aligned} \tag{4.8}$$

$$h^l(X,TY) + D^l(X,LY) + \nabla^l_X\bar{\phi}f_1Y = \bar{\phi}f_1\nabla^l_XY, \tag{4.9}$$

$$D^s(X,\bar{\phi}f_1Y) + h^s(X,TY) = P_2h^s(X,Y) - \nabla^s_XLY + L\nabla_XY. \tag{4.10}$$

Theorem 4.1. *Let M be a $2q-$null submanifold of a para-Sasakian manifold $\bar{M}$ with $\xi \in \Gamma(TM)$ such that $\bar{\phi}(RadTM) = ltr(TM)$. Then M is a screen slant radical transversal null submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2X = \lambda(X - \eta(X)\xi), \forall X \in \Gamma(D).$$

Proof. Assume that there exists a constant λ such that for all $X \in \Gamma(D)$, $P^2X = \lambda(X - \eta(X)\xi) = \lambda\bar{\phi}^2X$. Then we have

$$\begin{aligned} \cos\theta(X) &= \frac{g(\bar{\phi}X, PX)}{\|PX\|\|\bar{\phi}X\|} \\ &= -\frac{g(X, \bar{\phi}PX)}{\|PX\|\|\bar{\phi}X\|} \\ &= -\frac{g(X, P^2X)}{\|PX\|\|\bar{\phi}X\|} \\ &= -\frac{g(X, \bar{\phi}^2X)}{\|PX\|\|\bar{\phi}X\|} \\ &= \lambda\frac{g(\bar{\phi}X, \bar{\phi}X)}{\|PX\|\|\bar{\phi}X\|}. \end{aligned}$$

From above equation, we find

$$\cos\theta(X) = \lambda\frac{\|\bar{\phi}X\|}{\|PX\|}. \tag{4.11}$$

On the other hand, $\|PX\| = \|\bar{\phi}X\| \cos\theta(X)$, implies

$$\cos\theta(X) = \frac{\|PX\|}{\|\bar{\phi}X\|}. \tag{4.12}$$

From (4.11) and (4.12), we get

$$\cos^2\theta(X) = \lambda \text{ (constant)}.$$

Thus M is a screen slant radical transversal null submanifold.

On the other hand, suppose that M is a screen slant radical transversal null submanifold. Then $\cos^2\theta(X) = \lambda$ (constant). From (4.12), we have

$$\frac{\|PX\|^2}{\|\bar{\phi}X\|^2} = \lambda.$$

Now $g(PX, PX) = \lambda g(\bar{\phi}X, \bar{\phi}X)$, which gives $g(X, P^2X) = \lambda g(X, \bar{\phi}^2X)$. Therefore $g(X, (P^2 - \lambda\bar{\phi}^2)X) = 0$. Since X is a non-null vector, we obtain

$$\begin{aligned} P^2 - \lambda\bar{\phi}^2X &= 0, \\ P^2 &= \lambda\bar{\phi}^2X = \lambda(X - \eta(X)\xi). \end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then $RadTM$ is integrable if and only if*

$$D^s(Y, \bar{\phi}X) = D^s(X, \bar{\phi}Y) \quad \textit{and} \quad A_{\bar{\phi}X}Y = A_{\bar{\phi}Y}X,$$

for all $X, Y \in \Gamma(RadTM)$.

Proof. Suppose that M is a screen slant radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. From (4.10), we get

$$D^s(X, \bar{\phi}Y) = P_2h^s(X, Y) + L\nabla_XY, \quad X, Y \in \Gamma(RadTM). \tag{4.13}$$

Interchanging the roles of X and Y in (4.13), we obtain

$$D^s(Y, \bar{\phi}X) = P_2h^s(Y, X) + L\nabla_YX. \tag{4.14}$$

From (4.13) with (4.14), we have

$$D^s(X, \bar{\phi}Y) - D^s(Y, \bar{\phi}X) = L(\nabla_XY - \nabla_YX) = L[X, Y]. \tag{4.15}$$

By using (4.8), we get

$$A_{\bar{\phi}Y}X + T\nabla_YX = P_1h^s(X, Y) + \bar{\phi}h^l(X, Y). \tag{4.16}$$

