WEAKLY $\alpha_\gamma$-REGULAR AND WEAKLY $\alpha_\gamma$-NORMAL SPACES

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Abstract. The aim of this paper is to introduce and study two new classes of spaces, called weakly $\alpha_\gamma$-regular and weakly $\alpha_\gamma$-normal spaces. Some basic properties of these separation axioms are studied by utilizing $\alpha_\gamma$-open and $\alpha_\gamma$-closed sets.

Keywords: operation; $\alpha_\gamma$-open set; weakly $\alpha_\gamma$-regular; weakly $\alpha_\gamma$-normal.

1. Introduction

Topology is the branch of mathematics, whose concepts exist not only in all branches of mathematics, but also in many real life applications. Several researchers are working on different structures of topological spaces. Njastad [6] initiated and explored the notion of $\alpha$-open sets. Ibrahim [5] introduced and discussed an operation of a topology $\alpha O(X)$ into the power set $P(X)$ of a space $X$ and also he introduced the concept of $\alpha_\gamma$-open sets and investigated the related topological properties of the associated topology $\alpha O(X,\tau)\gamma$ and $\alpha O(X,\tau)$ by using operation $\gamma$.

2. Preliminaries

Let $(X,\tau)$ be a topological space and $A$ a subset of $X$. The closure and the interior of $A$ are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset $A$ of a topological space $(X,\tau)$ is said to be $\alpha$-open [6] if $A \subseteq Int(Cl(Int(A)))$. The complement of an $\alpha$-open set is said to be $\alpha$-closed. The intersection of all $\alpha$-closed sets containing $A$ is called the $\alpha$-closure of $A$ and is denoted by $\alpha Cl(A)$. The family of all $\alpha$-open (resp., $\alpha$-closed) sets in a topological space $(X,\tau)$ is denoted by $\alpha O(X,\tau)$ (resp., $\alpha C(X,\tau)$). An operation $\gamma : \alpha O(X,\tau) \rightarrow P(X)$ [5] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X,\tau)$. We call the mapping $\gamma$ an operation on $\alpha O(X,\tau)$. A subset $A$ of $X$ is called an $\alpha_\gamma$-open set [5] if for each point $x \in A$, $x$ is not in the boundary of $A$. This is denoted by $A \subseteq \alpha Cl(A)$.
there exists an $\alpha$-open set $U$ of $X$ containing $x$ such that $U^\gamma \subseteq A$. The complement of an $\alpha$-open set is said to be $\alpha$-closed. We denote the set of all $\alpha$-open (resp., $\alpha$-closed) sets of $(X, \tau)$ by $\alpha O(X, \tau)$ (resp., $\alpha C(X, \tau)$). The $\alpha$-closure [5] of a subset $A$ of $X$ with an operation $\gamma$ on $\alpha O(X)$ is denoted by $\alpha Cl(A)$ and is defined to be the intersection of all $\alpha$-closed sets containing $A$. A point $x \in X$ is in $\alpha Cl_\gamma$-closure [5] of a set $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$ for each $\alpha$-open set $U$ containing $x$. The $\alpha Cl_\gamma$-closure of $A$ is denoted by $\alpha Cl_\gamma(A)$. The union of all $\alpha$-open sets contained in $A$ is called the $\alpha$-interior of $A$ and denoted by $\alpha Int(A)$ [3]. An operation $\gamma$ on $\alpha O(X, \tau)$ is said to be $\alpha$-regular [5] if for every $\alpha$-open sets $U$ and $V$ of each $x \in X$, there exists an $\alpha$-open set $W$ of $x$ such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. A subset $A$ of the space $X$ is said to be $\alpha$-generalized closed (Briefly, $\alpha$-g.closed) [5] if $\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an $\alpha$-open set in $X$. The complement of an $\alpha$-g.closed set is called an $\alpha$-g.open set [5].

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Let $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ and $\beta : \alpha O(Y, \sigma) \rightarrow P(Y)$ be two operations on $\alpha O(X, \tau)$ and $\alpha O(Y, \sigma)$, respectively.

**Remark 2.1.** [2] For an operation $\gamma : \alpha O(X) \rightarrow P(X)$ and a subset $H$ of $X$, we have $\alpha O(X) \upharpoonright H = \{V \cap H \in P(H) : V \text{ is } \alpha$-open in $X\}.$

**Definition 2.1.** [3] Let $(X, \tau)$ be a topological space and $x \in X$, then a subset $N$ of $X$ is said to be $\alpha$-$\gamma$-neighbourhood of $x$, if there exists an $\alpha$-open set $U$ in $X$ such that $x \in U \subseteq N$.

