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ON A CLASS OF β-KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to study globally ϕ-quasiconformally symmetric β-Kenmotsu manifolds. It has been shown that a globally ϕ-quasiconformally symmetric β-Kenmotsu manifold is globally ϕ-symmetric. Also we study 3-dimensional locally ϕ-symmetric β-Kenmotsu manifolds. Next we study second order parallel tensor and Ricci soliton on 3-dimensional β-Kenmotsu manifolds. Finally, we give some examples of 3-dimensional β-Kenmotsu manifolds which verifies our result.

1. Introduction

In [25] Tanno classified connected almost contact metric manifolds whose automorphism groups have the maximum dimension. For such a manifold M, the sectional curvature of plane section containing ξ is a constant, say c. If $c > 0$, M is a homogeneous Sasakian manifold of constant ϕ-sectional curvature. If $c = 0$, M is the product of a line or circle with a Kaehler manifold of constant holomorphic curvature. If $c < 0$, M is a warped product space $\mathbb{R} \times_f C^n$. In [13] Kenmotsu abstracted the differential geometric properties of the third case. In particular the almost contact metric structure in this case satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

and an almost contact metric manifold satisfying this condition is called a Kenmotsu manifold ([11],[13]). Again one has the more general notion of a β-Kenmotsu structure [11] which may be defined by

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.1}$$

where β is a non-zero constant. From the condition one may readily deduce that

$$\nabla_X \xi = \beta(X - \eta(X)\xi). \tag{1.2}$$

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Kenmotsu manifolds appear as examples of β-Kenmotsu manifolds, with $\beta = 1$. β-Kenmotsu manifolds have been studied by several authors such as Matamba [26], Janssens, and Vanhecke [11] and many others.
In the classification of Gray and Hervella [9] of almost Hermitian manifolds there appears a class, W_4,of Hermitian manifolds which are closely related to locally conformally Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on M is trans-Sasakian [19] if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times$R defined by
J $(X, f\frac{d}{df}) = (\phi$X$-f\xi$, η(X)$\frac{d}{dt}$), for all vector fields X on M, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [5]

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.3}$$

for smooth functions α and β on M. Hence we say that the trans-Sasakian structure is of type (α,β). In particular, it is normal and it generalizes both α-Sasakian and β-Kenmotsu structures. From the formula one easily obtains

$$\nabla_X\xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi). \tag{1.4}$$

Hence a trans-Sasakian structure of type (α, β) with $\alpha, \beta \in \mathbb{R}$ and $\alpha = 0$ is a β-Kenmotsu structure. The relation between trans-Sasakian, α-Sasakian and β-Kenmotsu structures was recently discussed by Marrero [15].

Proposition1.1(Marrero[15]): A trans-Sasakian manifold of dimension $\geq$5 is either α-Sasakian, β-Kenmotsu or Cosymplectic.

Let M_1 and M_2 be almost contact metric manifolds with structure tensors $(\phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$. Define an almost complex structure J on $M_1 \times M_2$ by

$$J(X_1, X_2) = (\phi_1X_1 - e^{-2\mu}\eta_2(X_2)\xi_1, \phi_2X_2 + e^{2\mu}\eta_1(X_1)\xi_2), \tag{1.5}$$

where μ is a function on $M_1 \times M_2$. Let $\widetilde{g}$ be the Riemannian metric on $M_1 \times M_2$ defined by

$$\widetilde{g}((X_1, X_2), (Y_1, Y_2)) = e^{2\rho}g_1(X_1, Y_1) + e^{2\tau}g_2(X_2, Y_2), \tag{1.6}$$

where ρ and τ are function on $M_1 \times M_2$. Blair and Oubina [5] proved that if $(M_1 \times M_2, J, \widetilde{g})$ is Kaehlerian, then M_2 is β- Kenmotsu if and only if $\xi_1\tau = 0$ and $grad^2\tau = -\beta\xi_2$.
Kenmotsu manifolds have been studied by several authors such as G.Pitis ([21],[22]), Jun, De and Pathak [12], De and Pathak ([8], [6]), Binh, Tamassy, De and Tarafdar [1], Sulgar, Özgür, and De [23] and many others.