Again interchanging roles of X and Y in (4.16), we get

$$A_{\bar{\phi}X}Y + T\nabla_X Y = P_1 h^s(Y,X) + \bar{\phi}h^l(Y,X). \tag{4.17}$$

From (4.16) and (4.17), we obtain

$$A_{\bar{\phi}X}Y - A_{\bar{\phi}Y}X = T[X,Y]. \tag{4.18}$$

In view of (4.15) and (4.18), the proof completes. □

Theorem 4.3. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then the distribution D is integrable if and only if*

$$h^l(X,TY) + D^l(X,LY) = h^l(Y,TX) + D^l(Y,LX),$$

for all $X, Y \in \Gamma(D)$.

Proof. From (4.9), we get

$$h^l(X,TY) + D^l(X,LY) = \bar{\phi}f_1\nabla_X Y, \quad X, Y \in \Gamma(D). \tag{4.19}$$

If we change the roles of X and Y in (4.19), we find

$$h^l(Y,TX) + D^l(Y,LX) = \bar{\phi}f_1\nabla_Y X. \tag{4.20}$$

From (4.19) and (4.20), we have

$$h^l(X,TY) - h^l(Y,TX) + D^l(X,LY) - D^l(Y,LX) = \bar{\phi}f_1[X,Y]. \tag{4.21}$$

The proof follows from (4.21). □

Theorem 4.4. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then $S(TM)$ defines a totally geodesic foliation if and only if*

$$\bar{g}(A^*_{\bar{\phi}N}X, TY) = -\bar{g}(D^l(X,LY), \bar{\phi}N),$$

for all $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. We know that, $S(TM)$ defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X Y, N) = 0, \quad X, Y \in \Gamma(S(TM)), \;\; N \in \Gamma(ltr(TM)).$$

From (2.7), it can be easily seen that

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X Y, N). \tag{4.22}$$

From (2.20), (2.23) and (4.22), we obtain

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\bar{\nabla}_X \bar{\phi}Y, \bar{\phi}N). \tag{4.23}$$

By use of (2.7), (2.8), (4.4) and (4.23), we get

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(h^l(X, TY) + D^l(X, LY), \bar{\phi}N).$$

From (2.13), we have

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(A^*_{\bar{\phi}N} X, TY) - \bar{g}(D^l(X, LY), \bar{\phi}N).$$

Thus the proof completes. □

Theorem 4.5. *Let M be a screen slant radical transversal null submanifold of a para-Sasakian manifold $\bar{M}$. Then the distribution $\mathring{D} = RadTM \perp \{\xi\}$ defines a totally geodesic foliation on M if and only if*

$$A_{LZ} = h^*(X, TV) + \eta(V)X,$$

for all $X \in \Gamma(\mathring{D})$ and $V \in \Gamma(D)$.

Proof. By using the definition of radical transversal null submanifold, $\mathring{D}$ is a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X Y, V) = 0, \quad X, Y \in \Gamma(\mathring{D}), V \in \Gamma(S(TM)).$$

From (2.7), we obtain

$$\bar{g}(\nabla_X Y, V) = -\bar{g}(Y, \bar{\nabla}_X V). \tag{4.24}$$

In view of equations (2.7), (2.20), (2.23) and (4.24), we get

$$\bar{g}(\nabla_X Y, V) = \bar{g}(\bar{\phi}Y, X)\eta(V) + \bar{g}(\bar{\phi}Y, \nabla_X \bar{\phi}V).$$

By using (2.7),(2.8), (2.10), (4.4) and (4.23), we can write

$$\bar{g}(\nabla_X Y, V) = \eta(V)\bar{g}(\bar{\phi}Y, X) + \bar{g}(\bar{\phi}Y, h^*(X, TV)) - \bar{g}(\bar{\phi}Y, A_{LV}),$$

which implies

$$\bar{g}(\nabla_X Y, Z) = \bar{g}(\bar{\phi}Y, -A_{LZ} + h^*(X, TZ) + \eta(Z)X). \tag{4.25}$$

The proof follows from (4.25). □

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