APPROXIMATION THEOREMS FOR LIMIT 
(p, q)–BERNSTEIN-DURRMEYER OPERATOR

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Abstract. In the present paper, using the method developed in [6], we prove the existence of the limit operator of the slight modification of the sequence of (p, q)-Bernstein-Durrmeyer operators introduced recently in [10]. We also establish the rate of convergence of this limit operator.

Keywords: Approximation theorems; (p, q)–Bernstein-Durrmeyer operator; Rate of convergence

1. Introduction

The applications of q-calculus in the field of approximation theory have led to the discovery of new generalizations of Bernstein operators. The first generalization involving q-integers was obtained by Lupaş [13] in 1987. Ten years later Phillips [18] gave another generalization of the Bernstein operators introducing the so-called q-Bernstein operators. After that, several well-known positive linear operators and other new operators have been generalized to their q-variants and their approximation behavior have been studied (see e.g. [3] and [11]). The concept of the limit q-Bernstein operator was introduced by Il’inskii and Ostrovska [12], and its rate of convergence was established by Wang and Meng [23], and Finta [7], respectively. Nowadays, the (p, q)-calculus renders to find new generalizations of q-Bernstein operators possible (see [1], [2], [4], [16], [22], [15], [8], [14]). Some basic definitions and theorems of (p, q)-calculus may be found in the papers [9], [21], [19] and [20].

The (p, q)-integers \([n]_{p,q}\) are defined by

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad q \neq p.
\]

For \(p = 1\), we recover the well-known q-integers \([n]_q = (1 - q^n)/(1 - q)\). Obviously

\[(1.1) \quad [n]_{p,q} = p^{n-1}[n]_{q/p}.\]

Received September 16, 2016; accepted January 05, 2017

2010 Mathematics Subject Classification. Primary 41A25; Secondary 41A30, 41A36

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The \((p, q)\)-factorials \([n]_{p,q}\) are defined by

\[
[n]_{p,q}! = \begin{cases} \left[1\right]_{p,q} \left[2\right]_{p,q} \cdots \left[n\right]_{p,q}, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0, \end{cases}
\]

and the \((p, q)\)-binomial coefficients are given by

\[
\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}, \quad 0 \leq k \leq n.
\]

Further, the \((p, q)\)-power basis is defined by

\[
(x \odot a)^n_{p,q} = \left\{ \begin{array}{ll} (x-a)(px-qa)(p^2x-q^2a) \cdots (p^{n-1}x-q^{n-1}a), & \text{if } n \geq 1 \\ 1, & \text{if } n = 0, \end{array} \right.
\]

and the \((p, q)\)-integral of \(f\) over the interval \([0, a]\) is defined as

\[
\int_0^a f(t) \, dp,q t = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f \left( \frac{aq^k}{p^{k+1}} \right) \quad \text{for } 0 < q < p \leq 1.
\]

In [10] the \((p, q)\)-analogue of the Bernstein-Durrmeyer operators was introduced in the following way:

\[
D_{n}^{p,q}(f; x)
\]

\[
= [n+1]_{p,q} \sum_{k=0}^{n} p^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p,q}(1, x) \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) f(t) \, dp,q t,
\]

where \(f \in C[0,1], x \in [0, 1], 0 < q < p \leq 1,\)

\[
b_{n,k}^{p,q}(1, x) = \binom{n}{k}_{p,q} p^{(k(k-1)-n(n-1))/2} x^k (1 \odot x)^{n-k}
\]

and

\[
\tilde{b}_{n,k}^{p,q}(p, pqt) = \binom{n}{k}_{p,q} (pt)^k (p \odot pqt)^{n-k}_{p,q}.
\]

