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h-EXPONENTIAL CHANGE OF FINSLER METRIC

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Abstract. In this paper, we studied a Finsler space whose metric is given by an h-exponential change and obtain the Cartan connection coefficients for the change. We also find the necessary and sufficient condition for an h-exponential change of Finsler metric to be projective.

Keywords:Finsler space; h-exponential change; Projective change

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with the fundamental function $L(x, y)$. The metric tensor, angular metric tensor and Cartan tensor are defined by $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$, $h_{ij} = g_{ij} - l_i l_j$ and $C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk}$ respectively, where $\dot{\partial}_k = \frac{\partial}{\partial y^k}$. The Cartan connection is given by $C\Gamma = (F^i_{jk}, N^i_k, C^i_{jk})$. The h- and v-covariant derivatives $X_{i|j}$ and $X_i|_j$ of a covariant vector field X_i are defined by [10, 14]

$$X_{i|j} = \partial_j X_i - N^r_j\,\dot{\partial}_r X_i - X_r F^r_{ij}\,, \tag{1.1}$$

and

$$X_i|_j = \dot{\partial}_j X_i - X_r C^r_{ij}\,, \tag{1.2}$$

where $\partial_k = \frac{\partial}{\partial x^k}$.

In 2012, H. S. Shukla et.al.[15] considered a Finsler space $\overline{F}^n = (M^n, \overline{L})$, whose Fundamental metric function is an exponential change of Finsler metric function given by

$$\overline{L} = L\,e^{\frac{\beta}{L}}\,, \tag{1.3}$$

where $\beta = b_i(x)y^i$ is 1-form on manifold M^n.

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H. Izumi [6] introduced the concept of an h-vector $b_i(x,y)$ which is v-covariant constant with respect to the Cartan connection and satisfies $L\,C^h_{ij}\,b_h = \rho\,h_{ij}$, where ρ is a non-zero scalar function and C^i_{jk} are components of Cartan tensor. Thus if b_i is an h-vector then

$$(i)\, b_i|_k = 0, \qquad (ii)\, L\,C^h_{ij} b_h = \rho h_{ij}\,. \tag{1.4}$$

From the above definition, we have

$$L\,\dot\partial_j b_i = \rho h_{ij}\,, \tag{1.5}$$

which shows that b_i is a function of directional argument also. H. Izumi [6] proved that the scalar ρ is independent of directional argument. Gupta and Pandey [4] proved that if the h-vector b_i is gradient then the scalar ρ is constant. M. Matsumoto [9] discussed the Cartan connection of Randers change of Finsler metric, while B. N. Prasad [13] obtained the Cartan connection of $(M^n, {}^*L)$ where ${}^*L(x,y)$ is given by ${}^*L(x,y) = L(x,y) + b_i(x,y)y^i$, and $b_i(x,y)$ is an h-vector. Gupta and Pandey [2, 3] discussed the hypersurface of a Finsler space whose metric is given by certain transformation with an h-vector.

A geodesic of $F^n = (M^n, L)$ is given by system of differential equation

$$\frac{d^2x^i}{dx^2} + 2G^i(x, \frac{dx}{ds}) = 0\,,$$

where $G^i(x,y)$ are positively homogeneous of degree two in y^i and are given by

$$2G^i = g^{ij}(y^r \dot\partial_j \partial_r F - \partial_j F), \quad F = \frac{L^2}{2}\,,$$

where g^{ij} is the inverse of g_{ij}.

A transformation from $F^n = (M^n, L)$ to $\overline{F}^n = (M^n, \overline{L})$ is called projective change if a geodesic on F^n is also a geodesic on $\overline{F}^n$ and vice-versa. The change $L \mapsto \overline{L}$ is projective change if and only if there exits a scaler function $P(x,y)$ which is positive homogeneous of degree one in y^i and satisfies [1]

$$\overline{G}^i(x,y) = G^i(x,y) + P(x,y)y^i\,.$$

I. Y. Lee and H. S. Park [12] studied projective change between a Finsler space with (α, β)-metric and associated Riemannian space. M. Hashiguchi and Y. Ichijyo [5] obtained the condition for Randers change to be projective whereas Gupta and Pandey[4] derived the condition for Kropina change to be projective. Many authors [7, 11, 15] studied the projective change in different spaces.