Let (M^n, g), $n > 3$, be a Riemannian manifold. The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [28]. According to them a quasi-conformal curvature tensor is defined by

$$\begin{aligned} C^*(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad -g(X, Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{1.7}$$

where a and b are constants, S is the Ricci tensor, Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold M^n. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.7) takes the form

$$\begin{aligned} C^*(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z, \end{aligned}$$

where C is the conformal curvature tensor [27]. In [7], De and Matsuyama studied a quasi-conformally flat Riemannian manifold satisfying a certain condition on the Ricci tensor. From Theorem 5 of [7], it can be proved that a 4-dimensional quasi-conformally flat semi-Riemannian manifold is the Robertson-Walker space time. Robertson-Walker spacetime is the warped product $I \times_f M^*$, where M^* is a space of constant curvature and I is an open interval [16]. Thus quasi-conformal curvature tensor has some importance in general theory of relativity also. From (1.7), we obtain

$$\begin{aligned} (\nabla_W C^*)(X,Y)Z &= a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \\ &\quad +g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)] \\ &\quad -\frac{dr(W)}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.8}$$

where ∇ denotes the Levi-Civita connection . If the condition

$$\nabla R = 0 \tag{1.9}$$

holds on M, then M is called locally symmetric. A β-Kenmotsu manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_X R)(Y,Z)W) = 0, \tag{1.10}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [24]. Later in [4], Blair, Koufogiorgos and Sharma studied locally ϕ-symmetric contact metric manifolds.

In (1.10), if X, Y, Z and W are not horizontal vectors then we call the manifold globally ϕ-symmetric.

In this paper, we define locally ϕ-quasiconformally symmetric and globally ϕ-quasiconformally symmetric contact metric manifolds. A contact metric manifold (M, g) is called locally ϕ-quasiconformally symmetric if the condition

$$\phi^2((\nabla_X C^*)(Y,Z)W) = 0 \tag{1.11}$$

holds on M, where X, Y, Z and W are horizontal vectors. If X, Y, Z and W are arbitrary vectors then the manifold is called globally ϕ-quasiconformally symmetric. Quasi-conformal curvature tensor have been studied by several authors such as Yano and Sawaki [28], Ghosh and De [10], De and Matsuyama [7], Ozgur and de [20] and many others. Motivated by the above studies in the present paper we like to study ϕ-quasi-conformally symmetric β-Kenmotsu manifolds.

In a Riemannian manifold a tensor α of **second order** is said to be **parallel** if

$$\nabla\alpha = 0,$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

In 1926 H. Levy [14] proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In recent papers R. Sharma [18], generalized Levy's result and also studied a second order parallel tensor on Kaehler space of constant holomorphic sectional curvature as well as on contact manifolds.

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [17]. On the manifold M, a **Ricci soliton** is a triple (g, V, λ) with g, a Riemannian metric, V a vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.12}$$

where £ is a Lie derivative. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive.
A Kenmotsu manifold M of dimension $n > 2$ is called an **Einstein manifold** if the Ricci tensor S can be expressed as

$$S(X, Y) = \lambda g(X, Y), \tag{1.13}$$

where λ is a constant and also called an η**-Einstein manifold** if

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.14}$$

where a and b are smooth functions on the manifold.