**Remark 2.2.** [3] Let $(X, \tau)$ be a topological space and $\gamma$ an $\alpha$-regular operation on $\alpha O(X)$. If $A$ is a subset of $X$, then

1. For every $\alpha$-open set $G$ of $X$, we have that $\alpha Cl(A) \cap G \subseteq \alpha Cl(A \cap G)$.
2. For every $\alpha$-$\gamma$-closed set $F$ of $X$, we have that $\alpha Int(A \cup F) \subseteq \alpha Int(A) \cup F$.

**Proposition 2.1.** [3] A subset $A$ of $X$ is $\alpha$-$\gamma$-open if and only if $F \subseteq \alpha$-$\gamma$-closed in $X$.

**Theorem 2.1.** [4] Let $(X, \tau)$ be a topological space and $\gamma$ an operation on $\alpha O(X)$. Then, the following properties are equivalent:

1. $(X, \tau)$ is $\alpha$-$\gamma$-$T_1$.
2. For every point $x \in X$, $\{x\}$ is an $\alpha$-$\gamma$-closed set.
3. $(X, \tau)$ is $\alpha$-$T_1$.

**Definition 2.2.** [1] A topological space $(X, \tau)$ is said to be $\alpha$-$\gamma$-regular if for each $x \in X$ and for each $\alpha$-open set $V$ in $X$ containing $x$, there exists an $\alpha$-open set $U$ in $X$ containing $x$ such that $U^\gamma \subseteq V$.
Theorem 2.2. [3] Let \( (X, \tau) \) be a topological space and \( \gamma \) an operation on \( \alpha O(X) \). Then the following statements are equivalent.

1. \( \alpha O(X, \tau) = \alpha O(X, \tau)_{\gamma} \).
2. \((X, \tau)\) is an \( \alpha_{\gamma} \)-regular space.
3. For every \( x \in X \) and every \( \alpha \)-open set \( U \) of \( X \) containing \( x \) there exists an \( \alpha_{\gamma} \)-open set \( W \) of \( X \) such that \( x \in W \) and \( W \subseteq U \).

Definition 2.3. [5] A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \alpha(\gamma, \beta) \)-continuous if for each \( x \) of \( X \) and each \( \alpha \beta \)-open set \( V \) containing \( f(x) \), there exists an \( \alpha \gamma \)-open set \( U \) such that \( x \in U \) and \( f(U) \subseteq V \).

Definition 2.4. [5] A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \alpha(\gamma, \beta) \)-closed if for any \( \alpha \gamma \)-closed set \( A \) of \((X, \tau)\), \( f(A) \) is \( \alpha \beta \)-closed in \((Y, \sigma)\).

3. Weakly \( \alpha_{\gamma} \)-Regular and Weakly \( \alpha_{\gamma} \)-Normal Spaces

Definition 3.1. A space \( X \) is said to be weakly \( \alpha_{\gamma} \)-regular space, if for any \( \alpha_{\gamma} \)-closed set \( A \) and \( x \notin A \), there exist \( \alpha_{\gamma} \)-open sets \( U \) and \( V \) such that \( x \in U \), \( A \subseteq V \) and \( U \cap V = \emptyset \).

From the following examples, it can be easily seen that weakly \( \alpha_{\gamma} \)-regular and regular spaces are incomparable in general.

Example 3.1. Consider \( X = \{a, b, c\} \) with the discrete topology \( \tau \) on \( X \). For a non-empty set \( A \), we define an operation \( \gamma \) on \( \alpha O(X) \) by

\[
A^\gamma = \begin{cases} 
A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\
X & \text{otherwise.}
\end{cases}
\]

Then, the space \( X \) is a regular but it is not weakly \( \alpha_{\gamma} \)-regular.

Example 3.2. Consider \( X = \{a, b, c\} \) with the topology
\( \tau = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} \). For a non-empty set \( A \), we define an operation \( \gamma \) on \( \alpha O(X) \) by

\[
A^\gamma = \begin{cases} 
A & \text{if } A = \{c\} \text{ or } \{a, b\} \\
X & \text{otherwise.}
\end{cases}
\]

Then, \( X \) is a weakly \( \alpha_{\gamma} \)-regular space but it is not regular.

Remark 3.1. The indiscrete space is a weakly \( \alpha_{\gamma} \)-regular space, for if any \( \alpha_{\gamma} \)-closed set \( A \) and any point \( x \notin A \), there are \( \alpha_{\gamma} \)-open sets \( U \) and \( V \) such that \( x \in U \), \( A \subseteq V \) and \( U \cap V = \emptyset \).

In the next result, we give some characterizations of weakly \( \alpha_{\gamma} \)-regular spaces.
Theorem 3.1. Consider the following statements on a space $X$.

1. $X$ is weakly $\alpha_\gamma$-regular.