In the present paper, we consider a Finsler space ${}^*F^n = (M^n, {}^*L)$, whose metric function ${}^*L$, an h-exponential change of metric, is given by

$${}^*L = L\,e^{\frac{\beta}{L}}\,, \tag{1.6}$$

where $\beta = b_i(x,y)y^i$ and b_i is an h-vector. And we obtain the relation between Cartan connection coefficients of F^n and ${}^*F^n$. We also derive the condition for an h-exponential change of metric to be projective.

2. Finsler space ${}^*F^n = (M^n, {}^*L)$

We shall use the following notations $L_i = \dot{\partial}_i L = l_i$, $L_{ij} = \dot{\partial}_i \dot{\partial}_j L$, $L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L$. The quantities corresponding to ${}^*F^n$ is denoted by asterisk over that quantity. From (1.6), we have

$$ {}^*L_i = e^{\tau}\big(m_i + l_i\big). \tag{2.1} $$

$$ {}^*L_{ij} = e^{\tau}(1+\rho-\tau)L_{ij} + \frac{e^{\tau}}{L} m_i m_j \,. \tag{2.2} $$

$$ \begin{aligned} {}^*L_{ijk} = {} & e^{\tau}\big(1+\rho-\tau\big)L_{ijk} + \big(\rho-\tau\big)\frac{e^{\tau}}{L}\big[m_i L_{jk} + m_j L_{ik} + m_k L_{ij}\big] \\ & - \frac{e^{\tau}}{L^2}\big[m_j m_k l_i + m_i m_k l_j + m_i m_j l_k - m_i m_j m_k\big], \end{aligned} \tag{2.3} $$

where $\tau = \frac{\beta}{L}$, $m_i = b_i - \tau l_i$. The normalised supporting element, the metric tensor and Cartan tensor of ${}^*F$ are obtained as

$$ {}^*l_i = e^{\tau}\big(m_i + l_i\big), \tag{2.4} $$

$$ {}^*g_{ij} = \nu\, e^{2\tau} g_{ij} + e^{2\tau}\Big(2\tau^2 - \tau - \rho\Big) l_i l_j + e^{2\tau}\Big(1 - 2\tau\Big)\big(b_i l_j + b_j l_i\big) + 2e^{2\tau} b_i b_j, \tag{2.5} $$

$$ {}^*C_{ijk} = \nu\, e^{2\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k + \frac{1}{2L} e^{2\tau}(2\nu - 1)\big(m_i h_{kj} + m_j h_{ki} + m_k h_{ij}\big), \tag{2.6} $$

where $\nu = 1 + \rho - \tau$.

For the computation of the inverse metric tensor, we use the following lemma [8] :

Lemma 2.1. *Let (m_{ij}) be a non-singular matrix and $l_{ij} = m_{ij} + n_i n_j$. The elements l^{ij} of the inverse matrix and determinant of the matrix (l_{ij}) are given by*

$$ l^{ij} = m^{ij} - \big(1 + n_k n^k\big)^{-1} n^i n^j, \quad det\big(l_{ij}\big) = \big(1 + n_k n^k\big) det\big(m_{ij}\big) $$

respectively, where m^{ij} are elements of inverse matrix $\big(m_{ij}\big)$ and $n^k = m^{ki} n_i$.

The inverse metric tensor of ${}^*F^n$ is derived as follows :

$$ {}^*g^{ij} = \frac{e^{-2\tau}}{\nu}\Big[g^{ij} - \frac{1}{m^2+\nu} b^i b^j + \frac{\tau-\nu}{m^2+\nu}\Big(b^i l^j + b^j l^i\Big) - l^i l^j \Big\{\frac{\tau-\nu}{m^2+\nu}(m^2+\tau) - \rho\Big\}\Big], \tag{2.7} $$

where b is magnitude of the vector $b^i = g^{ij} b_j$.
From (2.6) and (2.7), we obtain