In what follows we propose the following slight modification of (1.2):

\[
\tilde{D}_{n}^{p,q}(f; x)
\]

\[
= [n+1]_{p,q} \sum_{k=0}^{n} p^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p,q}(1, x) \int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) f(pt) \, dp,q t,
\]

where \(b_{n,k}^{p,q}(1, x)\) and \(\tilde{b}_{n,k}^{p,q}(p, pqt)\) have got the same expressions as above. For \(p = q = 1\), we recover the Durrmeyer operators (see [5]). The goal of the paper is to study the limit \((p, q)\)-Bernstein-Durrmeyer operator \(D_{\infty}^{p,q} : C[0,1] \to C[0,1]\) defined.
by $\tilde{D}_{n}^{p,q}(f;x) = \lim_{n \to \infty} \tilde{D}_{n}^{p,q}(f;x)$, where $f \in C[0,1]$ is arbitrary. Throughout the paper we fix the parameters $p,q$ such that $0 < q < p \leq 1$. We establish the rate of convergence of $\tilde{D}_{n}^{p,q}(f;x)$ using the modulus of continuity of $f \in C[0,1]$ given by

$$
\omega(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x - y| \leq \delta\}, \quad \delta > 0.
$$

The existence of $\tilde{D}_{n}^{p,q}$ is proven with the aid of the method developed in [6]. More precisely, we shall apply the following result (see [6, p. 393, Theorem 2.1] and [6, p. 394, Corollary 2.1]):

**Theorem 1.1.** Let $\Lambda$ be a set of parameters and for $\lambda \in \Lambda$ let $(\hat{L}_{n}^{\lambda})_{n \geq 1}$ be a sequence of positive linear operators on $C[0,1]$. If there exist the positive sequences $(\alpha_{n})_{n \geq 1}$ and $(\beta_{n})_{n \geq 1}$ such that

a) $\alpha_{n} \to 0$ as $n \to \infty$,

b) there exists $C_{1} > 0$ with $\beta_{n} + \beta_{n+1} + \ldots + \beta_{n+m-1} \leq C_{1}\alpha_{n}$ for all $n, m \geq 1$,

c) there exists $C_{2} > 0$ with $\|L_{n}^{\lambda}g - L_{n+1}^{\lambda}g\| \leq C_{2}\beta_{n}\|g\|$ for all $n \geq 1$ and $g \in C^{1}[0,1]$,

then there exists $C_{3} = C_{3}(\|L_{1}^{\lambda}e_{0}\|) > 0$ and a positive linear operator $L_{n}^{\lambda} : C[0,1] \to C[0,1]$ such that

$$
\|L_{n}^{\lambda}f - L_{n}^{\lambda}\| \leq C_{3}\omega(f, \alpha_{n})
$$

for all $f \in C[0,1]$ and $n = 1, 2, \ldots$

We mention that $\|\cdot\|$ denotes the uniform norm on $C[0,1]$, $e_{0}(x) = 1$ for $x \in [0,1]$, and the sequences $(\alpha_{n})_{n \geq 1}$ and $(\beta_{n})_{n \geq 1}$ may depend on $\lambda$.

## 2. Main results

First, we establish some auxiliary results.

**Lemma 2.1.** With the notation

$$
\lambda_{n,k}^{p,q}(f) = [n+1]_{p,q}p^{-(n^2+3n-k^2-k)/2} \int_{0}^{1} \tilde{\nu}_{n,k}^{p,q}(p,pq)t^{p}d_{p,q}t,
$$

where $k = 0, 1, \ldots, n$ and $f \in C[0,1]$, we have for $f \in [0,1]$ that

$$
\tilde{D}_{n}^{p,q}(f;x) - \tilde{D}_{n+1}^{p,q}(f;x)
= b_{n+1,0}^{p,q}(1,x)\{\lambda_{n,0}^{p,q}(f) - \lambda_{n+1,0}^{p,q}(f)\} + \sum_{k=1}^{n} b_{n+1,k}^{p,q}(1,x)\left\{\lambda_{n,k}^{p,q}(f)\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}p^{k} + \lambda_{n,k-1}^{p,q}(f)\frac{[k]_{p,q}}{[n+1]_{p,q}}q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f)\right\}
+ b_{n+1,n+1}^{p,q}(1,x)\{\lambda_{n,n}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f)\}.
$$
Proof. We express the difference $\bar{D}_{n}^{p,q}(f; x) - \bar{D}_{n+1}^{p,q}(f; x)$ as follows (see also [17, pp. 411-412]):