2. For any $\alpha_\gamma$-open set $U$ in $X$ and $x \in U$, there is an $\alpha_\gamma$-open set $V$ containing $x$ such that $\alpha\text{Cl}_\gamma(V) \subseteq U$.

3. For every point $x \in X$ and every $\alpha_\gamma$-neighbourhood $N$ of $x$, there exists an $\alpha$-closed $B$ such that $x \in B \subseteq N$.

Then (1) implies (2) and (2) implies (3).

Proof. (1) $\Rightarrow$ (2): Let $U$ be $\alpha_\gamma$-open set and $x \in U$. Then $X \setminus U$ is an $\alpha_\gamma$-closed set such that $x \notin X \setminus U$. By the weakly $\alpha_\gamma$-regularity of $X$, there are $\alpha_\gamma$-open sets $V$ and $G$ such that $x \in V$, $X \setminus U \subseteq G$ and $V \cap G = \phi$. Clearly $X \setminus G$ is an $\alpha_\gamma$-closed set contained in $U$. Then $V \subseteq X \setminus G \subseteq U$. This gives $\alpha\text{Cl}_\gamma(V) \subseteq X \setminus G \subseteq U$.

Consequently, $x \in V$ and $\alpha\text{Cl}_\gamma(V) \subseteq U$.

(2) $\Rightarrow$ (3): Let $x \in X$ be any element and $N$ be any $\alpha_\gamma$-neighbourhood of $x$. By Definition 2.1, there exists an $\alpha_\gamma$-open set $U$ such that $x \in U \subseteq N$. By (2) there is an $\alpha_\gamma$-open set $V$ containing $x$ such that $\alpha\text{Cl}_\gamma(V) \subseteq U$. Then, $x \in \alpha\text{Cl}_\gamma(V) \subseteq N$ and since $\alpha\text{Cl}_\gamma(V)$ is $\alpha$-closed set by [5], Theorem 2.26 (1)]. Thus for each $x \in X$, the set $N$ form an $\alpha_\gamma$-neighbourhood consisting of $\alpha$-closed set of $X$. This proves (3).

In general, the statements in above theorem are not equivalent as it is shown in the following example.

Example 3.3. Consider $X = \{a, b, c\}$ with the discrete topology $\tau$ on $X$. For a non-empty set $A$, we define an operation $\gamma$ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{b, c\} \\ X & \text{otherwise}. \end{cases}$$

Then, the statement (3) is holds but $X$ is not weakly $\alpha_\gamma$-regular.

Remark 3.2. In Theorem 3.1, the statements (1), (2) and (3) are equivalent to each other if one of the following conditions is satisfied:

1. $\gamma$ is identity operation and $\alpha O(X, \tau) = \alpha C(X, \tau)$.

2. $\gamma$ is identity operation and every subsets of $X$ is $\alpha_\gamma$-g.closed.

3. $X$ is $\alpha_\gamma$-regular and $\alpha O(X, \tau) = \alpha C(X, \tau)$.

4. $X$ is $\alpha_\gamma$-regular and every subsets of $X$ is $\alpha_\gamma$-g.closed.

Proof. It follows from the proof of Theorem 3.1, that we know the following implications, (1) $\Rightarrow$ (2) $\Rightarrow$ (3), so it is sufficient to prove (3) $\Rightarrow$ (1). Let $A$ be an $\alpha_\gamma$-closed set such that $x \notin A$. Then, $X \setminus A$ is $\alpha_\gamma$-open and $\alpha_\gamma$-neighbourhood of $x$. By (3) there is an $\alpha$-closed set $B$ such that $x \in B$ and $B \subseteq X \setminus A$, implies that $A \subseteq X \setminus B$. Then, $x \in B$, $A \subseteq X \setminus B$ and $B \cap X \setminus B = \phi$. Therefore,
Theorem 3.2. The following statements are equivalent for a topological space \((X, \tau)\) with an operation \(\gamma\) on \(aO(X, \tau)\):

1. \(X\) is weakly \(\alpha_\gamma\)-regular.

2. For each \(x \in X\) and each \(\alpha_\gamma\)-open set \(U\) containing \(x\), there exists an \(\alpha_\gamma\)-open set \(V\) such that \(x \in V \subseteq a_\gamma Cl(V) \subseteq U\).

3. For each \(\alpha_\gamma\)-closed subset \(F\) of \(X\),
   \[
   F = \bigcap\{a_\gamma Cl(V) : F \subseteq V \text{ and } V \in aO(X, \tau)_\gamma\}.
   \]

4. For each \(A\) subset of \(X\) and each \(U \in aO(X, \tau)_\gamma\) with \(A \cap U \neq \phi\), there exists \(V \in aO(X, \tau)_\gamma\) such that \(A \cap V \neq \phi\) and \(\alpha_\gamma Cl(V) \subseteq U\).