$$
\begin{aligned}
{}^{*}C^h_{ij} = & C^h_{ij} + \frac{1}{m^2+\nu} C_{ijk} b^k (-b^h + 2\tau\, l^h - \rho l^h - l^h) \\
& + \frac{2}{\nu\, L}\Big[m_i m_j m^h + \frac{1}{m^2+\nu} m_i m_j m^2 (-b^h + 2\tau\, l^h - \rho l^h - l^h)\Big] \\
& + \frac{1}{2\nu\, L}(2\nu - 1)\Big[m_i h^h_j + m_j h^h_i + m^h h_{ij} \\
& + \frac{1}{m^2+\nu}(-b^h + 2\,\tau\, l^h - \rho l^h - l^h)\,(2 m_i m_j + m^2 h_{ij})\Big].
\end{aligned} \tag{2.8}
$$

3. Cartan connection of the space ${}^{*}F^n$

Let $C^{*}\Gamma = ({}^{*}F^i_{jk}, {}^{*}N^i_j, {}^{*}C^i_{jk})$ be the Cartan connection for the Finsler space ${}^{*}F^n = (M^n, {}^{*}L)$. Since $L_{i|j} = 0$ for the Cartan connection, we have

$$
\partial_j L_i = L_r F^r_{ij} + \dot{\partial}_r L_i\, N^r_j\, . \tag{3.1}
$$

Differentiating (2.1) with respect to x^j, and using (1.1) and (3.1), we get

(3.2)
$$
{}^{*}L_{ir}{}^{*}N^r_j + {}^{*}L_r{}^{*}F^r_{ij} = \Big[e^\tau\, \nu\, L_{ir} + \frac{e^\tau}{L} m_r m_i \Big] N^r_j + \Big[e^\tau \big(m_r + l_r \big) \Big] F^r_{ij} + \frac{e^\tau \beta_j\, m_i}{L} + e^\tau b_{i|j}\, .
$$

Equation (3.2) serves the purpose to find relation between a Cartan connection of ${}^{*}F^n$ and F^n. For this, we put

$$
D^i_{jk} = {}^{*}F^i_{jk} - F^i_{jk}\, . \tag{3.3}
$$

With the help of (3.3), the equation (3.2) becomes

$$
\Big[e^\tau\, \nu\, L_{ir} + \frac{e^\tau}{L} m_i m_r \Big] D^r_{0j} + \Big[e^\tau \big(m_r + l_r \big) \Big] D^r_{ij} = \frac{e^\tau \beta_{|j} m_i}{L} + e^\tau b_{i|j}\, , \tag{3.4}
$$

where the subscript '0' denote the contraction by y^i.
Differentiating (2.2) with respect to x^k, and using (1.1) and (3.1), we have

(3.5)
$$
\begin{aligned}
& e^\tau\, \nu \Big[L_{ijr} D^r_{0k} + L_{rj} D^r_{ik} + L_{ir} D^r_{jk} \Big] + \big(\nu - 1\big) \frac{e^\tau}{L} \Big[m_r L_{ij} + m_i L_{jr} + m_j L_{ir} \Big] D^r_{0k} \\
& - \frac{e^\tau}{L^2} \Big[m_i m_j l_r + m_j m_r l_i + m_r m_i l_r - m_i m_j m_r \Big] D^r_{0k} + \frac{e^\tau}{L} \Big[m_r m_j D^r_{ik} + m_i m_r D^r_{jk} \Big] \\
& - \frac{e^\tau \big(\nu - 1\big)}{L} L_{ij}\, \beta_{|k} - \frac{e^\tau}{L^2} \beta_{|k} m_i m_j - e^\tau \rho_k L_{ij} = 0\, ,
\end{aligned}
$$

where $\rho_k = \rho_{|k} = \partial_k \rho$.

Theorem 3.1. *The Cartan connection of* ${}^*F^n$ *is completely determined by the equations (3.4) and (3.5) .*

To prove this, first we propose the following lemma :

Lemma 3.1. *The system of equations*

$$\begin{aligned} (i) \quad & {}^*L_{ir}\, A^r = B_i \\ (ii) \quad & {}^*L_r\, A^r = B \end{aligned}$$

has a unique solution A^r *for given* B *and* B_i.