\[ \prod_{l=0}^{n} (p^l - q^l x)^{-1} \{ \bar{D}_{n}^{p,q}(f; x) - \bar{D}_{n+1}^{p,q}(f; x) \} = \sum_{k=0}^{n} \frac{\lambda_{n,k}^{p,q}(f)}{p_q} \left[ \begin{array}{c} n \\ k \end{array} \right] p^{(k(k-1)-n(n-1))/2} x^k \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} \]

(2.1) \\

Because

\[ x^k \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} = x^k p^{-n+k} (p^{n-k} - q^{n-k} x + q^{n-k} x) \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} \]

we get from (2.1), that

\[ \prod_{l=0}^{n} (p^l - q^l x)^{-1} \{ \bar{D}_{n}^{p,q}(f; x) - \bar{D}_{n+1}^{p,q}(f; x) \} \]

\[ = \sum_{k=0}^{n} \frac{\lambda_{n,k}^{p,q}(f)}{p_q} \left[ \begin{array}{c} n \\ k \end{array} \right] p^{(k(k-1)-n(n-1))/2} x^k \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} \]

\[ + \sum_{k=0}^{n} \frac{\lambda_{n,k}^{p,q}(f)}{p_q} \left[ \begin{array}{c} n+1 \\ k \end{array} \right] p^{(k(k-1)-(n+1)n)/2} x^{k+1} \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} \]

\[ - \sum_{k=0}^{n+1} \frac{\lambda_{n+1,k}^{p,q}(f)}{p_q} \left[ \begin{array}{c} n+1 \\ k \end{array} \right] p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} \]

\[ \{ \lambda_{n,0}^{p,q}(f)p^{-n(n-1)/2} p^{-n} - \lambda_{n+1,0}^{p,q}(f)p^{-(n+1)n/2} \}

\[ + \sum_{k=1}^{n} \left[ \begin{array}{c} n+1 \\ k \end{array} \right] p^{(k(k-1)-(n+1)n)/2} x^k \prod_{l=n-k}^{n} (p^l - q^l x)^{-1} \]

\[ \times \left\{ \lambda_{n,k}^{p,q}(f) \left[ \begin{array}{c} n \\ k \end{array} \right] p^k + \frac{\lambda_{n,k}^{p,q}(f)}{p_q} \left[ \begin{array}{c} n \\ k \end{array} \right] p^{n+1-k} \frac{q}{p} \right\} \]

\[ + \{ \lambda_{n+1,n}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f) \} x^{n+1} \prod_{l=0}^{n} (p^l - q^l x)^{-1} \]
we have

\[ \{ \lambda_{n+1,0}^{p,q}(f) - \lambda_{n+1,0}^{p,q}(f) \} p^{-\frac{1}{2}(n+1)n^2} \]

\[ + \sum_{k=1}^{n} \left[ \frac{n+1}{k} \right]_{p,q} p^{(k(k-1)-(n+1)n)/2} \prod_{l=n+1-k}^{n} (p^l - q^l)^{-1} \]

\[ \times \left\{ \lambda_{n,k}^{p,q}(f) \left[ \frac{n+1-k}{n+1} \right]_{p,q} p^k + \lambda_{n,k-1}^{p,q}(f) \left[ \frac{k}{n+1} \right]_{p,q} q^{n+1-k} - \lambda_{n+1,k}^{p,q}(f) \right\} \]

\[ + \{ \lambda_{n+1,n+1}^{p,q}(f) - \lambda_{n+1,n+1}^{p,q}(f) \} x^{n+1} \prod_{l=0}^{n} (p^l - q^l)^{-1} \].