5. For each nonempty subset \(A\) of \(X\) and each \(\alpha_\gamma\)-closed subset \(F\) of \(X\) with \(A \cap F = \phi\), there exists \(V, W \in aO(X, \tau)_\gamma\) such that \(A \cap V \neq \phi\), \(F \subseteq W\) and \(W \cap V = \phi\).

6. For each \(\alpha_\gamma\)-closed set \(F\) and \(x \notin F\), there exists \(U \in aO(X, \tau)_\gamma\) and an \(\alpha_\gamma\)-open set \(V\) such that \(x \in U\), \(F \subseteq V\), and \(U \cap V = \phi\).

7. For each \(A \subseteq X\) and each \(\alpha_\gamma\)-closed set \(F\) with \(A \cap F = \phi\), there exists \(U \in aO(X, \tau)_\gamma\) and an \(\alpha_\gamma\)-open set \(V\) such that \(A \cap U \neq \phi\), \(F \subseteq V\), and \(U \cap V = \phi\).

8. For each \(\alpha_\gamma\)-closed set \(F\) of \(X\),
   \[
   F = \bigcap\{a_\gamma Cl(V) : F \subseteq V \text{ and } V \text{ is } \alpha_\gamma\text{-open}\}.
   \]

Proof. (1) \(\Rightarrow\) (2): Let \(x \notin X \setminus U\), where \(U\) is any \(\alpha_\gamma\)-open set containing \(x\). Then by (1), there exist \(G, V \in aO(X, \tau)_\gamma\) such that \((X \setminus U) \subseteq G\), \(x \in V\), and \(G \cap V = \phi\). Therefore \(V \subseteq (X \setminus G)\) and so \(x \in V \subseteq a_\gamma Cl(V) \subseteq (X \setminus G) \subseteq U\).

(2) \(\Rightarrow\) (3): Let \(X \setminus F\) be any \(\alpha_\gamma\)-open set containing \(x\). Then by (2), there exists an \(\alpha_\gamma\)-open set \(U\) containing \(x\) such that \(x \in U \subseteq a_\gamma Cl(U) \subseteq (X \setminus F)\). So, \(F \subseteq X \setminus a_\gamma Cl(U) = V\), \(V \in aO(X, \tau)_\gamma\), and \(V \cap U = \phi\). Then by [[5], Theorem 2.23], \(x \notin a_\gamma Cl(V)\). Hence, we obtain that \(F \supseteq \bigcap\{a_\gamma Cl(V) : F \subseteq V, V \in aO(X, \tau)_\gamma\}\).

(3) \(\Rightarrow\) (4): Let \(U \in aO(X, \tau)_\gamma\) with \(x \in U \cap A\). Then \(x \notin X \setminus U\) and hence
by (3), there exists an $\alpha_\gamma$-open set $W$ such that $X \setminus U \subseteq W$ and $x \notin \alpha_\gamma Cl(W)$. We put $V = X \setminus \alpha_\gamma Cl(W)$, which is an $\alpha_\gamma$-open set containing $x$ and hence $V \cap A \neq \phi$. Now $V \subseteq (X \setminus W)$ and so $\alpha_\gamma Cl(V) \subseteq (X \setminus W) \subseteq U$.

$(4) \Rightarrow (5)$: Let $F$ be a set as in the hypothesis of (5). Then, $X \setminus F$ is $\alpha_\gamma$-open and $(X \setminus F) \cap A \neq \phi$. Then, there exists $V \in \alpha O(X, \tau)_\gamma$ such that $A \cap V \neq \phi$ and $\alpha_\gamma Cl(V) \subseteq X \setminus F$. If we put $W = X \setminus \alpha_\gamma Cl(V)$, then $F \subseteq W$ and $W \cap V = \phi$.

$(5) \Rightarrow (1)$: Let $F$ be an $\alpha_\gamma$-closed set not containing $x$. Then by (5), there exist $W, V \in \alpha O(X, \tau)_\gamma$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \phi$.

$(1) \Rightarrow (6)$: Obvious.

$(6) \Rightarrow (7)$: For $a \in A$, $a \notin F$ and hence by (6), there exist $U \in \alpha O(X, \tau)_\gamma$ and an $\alpha_\gamma$-g.open set $V$ such that $a \in U$, $F \subseteq V$ and $U \cap V = \phi$. So, $A \cap U \neq \phi$.

$(7) \Rightarrow (1)$: Let $x \notin F$, where $F$ is $\alpha_\gamma$-closed. Since $\{x\} \cap F = \phi$, by (7), there exist $U \in \alpha O(X, \tau)_\gamma$ and an $\alpha_\gamma$-g.open set $W$ such that $x \in U$, $F \subseteq W$ and $U \cap W = \phi$. Now put $V = \alpha_\gamma Int(W)$. Using Proposition 2.1 of $\alpha_\gamma$-g.open sets we get $F \subseteq V$ and $V \cap U = \phi$.