Proof. Using (2.2), equation (i) becomes

$$\frac{e^\tau}{L}\Big[\nu\big(g_{ir} - l_i l_r\big) + m_i m_r\Big] A^r = B_i\,. \tag{3.6}$$

Contracting by b^i, we get

$$m_r A^r = \frac{L B_\beta}{e^\tau}\big(m^2+\nu\big)^{-1}\,, \tag{3.7}$$

here we used subscript β to denote the contraction by b^i, i.e. $B_\beta = B_i b^i$.
From (2.1) and *(ii)* , we have

$$l_r A_r = \frac{B}{e^\tau} - \frac{L B_\beta}{e^\tau}\big(m^2+\nu\big)^{-1}\,. \tag{3.8}$$

Using (3.7) and (3.8), equation(3.6) becomes

$$g_{ir}A^r = \frac{L B_i}{\nu\, e^\tau} + l_i\Big[\frac{B}{e^\tau} - \frac{L B_\beta}{e^\tau}\big(m^2+\nu\big)^{-1}\Big] - \frac{m_i L B_\beta}{\nu\, e^\tau}\big(m^2+\nu\big)^{-1}\,,$$

contracting by g^{ij} , we have

$$A^j = \frac{L B^j}{\nu\, e^\tau} + l^j\Big[\frac{B}{e^\tau} - \frac{L B_\beta}{e^\tau}\big(m^2+\nu\big)^{-1}\Big] - \frac{m^j L B_\beta}{\nu\, e^\tau}\big(m^2+\nu\big)^{-1}\,, \tag{3.9}$$

which is concrete form of the solution A^j. □

Now we are in the position to prove the theorem. We will find an explicit expression of difference tensor D^i_{jk} in three steps. Firstly, we will find D^i_{00} and then D^i_{0k} and in the last D^i_{jk}.

Taking symmetric and skew-symmetric part of (3.4), we have

(3.10)
$$\begin{aligned} 2e^\tau\big(m_r + l_r\big)D^r_{ij} + \Big[\nu\, e^\tau L_{ir} + \frac{e^\tau}{L} m_i m_r\Big]D^r_{0j} + \Big[\nu\, e^\tau L_{jr} + \frac{e^\tau}{L} m_j m_r\Big]D^r_{0i} \\ = \frac{e^\tau}{L}\big(\beta_{|j} m_i + \beta_{|i} m_j\big) + 2e^\tau E_{ij}\,, \end{aligned}$$

and

$$\Big[\nu\, e^{\tau} L_{ir} + \frac{e^{\tau}}{L} m_i m_r\Big] D^r_{0j} - \Big[\nu\, e^{\tau} L_{jr} + \frac{e^{\tau}}{L} m_j m_r\Big] D^r_{0i} = \frac{e^{\tau}}{L}\big(\beta_{|j} m_i - \beta_{|i} m_j\big) + 2e^{\tau} F_{ij}, \tag{3.11}$$

where $2E_{ij} = b_{j|i} + b_{i|j}, \quad 2F_{ij} = b_{i|j} - b_{j|i}$.
Contracting (3.10) and (3.11) by y^j, we get

$$2e^{\tau}\big(m_r + l_r\big) D^r_{0i} + \Big[\nu\, e^{\tau} L_{ir} + \frac{e^{\tau}}{L} m_i m_r\Big] D^r_{00} = \frac{e^{\tau}}{L}\beta_{|0} m_i + 2e^{\tau} E_{i0}\,, \tag{3.12}$$

and

$$\Big[\nu\, e^{\tau} L_{ir} + \frac{e^{\tau}}{L} m_i m_r\Big] D^r_{00} = \frac{e^{\tau}}{L}\beta_{|0} m_i + 2e^{\tau} F_{i0}\,, \tag{3.13}$$

which may be re-written as

$$^{*}L_{ir} D^r_{00} = \frac{e^{\tau}}{L}\beta_{|0} m_i + 2e^{\tau} F_{i0}\,, \tag{3.14}$$

where $\beta_{|0} = \beta_{|j} y^j$. Transvecting (3.13) by m^i, we obtain

$$m_r D^r_{00} = \big(m^2 + \nu\big)^{-1}\big(\beta_{|0} m^2 + 2LF_{\beta 0}\big)\,. \tag{3.15}$$