Multiplying with \( \prod_{l=0}^{n} (p^l - q^l) \), we get the assertion of the lemma. \( \square \)

**Lemma 2.2.** For

\[ \tilde{\varphi}_{n,k}^{p,q}(p, pqt) = \left[ \frac{n}{k} \right]_{p,q} (pt)^k (p \circ pqt)^{n-k}, \quad k = 0, 1, \ldots, n, \]

we have

\[ [n+1]_{p,q} p^{-(n^2+3n-k^2-k)} \int_{0}^{1} \tilde{\varphi}_{n,k}^{p,q}(p, pqt) d_{p,q}t = 1, \]

\[ [n+1]_{p,q} p^{-(n^2+3n-k^2-k)} \int_{0}^{1} \tilde{\varphi}_{n,k}^{p,q}(p, pqt) dt d_{p,q}t = p^{n-k} \frac{[k+1]_{p,q}}{[n+2]_{p,q}}, \]

\[ [n+1]_{p,q} p^{-(n^2+3n-k^2-k)} \int_{0}^{1} \tilde{\varphi}_{n,k}^{p,q}(p, pqt)t^2 d_{p,q}t = p^{2(n-k)} \frac{[k+1]_{p,q}[k+2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}. \]

**Proof.** The equalities follow from the computations used in [10, Lemma 3.1]. \( \square \)

**Lemma 2.3.** For \( x \in [0, 1] \), we have

\[ \tilde{D}_n^{p,q}(1; x) = 1, \quad \tilde{D}_n^{p,q}(t; x) = \frac{p^{n+1} + pq[n]_{p,q}x}{[n+2]_{p,q}}, \]

\[ \tilde{D}_n^{p,q}(t^2; x) = \frac{p^{2n+2}[2x]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} + \frac{(2q^2 + q)p^{n+2}[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}} \]

\[ + \frac{q^2[n]_{p,q}[2x^2 + p^{n+1}x(1-x)]}{[n+2]_{p,q}[n+3]_{p,q}}. \]

**Proof.** Analogously to the proof of [10, Lemma 3.1], we get the statements of the lemma. \( \square \)

**Remark 2.1.** If \( p = p(n) \) and \( q = q(n) \) such that \( 0 < q(n) < p(n) \leq 1 \) and \( q(n) \to 1 \) as \( n \to \infty \), then, by Korovkin’s theorem, \( \tilde{D}_n^{p,q}(f; x) \) converges uniformly to \( f(x) \) for \( x \in [0, 1] \).
as \( n \to \infty \). Indeed, for each \( n \) the estimates

\[
|\dot{D}_n^{p,q}(t; x) - x| \leq \frac{1}{[n + 2]_q/p} + \frac{q[n]_{q/p}}{[n + 2]_{q/p} - 1},
\]

\[
|\dot{D}_n^{p,q}(t^2; x^2) - x^2| \leq \frac{2[n]_{q/p}}{[n + 2]_{q/p} + 3} + \left( 2 \frac{q}{p} \right)^3 + \frac{1}{[n + 2]_{q/p} - 1} + \left( \frac{q}{p} \right)^3 \frac{n}{[n + 2]_{q/p} - 1}.
\]

and the facts that \( [n]_{q/p} \to \infty \) and \( \frac{n}{n + 2} \to 1 \) as \( n \to \infty \), imply our statement.

In the next theorem we prove the existence of the limit \((p, q)\)-Bernstein-Durrmeyer operator.