$(3) \Rightarrow (8)$: We have $F \subseteq \bigcap \{\alpha_\gamma Cl(V): F \subseteq V$ and $V$ is $\alpha_\gamma$-open $\} \subseteq \bigcap \{\alpha_\gamma Cl(V): F \subseteq V$ and $V$ is $\alpha_\gamma$-open $\} = F$.

$(8) \Rightarrow (1)$: Let $F$ be an $\alpha_\gamma$-closed set in $X$ not containing $x$. Then by (8), there exists an $\alpha_\gamma$-g.open set $W$ such that $F \subseteq W$ and $x \in X \setminus \alpha_\gamma Cl(W)$. Since $F$ is $\alpha_\gamma$-closed and $W$ is $\alpha_\gamma$-g.open, then $F \subseteq \alpha_\gamma Int(W)$. Take $V = \alpha_\gamma Int(W)$. Then $F \subseteq V$, $x \in U = X \setminus \alpha_\gamma Cl(V)$ and $U \cap V = \phi$. □

The following theorem shows that weakly $\alpha_\gamma$-regularity is a hereditary property.

**Theorem 3.3.** Let $\gamma$ be an operation on $\alpha O(X)$ and $(H, \alpha O(X)_\gamma(H)$ be a subspace of a topological space $(X, \tau)$. If $X$ is weakly $\alpha_\gamma$-regular, then $H$ is also weakly $\alpha_\gamma$-regular.

**Proof.** Suppose that $A$ is $\alpha_\gamma$-closed set in $H$ and $y \in H$ such that $y \notin A$. Then, there exists an $\alpha_\gamma$-open $U$ in $X$ such that $H \setminus A = U \cap H$. This implies that $A = B \cap H$, where $B = X \setminus U$ is $\alpha_\gamma$-closed in $X$. Then $y \notin B$. Since $X$ is weakly $\alpha_\gamma$-regular, there exist disjoint $\alpha_\gamma$-open sets $U, V$ in $X$ such that $y \in U$, $B \subseteq V$. Then, $U \cap H$ and $V \cap H$ are disjoint $\alpha_\gamma$-open sets in $H$ containing $y$ and $A$ respectively. □

**Theorem 3.4.** Let $(X, \tau)$ be a topological space and $\gamma$ an $\alpha$-regular operation on $\alpha O(X)$. Then, $X$ is weakly $\alpha_\gamma$-regular if and only if for each $x \in X$ and an $\alpha_\gamma$-closed set $A$ such that $x \notin A$, there exist $\alpha_\gamma$-open sets $U, V$ in $X$ such that $x \in U$ and $A \subseteq V$ and $\alpha_\gamma Cl(U) \cap \alpha_\gamma Cl(V) = \phi$. 


2.2). Thus sufficiency is immediate.

Definition 3.2. A space $X$ is said to be weakly $\alpha$-normal space, if for any disjoint $\alpha$-closed sets $A, B$ of $X$, there exist $\alpha$-open sets $U, V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

From the following examples, we can observe that weakly $\alpha$-normal and normal spaces are incomparable in general.

Example 3.4. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. For any non-empty subset $A$ of $X$, we define an operation $\gamma$ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise.} \end{cases}$$

Then, the space $X$ is a normal but it is not weakly $\alpha$-normal.

Example 3.5. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. For any non-empty subset $A$ of $X$, we define an operation $\gamma$ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \\ X & \text{otherwise.} \end{cases}$$

Then, $X$ is a weakly $\alpha$-normal space but it is not normal.

In the next results, we give several characterizations of weakly $\alpha$-normal space.

Theorem 3.5. Let $(X, \tau)$ be a topological space and $\gamma$ an operation on $\alpha O(X)$. Then, $X$ is weakly $\alpha$-normal if and only if for any $\alpha$-closed set $A$ and $\alpha$-open set $U$ containing $A$, there is an $\alpha$-open set $V$ such that $A \subseteq V \subseteq \alpha Cl(V) \subseteq U$.

Proof. Since $U$ is $\alpha$-open set containing $A$, then $X \setminus U$ is $\alpha$-closed and $A \setminus (X \setminus U) = \phi$. Since $X$ is weakly $\alpha$-normal, there exist $\alpha$-open sets $V$ and $V_1$ such that $A \subseteq V$, $X \setminus U \subseteq V_1$ and $V \cap V_1 = \phi$. Hence, $A \subseteq V \subseteq \alpha Cl(V) \subseteq \alpha Cl(X \setminus V_1) = X \setminus V_1 \subseteq U$, or $A \subseteq V \subseteq \alpha Cl(V) \subseteq U$.