Contracting (3.12) by y^i, we get

$$2e^{\tau}\big(m_r + l_r\big) D^r_{00} = 2e^{\tau} E_{00}.$$

i.e.

$$^{*}L_r D^r_{00} = e^{\tau} E_{00}\,. \tag{3.16}$$

Applying Lemma 3.1 in equation (3.14) and (3.16), we have

$$\begin{aligned} D^i_{00} = & \frac{L}{\nu\, e^{\tau}}\Big[\frac{e^{\tau}}{L}\beta_{|0} m^i + 2e^{\tau} F^i_0\Big] + l^i\Big[E_{00} - \frac{L}{e^{\tau}}\big(m^2+\nu\big)^{-1}\big(\frac{e^{\tau}}{L}\beta_{|0} m^2 + 2e^{\tau} F_{\beta 0}\big)\Big] \\ & - \frac{m^i L}{\nu\, e^{\tau}}\big(m^2+\nu\big)^{-1}\Big[\frac{e^{\tau}}{L}\beta_{|0} m^2 + 2e^{\tau} F_{\beta 0}\Big]\,. \end{aligned} \tag{3.17}$$

Here we used $m^i b_i = m_i m^i = m^2$. Also we note that $E_{00} = E_{ij} y^i y^j = b_{i|j} y^i y^j = (b_i y^i)_{|j} y^j = \beta_{|0}$, $F^i_0 = g^{ij} F_{j0}$.

Secondly, applying Christoffel process with respect to indices i, j, k in equation

(3.5), we have

(3.18)

$$\begin{aligned}&\nu\, e^{\tau}\Big[L_{ijr}D^r_{0k}+L_{jkr}D^r_{0i}-L_{kir}D^r_{0j}\Big]+2D^r_{ik}\Big[\nu\, e^{\tau}\,L_{jr}+\frac{e^{\tau}}{L}m_r m_j\Big]\\ &+\frac{e^{\tau}}{L}D^r_{0k}\mathfrak{S}_{(rij)}\Big[(\nu-1)m_rL_{ij}-\frac{m_im_jl_r}{L}\Big]+\frac{e^{\tau}}{L}D^r_{0i}\mathfrak{S}_{(rjk)}\Big[(\nu-1)m_rL_{jk}-\frac{m_jm_kl_r}{L}\Big]\\ &-\frac{e^{\tau}}{L}D^r_{0j}\mathfrak{S}_{(rki)}\Big[(\nu-1)m_rL_{ki}-\frac{m_km_il_r}{L}\Big]-e^{\tau}\Big[\rho_kL_{ij}+\rho_iL_{jk}-\rho_jL_{ki}\Big]\\ &-\big(\nu-1\big)\frac{e^{\tau}}{L}\big(\beta_{|k}L_{ij}+\beta_{|i}L_{jk}-\beta_{|j}L_{ki}\big)-\frac{e^{\tau}}{L^2}\Big[\beta_{|k}m_im_j+\beta_{|i}m_jm_k-\beta_{|j}m_km_i\Big]\\ &+\frac{e^{\tau}}{L^2}\Big[m_im_jm_rD^r_{0k}+m_jm_km_rD^r_{0i}-m_km_im_rD^r_{0j}\Big]=0\,,\end{aligned}$$

where $\mathfrak{S}_{(ijk)}$ denote cyclic interchange of indices i,j,k and summation. Contracting by y^k, above equation becomes

$$\begin{aligned}&\nu\, e^{\tau}\Big[L_{ijr}D^r_{00}-L_{jr}D^r_{0i}+L_{ir}D^r_{0j}\Big]+2D^r_{0i}\Big[\nu\, e^{\tau}\,L_{jr}+\frac{e^{\tau}}{L}m_rm_j\Big]\\ &+\frac{e^{\tau}}{L}D^r_{00}\mathfrak{S}_{(rij)}\Big[(\nu-1)m_rL_{ij}-\frac{m_im_jl_r}{L}\Big]-\frac{e^{\tau}}{L}D^r_{0i}\frac{m_rm_jl_k}{L}y^k\\ &+\frac{e^{\tau}}{L}D^r_{0j}\frac{m_rm_il_k}{L}y^k+\frac{e^{\tau}}{L^2}m_im_jm_rD^r_{00}-(\nu-1)\frac{e^{\tau}}{L}\beta_{|0}L_{ij}\\ &-\frac{e^{\tau}}{L^2}\beta_{|0}m_im_j-e^{\tau}\rho_0L_{ij}=0\,.\end{aligned}\tag{3.19}$$