The paper is organized as follows:
In section 1, we give a brief account of β-Kenmotsu manifolds. After preliminaries, in the next section , we study globally ϕ-quasi-conformally symmetric β-Kenmotsu manifolds. We prove that if a β-Kenmotsu manifold is globally ϕ-quasi-conformally symmetric, then the manifold is an Einstein manifold. We also show that a globally ϕ-quasi-conformally symmetric β-Kenmotsu manifold is globally ϕ-symmetric. In Section 4, we study 3-dimensional locally ϕ-quasi-conformally symmetric β-Kenmotsu manifolds. We prove that a 3-dimensional β-Kenmotsu manifold is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant if $a + b \neq 0$ and $r \neq -6\beta$. In the next section we prove that a parallel symmetric (0,2) tensor field in a 3-dimensional non-cosympletic β-Kenmotsu manifold is a

constant multiple of the associated metric tensor. In section 6, I prove that in a 3-dimensional non-cosymplectic β-Kenmotsu manifold, the Ricci soliton (g, ξ, λ) is shrinking and the manifold is an η-Einstein manifold. We also give some examples of 3-dimensional β-Kenmotsu manifolds.

2. Priliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g), that is, ϕ is an (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X,\ Y \in T(M)$([2],[3]).
If an almost contact metric manifold satisfies

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

then M is called a β-Kenmotsu manifold, where ∇ is the Levi-Civita connection of g. From the above equation it follows that

$$\nabla_X \xi = \beta(X - \eta(X)\xi), \tag{2.5}$$

and

$$(\nabla_X \eta)Y = \beta(g(X, Y) - \eta(X)\eta(Y)). \tag{2.6}$$

Moreover, the curvature tensor R and the Ricci tensor S satisfy

$$R(X, Y)\xi = \beta(\eta(X)Y - \eta(Y)X) \tag{2.7}$$

and

$$S(X, \xi) = -\beta(n-1)\eta(X). \tag{2.8}$$

3. Globally ϕ-quasiconformally symmetric β-Kenmotsu manifolds

Definition 3.1: A β-Kenmotsu manifold M is said to be globally ϕ-quasiconformally symmetric if the quasi-conformal curvature tensor C^* satisfies

$$\phi^2((\nabla_X C^*)(Y, Z)W) = 0, \tag{3.1}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Let us suppose that M is a globally ϕ-quasiconformally symmetric β-Kenmotsu manifold. Then by definition

$$\phi^2((\nabla_W C^*)(X, Y)Z) = 0, \tag{3.2}$$

Using (2.1) we have

$$-(\nabla_W C^*)(X,Y)Z + \eta((\nabla_W C^*)(X,Y)Z)\,\xi = 0. \tag{3.3}$$

From (1.8) it follows that

$$\begin{aligned}
&-ag((\nabla_W R)(X,Y)Z,U) - bg(X,U)(\nabla_W S)(Y,Z) + bg(Y,U)(\nabla_W S)(X,Z)\\
&-bg(Y,Z)g((\nabla_W Q)X,U) + bg(X,Z)g((\nabla_W Q)Y,U)\\
&+\frac{1}{n}dr(W)\left[\frac{a}{n-1}+2b\right](g(Y,Z)g(X,U) - g(X,Z)g(Y,U))\\
&+a\eta((\nabla_W R)(X,Y)Z)\eta(U) + b(\nabla_W S)(Y,Z)\eta(U)\eta(X) - b(\nabla_W S)(X,Z)\eta(U)\eta(Y)\\
&+bg(Y,Z)\eta((\nabla_W Q)X)\eta(U) - bg(X,Z)\eta((\nabla_W Q)Y)\eta(U)\\
&-\frac{1}{n}dr(W)\left[\frac{a}{n-1}+2b\right](g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\eta(U) = 0.
\end{aligned} \tag{3.4}$$

Putting $X = U = e_i$, where $\{e_i\}$, $(i = 1, 2, ..., n)$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i, we get

$$\begin{aligned}
&-(a+nb-2b)(\nabla_W S)(Y,Z) - \{bg((\nabla_W Q)e_i,e_i) - \frac{n-1}{n}dr(W)\left(\frac{a}{n-1}+2b\right)\\
&-b\eta((\nabla_W Q)e_i)\eta(e_i) + \frac{1}{n}dr(W)\left(\frac{a}{n-1}+2b\right)\}g(Y,Z) + bg((\nabla_W Q)Y,Z)\\
&+a\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) - b(\nabla_W S)(\xi,Z)\eta(Y) - b\eta((\nabla_W Q)Y)\eta(Z)\\
&+\frac{1}{n}dr(W)\left(\frac{a}{n-1}+2b\right)\eta(Y)\eta(Z) = 0.
\end{aligned} \tag{3.5}$$