Conversely, let $A$ and $B$ be two disjoint $\alpha$-closed sets in $X$. Then $A \subseteq X \setminus B$, where $X \setminus B$ is $\alpha$-open in $X$. By hypothesis, there is an $\alpha$-open set $V$ such that $A \subseteq V \subseteq \alpha Cl(V) \subseteq X \setminus B$, implies that $B \subseteq X \setminus \alpha Cl(V)$ and $V \cap X \setminus \alpha Cl(V) = \phi$. Consequently, $A \subseteq V$, $B \subseteq X \setminus \alpha Cl(V)$. This proves that $X$ is weakly $\alpha$-normal.
\textbf{Theorem 3.6.} For a topological space \((X, \tau)\) with an operation \(\gamma\) on \(\alpha O(X, \tau)\), the following statements are equivalent:

1. \(X\) is weakly \(\alpha_{\gamma}\)-normal.

2. For each pair of disjoint \(\alpha_{\gamma}\)-closed sets \(A, B\) of \(X\), there exist disjoint \(\alpha_{\gamma}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

3. For each \(\alpha_{\gamma}\)-closed \(A\) and any \(\alpha_{\gamma}\)-open set \(V\) containing \(A\), there exists an \(\alpha_{\gamma}\)-open set \(U\) such that \(A \subseteq U \subseteq \alpha_{\gamma}\text{Cl}(U) \subseteq V\).

4. For each \(\alpha_{\gamma}\)-closed set \(A\) and any \(\alpha_{\gamma}\)-open set \(B\) containing \(A\), there exists an \(\alpha_{\gamma}\)-open set \(U\) such that \(A \subseteq U \subseteq \alpha_{\gamma}\text{Cl}(U) \subseteq \alpha_{\gamma}\text{Int}(B)\).

5. For each \(\alpha_{\gamma}\)-closed set \(A\) and any \(\alpha_{\gamma}\)-open set \(B\) containing \(A\), there exists an \(\alpha_{\gamma}\)-open set \(U\) such that \(\alpha_{\gamma}\text{Cl}(A) \subseteq U \subseteq \alpha_{\gamma}\text{Cl}(U) \subseteq B\).

6. For each \(\alpha_{\gamma}\)-closed set \(A\) and any \(\alpha_{\gamma}\)-open set \(B\) containing \(A\), there exists an \(\alpha_{\gamma}\)-open set \(G\) such that \(\alpha_{\gamma}\text{Cl}(A) \subseteq G \subseteq \alpha_{\gamma}\text{Cl}(G) \subseteq \alpha_{\gamma}\text{Int}(B)\).

7. For each \(\alpha_{\gamma}\)-closed set \(A\) and any \(\alpha_{\gamma}\)-open set \(B\) containing \(A\), there exists an \(\alpha_{\gamma}\)-open set \(G\) such that \(\alpha_{\gamma}\text{Cl}(A) \subseteq G \subseteq \alpha_{\gamma}\text{Cl}(G) \subseteq B\).

\textit{Proof.} (1) \(\Rightarrow\) (2): Follows from the fact that every \(\alpha_{\gamma}\)-open set is \(\alpha_{\gamma}\)-open.

(2) \(\Rightarrow\) (3): Let \(A\) be any \(\alpha_{\gamma}\)-closed set and \(V\) any \(\alpha_{\gamma}\)-open set containing \(A\). Since \(A\) and \(X \setminus V\) are disjoint \(\alpha_{\gamma}\)-closed sets, there exist \(\alpha_{\gamma}\)-open sets \(U\) and \(W\) such that \(A \subseteq U\), \(X \setminus V \subseteq W\) and \(U \cap W = \phi\). By Proposition 2.1, we get \(X \setminus V \subseteq \alpha_{\gamma}\text{Int}(W)\). Since \(U \cap \alpha_{\gamma}\text{Int}(W) = \phi\), we have \(\alpha_{\gamma}\text{Cl}(U) \cap \alpha_{\gamma}\text{Int}(W) = \phi\), and hence \(\alpha_{\gamma}\text{Cl}(U) \subseteq X \setminus \alpha_{\gamma}\text{Int}(W) \subseteq V\). Therefore \(A \subseteq U \subseteq \alpha_{\gamma}\text{Cl}(U) \subseteq V\).