Adding (3.11) and (3.19) , we have

$$^*L_{ir}D^r_{0j}=G_{ij}\,,\tag{3.20}$$

where

(3.21)

$$\begin{aligned}2G_{ij}=&\frac{e^{\tau}}{L}\big(\beta_{|j}m_i-\beta_{|i}m_j\big)-e^{\tau}\nu\,L_{ijr}D^r_{00}-\frac{e^{\tau}}{L}D^r_{00}\mathfrak{S}_{(rij)}\Big[(\nu-1)m_rL_{ij}-\frac{m_im_jm_r}{L}\Big]\\ &+2e^{\tau}F_{ij}-\frac{e^{\tau}}{L^2}m_rm_im_jD^r_{00}+\frac{(\nu-1)}{L}e^{\tau}\beta_{|0}L_{ij}+\frac{e^{\tau}}{L^2}B_0m_im_j+e^{\tau}\rho_0L_{ij}\,.\end{aligned}$$

Equation (3.12) can be written as

$$^*L_rD^r_{0j}=G_j\,,\tag{3.22}$$

where

$$2G_j=\frac{e^{\tau}}{L}\beta_{|0}m_j+2e^{\tau}E_{j0}+\Big[-e^{\tau}\nu\,L_{jr}-\frac{e^{\tau}m_jm_r}{L}\Big]D^r_{00}\,.$$

Using (3.13), the above equation may be written as

$$G_j=e^{\tau}\big(E_{j0}-F_{j0}\big).\tag{3.23}$$

Applying Lemma 3.1 in equation (3.20) and (3.22), we obtain

$$D^i_{0j} = \frac{LG^i_j}{\nu\, e^\tau} + \frac{l^i}{e^\tau}\Big[G_j - LG_{\beta j}\big(m^2+\nu\big)^{-1}\Big] - \frac{m^i LG_{\beta j}}{\nu\, e^\tau}\big(m^2+\nu\big)^{-1}. \tag{3.24}$$

Finally, the equation (3.10) may be written as

$$^*L_r D^r_{ik} = H_{ik}\,, \tag{3.25}$$

where

$$\begin{aligned} 2H_{ik} = {}& \frac{e^\tau}{L}\big(\beta_{|k} m_i + \beta_{|i} m_k\big) + e^\tau E_{ik} - \Big[e^\tau \nu\, L_{ir} + \frac{e^\tau}{L} m_i m_r\Big] D^r_{0k} \\ & - \Big[e^\tau \nu\, L_{kr} + \frac{e^\tau}{L} m_k m_r\Big] D^r_{0i}\,. \end{aligned} \tag{3.26}$$

Equation (3.18) may be written as

$$^*L_{rj} D^r_{ik} = H_{jik}\,, \tag{3.27}$$

where

(3.28)

$$\begin{aligned} 2H_{jik} = {}& -\nu\, e^\tau\Big[L_{ijr}D^r_{0k} + L_{jkr}D^r_{0i} - L_{kir}D^r_{0j}\Big] + e^\tau\Big[\rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki}\Big] \\ & - \frac{e^\tau}{L} D^r_{0k}\mathfrak{S}_{(rij)}\Big[(\nu-1) m_r L_{ij} - \frac{m_i m_j l_r}{L}\Big] - \frac{e^\tau}{L} D^r_{0i}\mathfrak{S}_{(rjk)}\Big[(\nu-1) m_r L_{jk} - \frac{m_j m_k l_r}{L}\Big] \\ & + \frac{e^\tau}{L} D^r_{0j}\mathfrak{S}_{(rki)}\Big[(\nu-1) m_r L_{ki} - \frac{m_k m_i l_r}{L}\Big] \\ & + (\nu-1)\frac{e^\tau}{L}\big(\beta_{|k}L_{ij} + \beta_{|i}L_{jk} - \beta_{|j}L_{ki}\big) + \frac{e^\tau}{L^2}\Big[\beta_{|k} m_i m_j + \beta_{|i} m_j m_k - \beta_{|j} m_k m_i\Big] \\ & - \frac{e^\tau}{L^2}\Big[m_i m_j m_r D^r_{0k} + m_j m_k m_r D^r_{0i} - m_k m_i m_r D^r_{0j}\Big]\,. \end{aligned}$$