Putting $Z = \xi$, we obtain

$$\begin{aligned}
&-(a+nb-2b)(\nabla_W S)(Y,\xi) - \eta(Y)\{bdr(W) - \frac{n-1}{n}dr(W)\left(\frac{a}{n-1}+2b\right)\\
&-b\eta((\nabla_W Q)e_i)\eta(e_i) + \frac{1}{n}dr(W)\left(\frac{a}{n-1}+2b\right)\} + a\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i)\\
&-b(\nabla_W S)(\xi,\xi)\eta(Y) + \frac{1}{n}dr(W)\left(\frac{a}{n-1}+2b\right)\eta(Y) = 0.
\end{aligned} \tag{3.6}$$

Now

$$\begin{aligned}
\eta((\nabla_W Q)e_i)\eta(e_i) &= g((\nabla_W Q)e_i,\xi)\eta(e_i)\\
&= \eta((\nabla_W Q)\xi) = g\left(Q\phi X,\xi\right)\\
&= S(\phi X,\xi) = 0.
\end{aligned} \tag{3.7}$$

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi). \tag{3.8}$$

$$\begin{aligned}
g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi)\\
&\quad - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi).
\end{aligned}$$

Since $\{e_i\}$ is an orthonormal basis $\nabla_X e_i = 0$ and using (2.7) we find

$$\begin{aligned} g\left(R(e_i, \nabla_W Y)\xi, \xi\right) &= \beta(g(\eta(e_i)\nabla_W Y - \eta(\nabla_W Y)e_i, \xi)) \\ &= \beta(\eta(e_i)\eta(\nabla_W Y) - \eta(\nabla_W Y)\eta(e_i)) \\ &= 0. \end{aligned}$$

As

$$g\left(R(e_i, Y)\xi, \xi\right) + g\left(R(\xi, \xi)\, Y, e_i\right) = 0 \tag{3.9}$$

we have

$$g\left(\nabla_W R(e_i, Y)\xi, \xi\right) + g\left(R(e_i, Y)\xi, \nabla_W \xi\right) = 0. \tag{3.10}$$

Using this we get

$$g\left(\left(\nabla_W R\right)(e_i, Y)\xi, \xi\right) = 0. \tag{3.11}$$

By the use of (3.7), (3.8) and (3.11), from (3.6) we obtain

$$\left(\nabla_W S\right)(Y, \xi) = \frac{1}{n} dr(W)\eta(Y), \tag{3.12}$$

since $a + (n-2)b \neq 0$. Because if $a + (n-2)b = 0$ then from (1.7), it follows that $C^* = aC$. So we can not take $a+(n-2)b = 0$. Putting $Y = \xi$ in (3.12) we get $dr(W) = 0$. This implies r is constant. So from (3.12), we have

$$\left(\nabla_W S\right)(Y, \xi) = 0. \tag{3.13}$$

Using (2.8), this implies

$$S(Y, W) = \lambda g(Y, W), \tag{3.14}$$

where $\lambda = -\beta(n-1)$. Hence we can state the following:

Theorem 3.1. *If a β-Kenmotsu manifold is globally ϕ-quasiconformally symmetric, then the manifold is an Einstein manifold.*

Next suppose $S(X, Y) = \lambda g(X, Y)$, i.e. $QX = \lambda X$. Then from (1.7) we have

$$\begin{aligned} C^*(X, Y)Z &= aR(X, Y)Z \\ &\quad + \left[2b\lambda - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right]\left[g(Y, Z)X - g(X, Z)Y\right], \end{aligned} \tag{3.15}$$

which gives us

$$\left(\nabla_W C^*\right)(X, Y)Z = a\left(\nabla_W R\right)(X, Y)Z. \tag{3.16}$$