(3) \(\Rightarrow\) (1): Let \(A\) and \(B\) be any two disjoint \(\alpha_{\gamma}\)-closed sets of \(X\). Since \(X \setminus B\) is an \(\alpha_{\gamma}\)-open set containing \(A\), there exists an \(\alpha_{\gamma}\)-open set \(G\) such that \(A \subseteq G \subseteq \alpha_{\gamma}\text{Cl}(G) \subseteq X \setminus B\). Since \(G\) is an \(\alpha_{\gamma}\)-open set, using Proposition 2.1, we have \(A \subseteq \alpha_{\gamma}\text{Int}(G)\). Taking \(U = \alpha_{\gamma}\text{Int}(G)\) and \(V = X \setminus \alpha_{\gamma}\text{Cl}(G)\), we have two disjoint \(\alpha_{\gamma}\)-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\). Hence, \(X\) is weakly \(\alpha_{\gamma}\)-normal.

(5) \(\Rightarrow\) (4) and (4) \(\Rightarrow\) (3): Obvious.

(5) \(\Rightarrow\) (3): Let \(A\) be any \(\alpha_{\gamma}\)-closed set and \(V\) any \(\alpha_{\gamma}\)-open set containing \(A\).

Since every \(\alpha_{\gamma}\)-open set is \(\alpha_{\gamma}\)-open, there exists an \(\alpha_{\gamma}\)-open set \(G\) such that \(A \subseteq G \subseteq \alpha_{\gamma}\text{Cl}(G) \subseteq \alpha_{\gamma}\text{Int}(V)\). Hence, we have an \(\alpha_{\gamma}\)-open set \(G\) such that \(A \subseteq G \subseteq \alpha_{\gamma}\text{Cl}(G) \subseteq \alpha_{\gamma}\text{Int}(V) \subseteq V\).

(3) \(\Rightarrow\) (5): Let \(A\) be any \(\alpha_{\gamma}\)-closed set and \(B\) any \(\alpha_{\gamma}\)-open set containing \(A\). Using Proposition 2.1, of an \(\alpha_{\gamma}\)-open set we get \(A \subseteq \alpha_{\gamma}\text{Int}(B) = V\), say. Then applying (3), we get an \(\alpha_{\gamma}\)-open set \(U\) such that \(A = \alpha_{\gamma}\text{Cl}(A) \subseteq U \subseteq \alpha_{\gamma}\text{Cl}(U) \subseteq V\). Again, using the same Proposition 2.1, we get \(A \subseteq \alpha_{\gamma}\text{Int}(U)\), and hence \(A \subseteq \alpha_{\gamma}\text{Int}(U) \subseteq V\).
This proves that $A \subseteq \alpha, Cl(U) \subseteq V$, which implies $A \subseteq \alpha, Int(U) \subseteq \alpha, Cl(\alpha, Int(U)) \subseteq \alpha, Cl(U) \subseteq V$, that is $A \subseteq G \subseteq \alpha, Cl(G) \subseteq \alpha, Int(B)$, where $G = \alpha, Int(U)$.

$(6) \Rightarrow (7)$ and $(7) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (7)$: Let $A$ be any $\alpha, g$-closed set and $B$ any $\alpha, g$-open set containing $A$. Since $A$ is an $\alpha, g$-closed set, we have $\alpha, Cl(A) \subseteq B$, therefore by $(3)$ we can find an $\alpha, g$-open set $U$ such that $\alpha, Cl(A) \subseteq U \subseteq \alpha, Cl(U) \subseteq B$.

$(7) \Rightarrow (6)$: Let $A$ be any $\alpha, g$-closed set and $B$ any $\alpha, g$-open set containing $A$, then by $(7)$, there exists an $\alpha, g$-open set $G$ such that $\alpha, Cl(A) \subseteq G \subseteq \alpha, Cl(G) \subseteq B$. Since $G$ is an $\alpha, g$-open set, then by Proposition 2.1, we get $\alpha, Cl(A) \subseteq \alpha, Int(G)$. If we take $U = \alpha, Int(G)$, the proof follows.

Remark 3.3. The space in Example 3.5, is a weakly $\alpha, g$-normal but it is not weakly $\alpha, g$-regular.

**Theorem 3.7.** Every weakly $\alpha, g$-normal $\alpha, g$-$T_1$ space is weakly $\alpha, g$-regular.

**Proof.** Suppose that $A$ is an $\alpha, g$-closed set and $x \notin A$. Since $X$ is an $\alpha, g$-$T_1$ space, then by Theorem 2.1, $\{x\}$ is $\alpha, g$-closed in $X$. Also since $X$ is weakly $\alpha, g$-normal, then there exist $\alpha, g$-open sets $U$ and $V$ such that $\{x\} \subseteq U$, $A \subseteq V$ and $U \cap V = \phi$, or $x \in U$, $A \subseteq V$ and $U \cap V = \phi$ give $X$ is weakly $\alpha, g$-regular.