Applying Lemma 3.1 in (3.25) and (3.27), we have

$$D^j_{ik} = \frac{LH^j_{ik}}{\nu\, e^\tau} + \frac{l^j}{e^\tau}\Big[H_{ik} - LH_{\beta ik}\big(m^2+\nu\big)^{-1}\Big] - \frac{m^j L}{\nu\, e^\tau} H_{\beta ik}\big(m^2+\nu\big)^{-1}, \tag{3.29}$$

where we put $H^j_{ik} = g^{jm}H_{mik}$.
Thus in view of (3.3), we get the Cartan connection coefficient $^*F^i_{jk}$. This completes the proof of theorem (3.1).

Now, suppose Cartan connection coefficients for both spaces F^n and $^*F^n$ are same, *i.e.* $^*F^i_{jk}{=}F^i_{jk}$. Then $D^i_{jk} = 0$. But then equations (3.12) and (3.13) implies that $E_{i0} = F_{i0}$, and hence

$$b_{0|i} = 0\,, \tag{3.30}$$

i.e. $\beta_{|i} = 0$. Differentiating $\beta_{|i} = 0$ partially with respect to y^j and applying commutation formulae $\dot\partial_j(\beta_{|i}) - (\dot\partial_j\beta)_{|i} = -(\dot\partial_r\beta)C^r_{ij|0}$, we get

$$b_{j|i} = b_r C^r_{ij|0}\,. \tag{3.31}$$

From the above equation, we conclude that $F_{ij} = 0$. M. K. Gupta and P. N. Pandey [4] has proved that if h-vector b_i is gradient, i.e. $F_{ij} = 0$ then ρ is constant, i.e. $\rho_i = \rho_{|i} = 0$. Taking h-covariant derivative of $LC^r_{ij} b_r = \rho h_{ij}$ and using $L_{|k} = 0$, $\rho_{|k} = 0$ and $h_{ij|k} = 0$, we have

$$(b_r C^r_{ij})_k = \frac{\rho}{L} h_{ij} = 0\,,$$

i.e.,

$$b_{r|k} C^r_{ij} + b_r C^r_{ij|k} = 0\,.$$

From (3.31), $b_{r|k} = b_{k|r}$ and hence above equation becomes

$$b_{k|r} C^r_{ij} + b_r C^r_{ij|k} = 0\,.$$

Transvecting by y^k, we have $b_{0|r} C^r_{ij} + b_r C^r_{ij|0} = 0$. Using (3.30) and (3.31), we conclude that $b_{i|j} = 0$.

Conversely, $b_{i|j} = 0$ implies that $E_{ij} = 0 = F_{ij}$ and $\beta_{|i} = \beta_{|i} = b_{j|i} = 0$. $F_{ij} = 0$ implies that $\rho_i = \rho_{|i} = 0$[4]. Therefore from (3.17), we get $D^i_{00} = 0$ and then $G_{ij} = 0$ and $G_j = 0$. This gives $D^i_{0j} = 0$ and then $H_{jik} = 0$ and $H_{ik} = 0$. Therefore (3.29) implies that $D^i_{jk} = 0$. Thus, we have:

Theorem 3.2. *For an h-exponential change of metric, the Cartan connection coefficients for both spaces F^n and $^*F^n$ are same if and only if the h-vector b_i is parallel with respect to Cartan connection of F^n.*

Now transvecting (3.3) by y^j and using $F^i_{jk}\, y^j = G^i_k$, we obtain

$$^*G^i_k = G^i_k + D^i_{0k}\,. \tag{3.32}$$

Transvecting again the above equation by y^k and using $G^i_k y^k = 2G^i$, we get

$$2\,^*G^i = 2G^i + D^i_{00}\,. \tag{3.33}$$

Differentiating (3.32) partially with respect to y^h and using $\dot{\partial}_h G^i_k = G^i_{kh}$, we have

$$^*G^i_{kh} = G^i_{kh} + \dot{\partial}_h D^i_{0k}\,, \tag{3.34}$$

where G^i_{kh} are Berwald connection coefficients.

