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ON NON-INVARIANT HYPERSURFACES OF AN ε-PARA SASAKIAN MANIFOLD

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Abstract. Non-invariant hypersurfaces of an $\varepsilon-$ para Sasakian manifold of an induced structure (f, g, u, v, λ) have been studied in this paper. Some properties followed by this structure have ben obtained. A necessary and sufficient condition for totally umbilical non-invariant hypersurfaces equipped with $(f, g, u, v, \lambda)-$ structure of $\varepsilon-$para Sasakian manifold to be totally geodesic has also been explored.

Keywords: $\varepsilon-$ Para Sasakian Manifold, totally umbilical, totally geodesic.

1. Introduction

In 1976, Sato [1] introduced a structure (ϕ, ξ, η) satisfying $\phi^2 = I - \eta \otimes \xi$ and $\eta(\xi) = 1$ on a differentiable manifold, which is now well known as an almost paracontact structure. The structure is an analogue of the almost contact structure [2, 3] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [4] introduced almost contact manifolds equipped with associated pseudo Riemannian metrics. In particular, he studied Sasakian manifolds equipped with an associated pseudo$-$ Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as $\varepsilon-$almost contact metric manifolds and $\varepsilon-$Sasakian manifolds respectively [5, 6, 7]. Also, in 1989, K. Matsumoto [8] replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure, calling it a Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold given by Matsumoto, the semi$-$Riemannian metric has only index 1 and the structure vector field ξ is always timelike. These circumstances motivated the authors in [9] to associate a semi$-$Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and they called this indefinite

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almost paracontact metric structure an $\varepsilon-$almost paracontact structure, where the structure vector field ξ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$.

In [9] the authors studied $\varepsilon-$almost paracontact manifolds, and in particular, $\varepsilon-$para Sasakian manifolds. They gave basic definitions, some examples of $\varepsilon-$almost paracontact manifolds and introduced the notion of an $\varepsilon-$para Sasakian structure. The basic properties, some typical identities for curvature tensor and Ricci tensor of the $\varepsilon-$para Sasakian manifolds were also studied in [9]. The authors in [9] proved that if a semi$-$Riemannian manifold is one of flat, proper recurrent or proper Ricci$-$recurrent, then it can not admit an $\varepsilon-$para Sasakian structure. Also. they showed that, for an $\varepsilon-$para Sasakian manifold, the conditions of being symmetric, semi$-$symmetric or of constant sectional curvature are all identical.

On the other hand In 1970, S. I. Goldberg et. al [10] introduced the notion of a non$-$invariant hypersurface of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the $(1,1)$ structure tensor field ϕ defining the almost contact structure is never tangent to the hypersurface.

The notion of $(f, g, u, v, \lambda)-$ structure was given by K.Yano [11]. It is well known [12, 13] that a hypersurface of an almost contact metric manifold always admits a $(f, g, u, v, \lambda)-$ structure. Authors [10] proved that there always exists a $(f, g, u, v, \lambda)-$ structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. R. Prasad [14] studied the non-invariant hypersurfaces of trans-Sasakian manifolds. Non-invariant hypersurfaces of nearly Trans-Sasakian manifold have been studied by S. Kishor et. al [15] . The present paper is devoted to the study of non-invariant hypersurfaces of $\varepsilon-$para Sasakian manifolds. The contents of the paper are organized as follows:

In section-2 some preliminaries are given. Section-3 deals with the study of non-invariant hypersurfaces of $\varepsilon-$para Sasakian manifolds. A necessary and sufficient condition for a totally umbilical non-invariant hypersurface of an $\varepsilon-$para Sasakian manifold to be totally geodesic is found.

2. Preliminaries

Let $\widetilde{M}$ be an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g) where ϕ is $(1,1)$ tensor field, η is $1-$ form and g is a compatible Riemannian metric such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta o \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = g(\phi X, Y), \quad g(X, \xi) = \epsilon \eta(X) \tag{2.3}$$

for all $X, Y \in T\widetilde{M}$.

An almost contact metric manifold is an $\varepsilon-$para Sasakian manifold if

$$(\widetilde{\nabla}_X \phi)(Y) = -g(\phi X, \phi Y) - \epsilon\eta(Y)\phi^2 X \tag{2.4}$$

for for all vector fields X, Y on $\widetilde{M}$ where $\widetilde{\nabla}$ is the operator of covariant differentiation with respect to g. From (2.4), we have

$$\widetilde{\nabla}_X \xi = \varepsilon\phi X \tag{2.5}$$

A hypersurface of an almost contact metric manifold $\widetilde{M}\,(\phi, \xi, \eta, g)$ is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of $(1,1)$ tensor field ϕ defining the contact structure is never tangent to the hypersurface. Let X be a tangent vector on a non-invariant hypersurface of an almost contact metic manifold $\widetilde{M}$, then $X\phi$ is never tangent to the hypersurface.