**Theorem 3.8.** Let $\gamma$ be an operation on $\alpha O(X)$ and $(A, \alpha O(X), |A|$ be a subspace of a topological space $(X, \tau)$. If $A$ is $\alpha, g$-closed in $X$ and $X$ is weakly $\alpha, g$-normal, then $A$ is weakly $\alpha, g$-normal.

**Proof.** Let $A_1$ and $A_2$ be two disjoint $\alpha, g$-closed sets of $A$. Then, there are $\alpha, g$-closed sets $B_1$ and $B_2$ in $X$ such that $A_1 = B_1 \cap A$, $A_2 = B_2 \cap A$. Since $A$ is $\alpha, g$-closed in $X$, then $A_1$ and $A_2$ are $\alpha, g$-closed in $X$. Also since $X$ is weakly $\alpha, g$-normal, there exist $\alpha, g$-open sets $U_1$ and $U_2$ in $X$ such that $A_1 \subseteq U_1$, $A_2 \subseteq U_2$ and $U_1 \cap U_2 = \phi$. But $A_1 \subseteq A \cap U_1$ and $A_2 \subseteq A \cap U_2$, where $A \cap U_1$, $A \cap U_2$ are disjoint $\alpha, g$-open sets in $A$. This proves that $A$ is weakly $\alpha, g$-normal.

**Theorem 3.9.** Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is a bijective, $\alpha, (\gamma, \beta)$-continuous and $\alpha, (\gamma, \beta)$-closed mapping. If $X$ is weakly $\alpha, g$-normal, then $Y$ is $\alpha, \beta$-normal.

**Proof.** Let $A_1$ and $B_1$ be two disjoint $\alpha, \beta$-closed subsets of $Y$. Then by $\alpha, (\gamma, \beta)$-continuity of $f$, $A = f^{-1}(A_1)$, $B = f^{-1}(B_1)$ are disjoint $\alpha, \gamma$-closed subsets of $X$. Since $X$ is weakly $\alpha, g$-normal, then there exist $\alpha, g$-open sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$. Since $f$ is $\alpha, (\gamma, \beta)$-closed, then $f(U \setminus U)$ and $f(X \setminus V)$ are $\alpha, \beta$-closed in $Y$. Then easy calculations give that $U_1 = Y \setminus f(X \setminus U)$, $U_2 = Y \setminus f(X \setminus V)$ are disjoint $\alpha, \beta$-open sets in $Y$ containing $A_1$, $B_1$ respectively. This proves that $Y$ is $\alpha, \beta$-normal.
Theorem 3.10. Let $(X, \tau)$ be a topological space and $\gamma$ an $\alpha$-regular operation on $\alpha O(X)$. Then, $X$ is weakly $\alpha_\gamma$-normal if and only if for each disjoint $\alpha_\gamma$-closed sets $A$ and $B$ in $X$, there exist $\alpha_\gamma$-open sets $U$ and $V$ in $X$ such that $A \subseteq U$, $B \subseteq V$ and $\alpha_\gamma \text{Cl}(U) \cap \alpha_\gamma \text{Cl}(V) = \emptyset$.

Proof. The sufficiency is clear. Let $A$ and $B$ be any two disjoint $\alpha_\gamma$-closed sets in $X$. Then $X \setminus B$ is $\alpha_\gamma$-open and $A \subseteq X \setminus B$. Then by Theorem 3.5, there is an $\alpha_\gamma$-open set $C$ such that $A \subseteq C \subseteq \alpha_\gamma \text{Cl}(C) \subseteq X \setminus B$. Consequently, $A \subseteq U \subseteq \alpha_\gamma \text{Cl}(U) \subseteq C$ and $\alpha_\gamma \text{Cl}(C) \subseteq X \setminus B$ implies that $B \subseteq X \setminus \alpha_\gamma \text{Cl}(C)$. Put $V = X \setminus \alpha_\gamma \text{Cl}(C)$. Then $V$ is $\alpha_\gamma$-open containing $B$ and moreover $\alpha_\gamma \text{Cl}(U) \cap \alpha_\gamma \text{Cl}(V) = \alpha_\gamma \text{Cl}(U) \cap \alpha_\gamma \text{Cl}(X \setminus \alpha_\gamma \text{Cl}(C)) \subseteq C \cap \alpha_\gamma \text{Cl}(X \setminus \alpha_\gamma \text{Cl}(C)) \subseteq \alpha_\gamma \text{Cl}(C \setminus X \setminus \alpha_\gamma \text{Cl}(C)) = \alpha_\gamma \text{Cl}(\emptyset) = \emptyset$ (by Remark 2.2). Thus, $U$ and $V$ are the required $\alpha_\gamma$-open sets in $X$. This proves the necessity. Hence the proof.

REFERENCES

2. A. B. Khalaf and H. Z. Ibrahim, Some operations defined on subspaces via $\alpha$-open sets, (Submitted).

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