Now, if the h-vector b_i is parallel with respect to Cartan connection of F^n then by Theorem (3.2), the Cartan connection coefficients for both spaces F^n and $^*F^n$ are same, therefore $D^i_{jk} = 0$. Hence from (3.34), we get $^*G^i_{kh} = G^i_{kh}$.
Thus, we have:

Theorem 3.3. *For an h-exponential change of metric, if an h-vector b_i is parallel with respect to Cartan connection of F^n then Barwald connection coefficients for both spaces F^n and $^*F^n$ are the same.*

4. Condition for h-exponential change of metric to be projective

Now, we find the condition for exponential change with h-vector to be projective. From (3.33), it follows that exponential change with h-vector to be projective if and only if $D^i_{00} = 2Py^i$. Then from (3.17), we get

(4.1)
$$\begin{aligned}2Py^i =&\frac{L}{\nu\, e^\tau}\Big[\frac{e^\tau}{L}\beta_{|0}m^i + 2e^\tau F^i_0\Big] + l^i\Big[E_{00} - \frac{L}{e^\tau}\big(m^2+\nu\big)^{-1}\big(\frac{e^\tau}{L}\beta_{|0}m^2 + 2e^\tau F_{\beta 0}\big)\Big]\\ &- \frac{m^i L}{\nu\, e^\tau}\big(m^2+\nu\big)^{-1}\Big[\frac{e^\tau}{L}\beta_{|0}m^2 + 2e^\tau F_{\beta 0}\Big].\end{aligned}$$

Transvecting (4.1) by y_i and using $m^i y_i = 0$, $F^i_0 y_i = 0$, we get

$$P = \frac{y_i l^i}{2L^2}\Big[E_{00} - \frac{L}{e^\tau}\big(m^2+\nu\big)^{-1}\big(\frac{e^\tau}{L}\beta_{|0}m^2 + 2e^\tau F_{\beta 0}\big)\Big]. \tag{4.2}$$

Substituting the value of P in(4.1), we get

$$F^i_0 = \frac{m^i}{2L}\big(m^2+\nu\big)^{-1}\big(\beta_{|0}m^2 + 2LF_{\beta 0}\big) - \frac{\beta_{|0}m^i}{2L}\,. \tag{4.3}$$

Using (3.15) in the above equation, we have

$$F^i_0 = \frac{m^i}{2L}m_r D^r_{00} - \frac{\beta_{|0}m^i}{2L}\,. \tag{4.4}$$

Transvecting by g_{ij} to above equation, we have

$$F_{i0} = \frac{m_i}{2L}m_r D^r_{00} - \frac{\beta_{|0}m_i}{2L}\,. \tag{4.5}$$

Using (4.5) in (3.13) and referring $\nu \neq 0$, we obtain $L_{ir}D^r_{00} = 0$, which transvecting by m^i and using $L_{ir}m^i = \frac{1}{L}m_r$, we get $m_r D^r_{00} = 0$, and then (4.5) becomes

$$F_{i0} = -\frac{\beta_{|0}m_i}{2L}\,. \tag{4.6}$$

The equation (4.6) is necessary condition for h-exponential change to be projective change.
Conversely, if (4.6) satisfied , the equation (3.13) yields

$$\Big[e^\tau \nu\, L_{ir} + \frac{e^\tau}{L}m_i m_r\Big]D^r_{00} = 0\,. \tag{4.7}$$

Transvecting by m^i and referring $(m^2+\nu) \neq 0$, we get $m_r D^r_{00} = 0$ and then (3.17) gives $D^i_{00} = E_{00}l^i$. Therefore ${}^*F^n$ is projective to F^n. Thus , we have :

Theorem 4.1. *The h-exponential change given by (1.6) is projective if and only if condition (4.6) is satisfied.*

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