Applying ϕ^2 on both sides of the above equation we have

$$\phi^2\left(\nabla_W C^*\right)(X, Y)Z = a\phi^2\left(\nabla_W R\right)(X, Y)Z. \tag{3.17}$$

Hence we can state:

Theorem 3.2. *A globally ϕ-quasiconformally symmetric β-Kenmotsu manifold is globally ϕ-symmetric.*

Remark 3.1. Since a globally ϕ-symmetric β-Kenmotsu manifold is always a globally ϕ-quasiconformally symmetric manifold, from Theorem 3.2 we conclude that on a β-Kenmotsu manifold, globally ϕ-symmetry and globally ϕ-quasiconformally symmetry are equivalent.

4. 3-dimensional locally ϕ-quasiconformally symmetric β-Kenmotsu manifolds

Let us consider a 3-dimensional β-Kenmotsu manifold. It is known that the conformal curvature tensor vanishes identically in the 3-dimensional Riemannian manifold. Thus we find

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{4.1}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Putting $Z = \xi$ in (4.1) and using (2.8) we have

$$\eta(Y)QX - \eta(X)QY = (\frac{r}{2} + \beta)[\eta(Y)X - \eta(X)Y]. \tag{4.2}$$

Putting $Y = \xi$ in (4.2) and using (2.1) and (2.8), we get

$$QX = \frac{1}{2}[(r + 2\beta)X - (r + 6\beta)\eta(X)\xi], \tag{4.3}$$

that is,

$$S(X,Y) = \frac{1}{2}[(r + 2\beta)g(X,Y) - (r + 6\beta)\eta(X)\eta(Y)]. \tag{4.4}$$

Using (4.3) in (4.1), we get

$$\begin{aligned} R(X,Y)Z &= (\frac{r+4\beta}{2})[g(Y,Z)X - g(X,Z)Y] - (\frac{r+6\beta}{2})[g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{4.5}$$

Putting (4.3), (4.4) and (4.5) into (1.7) we have

$$\begin{aligned} C^{*}(X,Y)Z &= (a+b)(r+6\beta)[\frac{1}{3}\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad -\frac{1}{2}\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}]. \end{aligned} \tag{4.6}$$

Thus we have

Lemma 4.1. *Let M be a 3-dimensional β-Kenmotsu manifold. If $a+b=0$ or $r=-6\beta$, then the quasi-conformal curvature tensor vanishes identically.*

Next, we assume that $a+b\neq 0$ or $r\neq -6\beta$. Taking the covariant differentiation of (4.6), we get

$$\begin{aligned}(\nabla_W C^*)(X,Y)Z &= \frac{dr(W)}{3}(a+b)\{g(Y,Z)X-g(X,Z)Y\}\\ &-\frac{dr(W)}{2}(a+b)\{g(Y,Z)\eta(X)\xi\\ &-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &-\frac{1}{2}(r+6\beta)(a+b)[g(Y,Z)(\nabla_W\eta)(X)\xi-g(X,Z)(\nabla_W\eta)(Y)\xi\\ &+g(Y,Z)\eta(X)\nabla_W\xi-g(X,Z)\eta(Y)\nabla_W\xi\\ &+g(Y,\nabla_W\xi)\eta(Z)X+g(Z,\nabla_W\xi)\eta(Y)X\\ &-g(X,\nabla_W\xi)\eta(Z)Y-g(Z,\nabla_W\xi)\eta(X)Y].\end{aligned}$$

If the vector fields X, Y and Z are horizontal, then the above equation is rewritten as follows:

$$\begin{aligned}(\nabla_W C^*)(X,Y)Z &= \frac{dr(W)}{3}(a+b)\{g(Y,Z)X-g(X,Z)Y\}\\ &-\frac{1}{2}(r+6\beta)(a+b)[g(Y,Z)(\nabla_W\eta)(X)\xi-g(X,Z)(\nabla_W\eta)(Y)\xi].\end{aligned}\tag{4.7}$$