Let M be a non-invariant hypersurface of an almost contact metric manifold, then defining

$$\phi X = fX + u(X)\hat{N}, \tag{2.6}$$

$$\phi\hat{N} = -U, \tag{2.7}$$

$$\xi = V + \lambda\hat{N}, \qquad \lambda = n(\hat{N}); \tag{2.8}$$

$$\eta(X) = \nu(X), \tag{2.9}$$

where f is a $(1,1)$ tensor field, u, v are $1-$forms, $\hat{N}$ is a unit normal to the hypersurface, $X \in TM$ and $u(X) \neq 0$, then we get an induced (f, g, u, v, λ) structure on M satisfying the conditions [11, 12] :

$$f^2 = -I + u \otimes U + v \otimes V, \tag{2.10}$$

$$fU = -\lambda V, \quad fV = \lambda U, \tag{2.11}$$

$$uof = \lambda v, \qquad vof = -\lambda u, \tag{2.12}$$

$$u(U) = 1 - \lambda^2, \;\; u(V) = v(U) = 0, \qquad v(V) = 1 - \lambda^2, \tag{2.13}$$

$$g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y), \tag{2.14}$$

$$g(X,fY)=-g(fX,Y),\qquad g(X,U)=u(X),\qquad g(X,V)=v(X),\tag{2.15}$$

for all $X,Y\in TM$, where $\lambda=n(\hat{N})$.

The Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X,Y)\hat{N},\tag{2.16}$$

$$\tilde{\nabla}_X\hat{N}=-A_{\hat{N}}X,\tag{2.17}$$

for all $X,Y\in TM$, where $\tilde{\nabla}$ and ∇ are the Riemannian and induced Riemannian connections on $\tilde{M}$ and M respectively and $\hat{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$. The second fundamental form σ on M is related to $A_{\hat{N}}$ by

$$\sigma(X,Y)=g(A_{\hat{N}}X,Y),\qquad \text{for all } X,Y\in TM.\tag{2.18}$$

3. Non-invariant Hypersurfaces

Lemma 3.1. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$structure of an $\varepsilon-$para Sasakian manifold $\tilde{M}$, then*

$$(\tilde{\nabla}_X\phi)Y=(\nabla_X f)Y-u(Y)A_{\hat{N}}X+\sigma(X,Y)U+((\nabla_X u)Y+\sigma(X,fY))\hat{N}\tag{3.1}$$

$$(\tilde{\nabla}_X\eta)Y=(\nabla_X u)Y-\lambda\sigma(X,Y)\tag{3.2}$$

$$\tilde{\nabla}_X\xi=\left(\nabla_X V-\lambda A_{\hat{N}}X\right)+(\sigma(X,V)+X\lambda)\hat{N}\tag{3.3}$$

for all $X,Y\in TM$.

Proof. : Consider

$$\begin{aligned}(\tilde{\nabla}_X\phi)Y &= \tilde{\nabla}_X(\phi Y)-\phi(\tilde{\nabla}_X Y)\\ &= \tilde{\nabla}_X(fY+u(Y)\hat{N})-\phi(\nabla_X Y+\sigma(X,Y)\hat{N})\\ &= \tilde{\nabla}_X(fY)+\tilde{\nabla}_X(u(Y)\hat{N})-f(\nabla_X Y)-u(\nabla_X Y)\hat{N}-\sigma(X,Y)(-U)\\ &= \nabla_X(fY)+\sigma(X,fY)\hat{N}+u(Y)(-A_{\hat{N}}X)+\nabla_X(u(Y))\hat{N}-f(\nabla_X Y)\\ &\quad -u(\nabla_X Y)\hat{N}+\sigma(X,Y)U\end{aligned}$$

which gives,

$$(\widetilde{\nabla}_X\phi)Y = (\nabla_X f)Y - u(Y)(A_{\hat{N}}X) + \sigma(X,Y)U + (\nabla_X u)Y + \sigma(X, fY)\hat{N}$$

Also,

$$\begin{aligned}(\widetilde{\nabla}_X\eta)Y &= \widetilde{\nabla}_X\eta(Y) - \eta(\widetilde{\nabla}_X Y)\\ &= \nabla_X(v(Y)) - v(\nabla_X Y) - \sigma(X,Y)\,\eta(\hat{N}).\end{aligned}$$