**Theorem 2.1.** Let \( \dot{D}_n^{p,q}(f; x) \) be defined by (1.3), where \( p \) and \( q \) are fixed with \( 0 < q < p \leq 1 \). Then there exist an absolute constant \( C > 0 \) and a positive linear operator \( \dot{D}_{\infty}^{p,q} : C[0,1] \to C[0,1] \) such that

\[
\| \dot{D}_n^{p,q}(f; x) - \dot{D}_{\infty}^{p,q}(f; x) \| \leq C \omega \left( f, \left( \frac{q}{p} \right)^{n/2} \right)
\]

for all \( f \in C[0,1] \) and \( n = 1, 2, \ldots \)

**Proof.** We have \([n + 1]_{p,q} = p^n[n + 1 - k]_{p,q} + q^{n+1-k}[k]_{p,q}\) for \( k = 0, 1, \ldots, n + 1 \). Using the notation of Lemma 2.1, we obtain for \( f \in C[0,1] \), that

\[
\lambda_{n,k}(f) = \frac{[n + 1 - k]_{p,q}}{[n + 1]_{p,q}} p^k + \lambda_{n,k-1}(f) \frac{[k]_{p,q}}{[n + 1]_{p,q}} q^{n+1-k} - \lambda_{n+1,k}(f)
\]

\[
= \frac{[n + 1 - k]_{p,q}}{[n + 1]_{p,q}} p^k \{ \lambda_{n,k}(f) - \lambda_{n+1,k}(f) \} + \frac{[k]_{p,q}}{[n + 1]_{p,q}} q^{n+1-k} \{ \lambda_{n,k-1}(f) - \lambda_{n+1,k}(f) \}.
\]  

(2.2)

Let \( g \in C^1[0,1] \) and \( x_k = p^{n+2-k} \frac{[k + 1]_{p,q}}{[n + 3]_{p,q}} \) for \( k = 0, 1, \ldots, n \). Obviously, by (1.1), we have

\[
x_k = p^{n+2-k} \frac{[k + 1]_{q/p}}{p^{n+2} + [n + 3]_{q/p}} \in [0,1],
\]

where \( k = 0, 1, \ldots, n \). Further \( g(pt) = g(x_k) + \int_{x_k}^{pt} g'(u) du \), where \( t \in [0,1] \) is arbitrary. Hence, by definition of \( \lambda_{n,k}^{p,q}(g) \) and Lemma 2.2, we obtain

\[
\lambda_{n,k}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)
\]
Analogously, the equality \(\int_0^1 \tilde{b}_{n,k}^{p,q}(p, pqt) \left[ g(x_k) + \int_{x_k}^{pt} g'(u) \, du \right] \, dp \, qt\) for \(k = 0, 1, \ldots, n\). On the other hand, using \([n + 2]_{p,q} = p^{k+1}[n + 1 - k]_{p,q} + q^{n+1-k}[k + 1]_{p,q}\), we have

\[
\frac{[n + 1]_{p,q} \, p^{n+2}}{[n + 2]_{p,q}} \frac{\tilde{b}_{n,k}^{p,q}(p, pqt)}{\tilde{b}_{n+1,k}^{p,q}(p, pqt)} - 1
\]

\[
= \frac{[n + 1]_{p,q} \, p^{n+2}}{[n + 2]_{p,q}} \left[ \frac{n}{k} \right]_{p,q} (pt)^k (p \odot pqt)^{n-k} - 1
\]

\[
= \frac{[n + 1]_{p,q} \, p^{n+2}}{[n + 2]_{p,q}} \frac{n + 1}{[n + 1]_{p,q}} \frac{1}{p^{n+1-k} - pq^{n+1-k}} - 1
\]

\[
= \frac{[n + 1]_{p,q} \, p^{n+2}}{[n + 2]_{p,q}} \frac{p^{n+1-k} - pq^{n+1-k}}{p^{n+1-k} - pq^{n+1-k}} - 1
\]

\[
= q^{n+1-k} \left( \frac{[n + 1]_{p,q} \, p^{k+2}}{[n + 2]_{p,q}} \frac{p^{k+2}}{p^{n+1-k} - pq^{n+1-k}} \right) \frac{[k + 1]_{p,q}}{[n + 2]_{p,q}}.
\]

The equality \([n + 2]_{p,q} = p^{k+1}[n + 1 - k]_{p,q} + q^{n+1-k}[k + 1]_{p,q}\) implies that

\[
\frac{[n + 1]_{p,q} \, p^{n+1-k} - pq^{n+1-k}}{[n + 2]_{p,q}} \leq p^{-(k+1)}.
\]