Operating ϕ^2 to the above equation, then we find

$$\phi^2((\nabla_W C^*)(X,Y)Z)=-\frac{dr(W)}{3}(a+b)\{g(Y,Z)X-g(X,Z)Y\}.\tag{4.8}$$

Hence we conclude the following theorem:

Theorem 4.1. *A 3-dimensional β-Kenmotsu manifold is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant if $a+b\neq 0$ and $r\neq -6\beta$.*

If $\beta=1$, then the manifold reduces to a Kenmotsu manifold. Thus from the above theorem we get the following:

Corollary 4.1. *A 3-dimensional Kenmotsu manifold is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant if $a+b\neq 0$ and $r\neq -6$.*

5. Second order parallel tensor

Let us consider a parallel symmetric (0,2)-tensor δ on a 3-dimensional β-Kenmotsu manifold M.
Then, by $\nabla\delta = 0$, we have

$$\delta(R(U,V)X,Y) + \delta(X, R(U,V)Y) = 0, \tag{5.1}$$

where U, V, X and Y are arbitrary vectors fields on M.
As δ is symmetric , putting $U = X = Y = \xi$ in (5.1), we obtain

$$\delta(\xi, R(\xi,X)\xi) = 0. \tag{5.2}$$

Now applying (2.7) in (5.2) we have

$$\beta\delta(Y,\xi) - \beta\eta(Y)\delta(\xi,\xi) = 0. \tag{5.3}$$

Differentiating (5.3) covariantly along X we find

$$\begin{aligned}&\beta\{\delta(\nabla_X Y,\xi) + \delta(Y,\nabla_X\xi)\} - \beta\{g(\nabla_X Y,\xi)\\ &+g(Y,\nabla_X\xi)\}\delta(\xi,\xi) - 2\beta g(Y,\xi)\delta(\nabla_X\xi,\xi) = 0.\end{aligned} \tag{5.4}$$

Putting $Y = \nabla_X Y$ in (5.2) we get

$$\beta\{\delta(\nabla_X Y,\xi) - \beta\eta(\nabla_X Y)\delta(\xi,\xi)\} = 0. \tag{5.5}$$

From (5.4) and (5.5) we have

$$\beta\delta(Y,\nabla_X\xi) - \beta g(Y,\nabla_X\xi)\delta(\xi,\xi) - 2\beta g(Y,\xi)\delta(\nabla_X\xi,\xi) = 0,$$

which implies that

$$\beta^2\{\delta(Y,X) - g(Y,X)\delta(\xi,\xi)\} = 0.$$

This implies either

$$\delta(Y,X) = \delta(\xi,\xi)g(Y,X), \quad or, \quad \beta = 0. \tag{5.6}$$

Since δ and g are parallel tensor fields, $\lambda = \delta(\xi,\xi)$ is constant on U. By the parallelity of δ and g it must be $\delta = \lambda g$ on whole of M. Thus we have the following:

Theorem 5.1. *A parallel symmetric (0,2) tensor in a 3-dimensional non-cosympletic β-Kenmotsu manifold is a constant multiple of the associated metric tensor.*

6. Ricci solitons

Suppose a 3-dimensional β-Kenmotsu manifold admits a Ricci soliton defined by (1.12). It is well known that $\nabla g = 0$. Since λ in the Ricci soliton equation

(1.12) is a constant, so $\nabla\lambda g = 0$. Thus $\pounds_V g + 2S$ is parallel. Hence using the previous theorem we have $\pounds_V g + 2S$ is a constant multiple of metric tensors g, that is, $\pounds_V g + 2S = ag$, where a is constant. Hence $\pounds_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$, that implies $\lambda = -a/2$. So we have the following:

Theorem 6.1. *In a 3-dimensional non-cosymplectic β-Kenmotsu manifold, the Ricci soliton (g, V, λ) is shrinking or expanding according as a is positive or negative.*