Therefore

$$(\widetilde{\nabla}_X\eta)Y = (\nabla_X u)Y - \lambda\sigma(X,Y)$$

Now consider

$$\begin{aligned}\widetilde{\nabla}_X\xi &= \nabla_X\xi + \sigma(X,\xi)\hat{N}\\ &= \nabla_X V + \nabla_X\lambda\hat{N} + \sigma(X,V)\hat{N}\\ &= \nabla_X V - \lambda\nabla_X\hat{N} + (X\lambda)\hat{N} + \sigma(X,V)\hat{N}\end{aligned}$$

which gives

$$\widetilde{\nabla}_X\xi = \left(\nabla_X V - \lambda A_{\hat{N}}X\right) + (\sigma(X,V) + X\lambda)\hat{N}.$$

□

Theorem 3.1. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$, then*

$$\sigma(X,\xi)U = -\varepsilon f^2X + \varepsilon u(X)U + f(\nabla_X\xi), \tag{3.4}$$

and

$$u(\nabla_X\xi) = -\varepsilon u(fX) \tag{3.5}$$

Proof. Consider

$$\begin{aligned}\left(\widetilde{\nabla}_X\phi\right)\xi &= \widetilde{\nabla}_X(\phi\xi) - \phi\left(\widetilde{\nabla}_X\xi\right)\\ &= -\varepsilon\phi(\phi X)\end{aligned}$$

or

$$\left(\widetilde{\nabla}_X\phi\right)\xi = -\varepsilon\phi\left(fX + u(X)\right)\hat{N} \tag{3.6}$$

Also we know that

$$\left(\widetilde{\nabla}_X\phi\right)\xi = -\phi\left(\nabla_X\xi\right) + \sigma(X,\xi)U \tag{3.7}$$

From (3.6) and (3.7) , we have

$$\begin{aligned} -\phi\left(\nabla_X \xi\right)+\sigma(X,\xi)U &= -\varepsilon\phi\left(fX+u(X)\right)\hat{N} \\ &= -\varepsilon\phi\left(fX\right)+\epsilon u(X)U \end{aligned}$$

Now from (2.6) & (2.7) , we have

$$-f\left(\nabla_X \xi\right)-u(\nabla_X \xi)\hat{N}+\sigma(X,\xi)U=-\varepsilon(f\left(fX\right)+u(fX)\hat{N})+\varepsilon u(X)U$$

Equating tangential and normal parts, we get

$$\sigma(X,\xi)U=-\varepsilon f^2X+\varepsilon u(X)U+f(\nabla_X \xi),$$

and

$$u\left(\nabla_X \xi\right)=-\varepsilon u(fX)$$

□

Theorem 3.2. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$, then*

$$(3.8)\qquad \left(\nabla_X f\right)Y=-g(X,Y)V+\varepsilon v(Y)X+\sigma(X,Y)U+u(Y)A_{\hat{N}}X$$

$$(3.9)\qquad \left(\nabla_X u\right)Y=-\lambda g\left(X,Y\right)-\sigma\left(X,fY\right)$$

Proof. From equations (3.1) & (2.4) , we have

$$\begin{aligned} &\left(\nabla_X f\right)Y-u(Y)A_{\hat{N}}X+\sigma(X,Y)U+\left(\left(\nabla_X u\right)Y+\sigma\left(X,fY\right)\right)\hat{N} \\ =\ & -g(X,Y)V-\lambda g\left(X,Y\right)\hat{N}+\varepsilon v(Y)X \end{aligned}$$

Equating tangential and normal parts in the above equation, we get (3.8) & (3.9) respectively. □

Theorem 3.3. *Let M be a non-invariant hypersurface with $(f,g,u,v,\lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$, then*

$$(3.10)\qquad \left(\widetilde{\nabla}_X\phi\right)Y=-g(X,Y)V-\lambda g\left(X,Y\right)\hat{N}+\varepsilon v(Y)X+2\sigma\left(X,Y\right)U$$

Proof. Consider

$$\begin{aligned} \left(\widetilde{\nabla}_X\phi\right)Y &= \widetilde{\nabla}_X\left(\phi Y\right)-\phi\left(\widetilde{\nabla}_X Y\right) \\ &= \widetilde{\nabla}_X\left(fY\right)+\widetilde{\nabla}_X\left(u(Y)\hat{N}\right)-f\left(\nabla_X Y\right)-u\left(\nabla_X Y\right)\hat{N}, \end{aligned}$$

This implies

$$(3.11)\quad \left(\tilde{\nabla}_X \phi\right) Y = (\nabla_X f) Y - u(Y) A_{\hat{N}} X + \sigma(X,Y)U + ((\nabla_X u) Y + \sigma(X, fY)) \hat{N}$$

Using (3.8) & (3.9) ,above equation reduces to

$$\left(\tilde{\nabla}_X \phi\right) Y = -g(X,Y) V - \lambda g(X,Y) \hat{N} + \epsilon v(Y) X + 2\sigma(X,Y)U$$

□

Furthur, we proceed for some results on totally geodesic non−invariant hypersurfaces.