Analogously, the equality \([n + 2]_{p,q} = q^{k+1}[n + 1 - k]_{p,q} + p^{n+1-k}[k + 1]_{p,q}\) implies that

\[
\frac{[k + 1]_{p,q}}{[n + 2]_{p,q}} \leq p^{-(n+1-k)}.
\]

Finally, the function \(t \rightarrow p^{k+2} / (p^{n+1-k} - pq^{n+1-k})\) is increasing on \([0, 1]\), therefore

\[
\frac{p^{k+2}}{p^{n+1-k} - pq^{n+1-k}} \leq \frac{p^{k+2}}{p^{n+1-k} - pq^{n+1-k}} = \frac{p^{k+2}}{p^{n+1-k} \left( 1 - p \left( \frac{2}{p} \right)^{n+1-k} \right)}
\]

\[
\leq \frac{p^{k+2}}{p^{n+1-k} \left( 1 - \frac{2}{p} \right)} = \frac{p^{k+2}}{p^{n+1-k}(1 - q)}.
\]
Combining (2.3)-(2.7), applying Lemma 2.2 and Hölder’s inequality, we find
\[ |\chi_{n,k}^{p,q}(g) - \chi_{n+1,k}^{p,q}(g)| \]
\[ \leq \|g'\|[n + 2]_{p,q}p_{-(n+1)^2 + 3(n+1) - k^2 - k}/2 \]
\[ \times \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p,pqt) pt - x_k |q^{n+1-k} \left( p^{-(k+1)} p^{k+2}_{p^{n+1-k}(1-q)} + p^{-(n+1-k)} \right) d_{p,q}t \]
\[ \leq \frac{1 + p - q}{1 - q} \|g'\| \left( \frac{q}{p} \right)^{n+1-k} \left[ n + 2 \right]_{p,q}p_{-(n+1)^2 + 3(n+1) - k^2 - k}/2 \]
\[ \left( \frac{q}{p} \right)^{n+1-k} \left[ n + 2 \right]_{p,q}p_{-(n+1)^2 + 3(n+1) - k^2 - k}/2 \]
\[ (2.8) \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p,pqt)(pt - x_k)^2 d_{p,q}t \right)^{1/2}. \]

Using Lemma 2.2, we get
\[ [n + 2]_{p,q}p_{-(n+1)^2 + 3(n+1) - k^2 - k}/2 \int_0^1 \tilde{b}_{n+1,k}^{p,q}(p,pqt)(pt - x_k)^2 d_{p,q}t \]
\[ = p^{2(n+2-k)} \frac{[k + 1]_{p,q}[k + 2]_{p,q}}{[n + 3]_{p,q}[n + 4]_{p,q}} - 2p^{n+2-k} \frac{[k + 1]_{p,q}p^{n+2-k} - [k + 1]_{p,q}}{[n + 3]_{p,q}} \]
\[ + \left( p^{n+2-k} \frac{[k + 1]_{p,q}}{[n + 3]_{p,q}} \right)^2 \]
\[ = p^{2(n+2-k)} \frac{[k + 1]_{p,q}[k + 2]_{p,q}}{[n + 3]_{p,q}[n + 4]_{p,q}} \left( \frac{[k + 1]_{p,q}}{[n + 3]_{p,q}} \right) \]
\[ = p^{2(n+2-k)} \left( \frac{[k + 1]_{p,q}}{[n + 3]_{p,q}} \right)^{k+1} \frac{n + 2 - k}{[n + 3]_{p,q}^2[n + 4]_{p,q}}. \]

Analogously to (2.6) and (2.5), we find that
\[ \frac{[k + 1]_{p,q}}{[n + 3]_{p,q}} \leq p^{-(n+2-k)} \quad \text{and} \quad \frac{n + 2 - k}{[n + 3]_{p,q}} \leq p^{-(k+1)}. \]