Now in particular we investigate the case $V = \xi$. Then (1.12) reduces to

$$\pounds_\xi g + 2S + 2\lambda g = 0. \tag{6.1}$$

Using (2.5) in a 3-dimensional β-Kenmotsu manifold we have

$$\pounds_\xi g(Y, Z) = 2\beta(g(Y, Z) - \eta(Y)\eta(Z)). \tag{6.2}$$

Then using (6.1) in (6.2) we get $\lambda = -S(\xi, \xi) = \beta(n - 1)$. Also from (6.1) it follows that the manifold is an η-Einstein manifold. Thus we have

Corollary 6.1. *In a 3-dimensional non-cosymplectic β-Kenmotsu manifold, the Ricci soliton (g, ξ, λ) is shrinking and the manifold is an η-Einstein manifold.*

7. Example of a 3-dimensional β- Kenmotsu manifold

Example 7.1: We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e_z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \quad e_3 = \alpha \frac{\partial}{\partial z}$$

are linearly independent at each point of M, where α is constant.

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have $[e_1, e_2] = 0$, $[e_1, e_3] = -\alpha e_1$ and $[e_2, e_3] = -\alpha e_2$.

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1} e_1 = \alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \tag{7.1}$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula (2.5) for $\beta = -\alpha$. Hence the manifold is a β-Kenmotsu manifold with $\beta =$ constant.

Example 7.2: We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$$

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$\begin{aligned} [e_1, e_3] &= e_1 e_3 - e_3 e_1 \\ &= z\frac{\partial}{\partial x}(z\frac{\partial}{\partial z}) - z\frac{\partial}{\partial z}(z\frac{\partial}{\partial x}) \\ &= z^2\frac{\partial^2}{\partial x \partial z} - z^2\frac{\partial^2}{\partial z \partial x} - z\frac{\partial}{\partial x} \\ &= -e_1. \end{aligned} \tag{7.2}$$

Similarly, $[e_1, e_2] = 0$ and $[e_2, e_3] = -e_2$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \tag{7.3}$$

which known as Koszul's formula.

Using (7.3) we have

$$\begin{aligned} 2g(\nabla_{e_1} e_3, e_1) &= -2g(e_1, e_1) \\ &= 2g(-e_1, e_1). \end{aligned} \tag{7.4}$$

Again by (7.3)

$$2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(-e_1, e_2) \tag{7.5}$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3). \tag{7.6}$$

From (7.4), (7.5) and (7.6) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X),$$

for all $X \varepsilon \chi(M)$.

Thus

$$\nabla_{e_1} e_3 = -e_1.$$

Therefore, (7.3) further yields

$$\nabla_{e_1}e_1 = e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = -e_1,$$

$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = -e_2,$$

$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0. \tag{7.7}$$

(7.7) tells us that the manifold satisfies (2.5) for $\beta = -1$ and $\xi = e_3$. Hence the manifold is a β-Kenmotsu manifold with β =constant.

It is known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{7.8}$$

With the help of the above results and using (7.8), it can be easily verified that

$$R(e_1,e_2)e_3 = 0, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_1,e_3)e_3 = -e_1,$$

$$R(e_1,e_2)e_2 = -e_1, \quad R(e_2,e_3)e_2 = e_3, \quad R(e_1,e_3)e_2 = 0,$$

$$R(e_1,e_2)e_1 = e_2, \quad R(e_2,e_3)e_1 = 0, \quad R(e_1,e_3)e_1 = e_3.$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned} S(e_1,e_1) &= g(R(e_1,e_2)e_2,e_1) + g(R(e_1,e_3)e_3,e_1) \\ &= -2. \end{aligned} \tag{7.9}$$

Similarly we have

$$S(e_2,e_2) = S(e_3,e_3) = -2.$$

Therefore,

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -6.$$

Thus the scalar curvature r is constant. Hence Theorem 4.1 is verified.

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