Theorem 3.4. *Let M be a totally umbilical non-invariant hypersurface with* $(f, g, u, v, \lambda)-$ *structure of an* $\varepsilon-$*para Sasakian manifold* $\tilde{M}$, *then it is totally geodesic if and only if*

$$(3.12)\quad \varepsilon u(X) - X\lambda = 0$$

Proof. Consider

$$\begin{aligned} \tilde{\nabla}_X \xi &= \nabla_X \xi + \sigma(X,\xi)\hat{N} \\ &= \nabla_X (V + \lambda \hat{N}) + \sigma(X,V)\hat{N} \\ &= \nabla_X V + \nabla_X \lambda \hat{N} + \sigma(X,V)\hat{N} \\ &= \nabla_X V + \lambda \nabla_X \hat{N} + (X\lambda)\hat{N} + \sigma(X,V)\hat{N} \end{aligned}$$

or

$$(3.13)\quad \tilde{\nabla}_X \xi = \left(\nabla_X V - \lambda A_{\hat{N}} X\right) + (\sigma(X,V) + X\lambda)\hat{N},$$

Now from (2.5) , the above equation is reduced to

$$\varepsilon(fX + u(X)\hat{N}) = \left(\nabla_X V - \lambda A_{\hat{N}} X\right) + (\sigma(X,V) + X\lambda)\hat{N},$$

Equating normal parts on both the sides, we get

$$(3.14)\quad \sigma(X,V) + X\lambda = \varepsilon u(X)$$

Now if M is totally umbilical, then $A_{\hat{N}} = \zeta I$, ζ is Kahlerian metric and equation (2.18) reduces to $\sigma(X,Y) = g\left(A_{\hat{N}} X, Y\right) = g(\zeta X, Y) = \zeta g(X,Y)$,

Therefore

$$\sigma(X,Y) = \zeta g(X,Y),$$

and equation (3.14) implies

$$\zeta g(X,Y) + X\lambda = \varepsilon u(X)$$

or

$$\varepsilon u(X) - X\lambda - \zeta g(X,Y) = 0 \tag{3.15}$$

Now if M is totally geodesic i.e. $\zeta = 0$, then (3.15) gives

$$\varepsilon u(X) - X\lambda = 0.$$

□

Theorem 3.5. *Let M be a non-invariant hypersurface with $(f, g, u, v, \lambda)-$ structure of an $\varepsilon-$para Sasakian manifold $\widetilde{M}$. If U is parallel, then we have*

$$\varepsilon\lambda X + f\left(A_{\hat{N}}X\right) - g(\phi X, U)V - \varepsilon\lambda v(X)V = 0 \tag{3.16}$$

Proof. Consider

$$\begin{aligned}
(\widetilde{\nabla}_X\phi)\hat{N} &= \widetilde{\nabla}_X\phi\hat{N} - \phi\left(\widetilde{\nabla}_X\hat{N}\right) \\
&= -\widetilde{\nabla}_XU - \phi(-A_{\hat{N}}X) \\
&= -\widetilde{\nabla}_XU - (-f(-A_{\hat{N}}X) - u(A_{\hat{N}}X)\hat{N})
\end{aligned}$$

This gives

$$(\widetilde{\nabla}_X\phi)\hat{N} = -\nabla_XU + f(A_{\hat{N}}X) \tag{3.17}$$

From equation (2.4), we have

$$(\widetilde{\nabla}_X\phi)Y = -g\left(\phi X, \phi Y\right)V - \lambda g\left(\phi X, \phi Y\right)\hat{N} + \varepsilon\eta\left(Y\right)X - \varepsilon\eta\left(X\right)\eta\left(Y\right)\xi \tag{3.18}$$

Substituting $Y = \hat{N}$, we have

$$(\widetilde{\nabla}_X\phi)\hat{N} = g(\phi X, U)V + \lambda g\left(\phi X, U\right)\hat{N} - \varepsilon\lambda X + \varepsilon\lambda v(X)\xi \tag{3.19}$$

Now from (3.17) and (3.19) , we have

$$-\nabla_XU + f(A_{\hat{N}}X) = g(\phi X, U)V + \lambda g\left(\phi X, U\right)\hat{N} - \varepsilon\lambda X + \varepsilon\lambda v(X)\xi$$

Equating tangential parts on both the sides, we have

$$\nabla_X U = f(A_{\hat{N}} X) - g(\phi X, U)V - \varepsilon\lambda X + \varepsilon\lambda v(X)V \tag{3.20}$$

Now if U is parellel, then

$$\varepsilon\lambda X - f(A_{\hat{N}} X) + g(\phi X, U)V - \varepsilon\lambda v(X)V = 0$$

□

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