Further, using (1.1), we get \([n + 4]_{p,q} = p^{n+3} [n + 4]_{q/p} \geq p^{n+3}. \) Hence, by (2.8)-(2.10), we have for \( k = 0, 1, \ldots, n, \) that
\[ |\chi_{n,k}^{p,q}(g) - \chi_{n+1,k}^{p,q}(g)| \leq \frac{1 + p - q}{1 - q} \|g'\| \]
\[ \left( \frac{q}{p} \right)^{n+1-k} \left( \frac{q}{p} \right)^{n+1-k} \left( \frac{q}{p} \right)^{(k+1)/2} \]
\[ = \frac{1 + p - q}{1 - q} \|g'\| \left( \frac{q}{p} \right)^{n+1-k} \left( \frac{q}{p} \right)^{(k+1)/2} \]
\[ = \frac{1 + p - q}{1 - q} \|g'\| \left( \frac{q}{p} \right)^{(2n-k+3)/2} \]
\[ \leq \frac{1 + p - q}{1 - q} \|g'\| \left( \frac{q}{p} \right)^{n/2} \left( \frac{q}{p} \right)^{3/2}. \]

(2.11)
we find that

Thus, we have

where

Analogously to (2.3), we obtain for

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for \(k = 1, 2, \ldots, n + 1\), hence, in view of (2.12), Lemma 2.2 and Hölder’s inequality, we find that

Thus, we have

\[
|\lambda_{n,k-1}^{p,q}(g) - \lambda_{n+1,k}^{p,q}(g)| = \|g\| \left(1 + p^{-(n+1-k)} \frac{[n+2]_p}{[k]_p} \left\{ p^{2(n+2-k)} \frac{[k]_p}{[n+2]_p} \left( \frac{[k]_p}{[n+3]_p} \right)^2 \right\} \right)^{1/2}
\]
(2.13) \[
\left(\frac{[n+2]_{p,q}(pq)^k}{[k]_{p,q}} \right) \left(\frac{[n+2]_{p,q}}{[n+2]_{p,q} [n+3]_{p,q}}\right)^{1/2}.
\]

On the other hand, by (1.1), we have \([n+2]_{p,q} = p^{n+1}[n+2]_{q/p} \geq p^{n+1}[k]_{q/p} = p^{n+2-k}[k]_{p,q}\) for \(k = 1, 2, \ldots, n+1\). Further \([n+3]_{p,q} = p^{k+1}[n+2-k]_{p,q} + q^{n+2-k}[k+1]_{p,q}\), \(k = 1, 2, \ldots, n+1\), thus
\[
\frac{[n+2-k]_{p,q}}{[n+3]_{p,q}} \leq p^{-(k+1)};
\]
morover \([k]_{p,q} = p^{k-1}[k]_{q/p} \geq p^{k-1}[1]_{q/p} = p^{k-1}\) for \(k = 1, 2, \ldots, n+1\). Hence, by (2.13), we have
\[
|\lambda_{p,q}^1(g) - \lambda_{n+1,k}^1(g)| \leq (1 + p)\|g'\| \left(pq\right)^{k-1}p^{-k(1+1)}^{1/2}
\]
(2.14) \[
= (1 + p)\|g'\| \left(\frac{q}{p}\right)^{k/2}.
\]

Because
\[
\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}} = p^{-k} \frac{[n+1-k]_{q/p}}{[n+1]_{q/p}} \leq p^{-(n+1-k)}
\]
for \(k = 0, 1, \ldots, n\), and
\[
\frac{[k]_{p,q}}{[n+1]_{p,q}} = p^{-(n+1-k)} \frac{[k]_{q/p}}{[n+1]_{q/p}} \leq p^{-(n+1-k)}
\]
for \(k = 1, 2, \ldots, n+1\), we obtain, in view of (2.2), (2.11) and (2.14), that
\[
|\lambda_{p,q}^1(g)\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}p^k + \lambda_{p,q}^1(g)\frac{[k]_{p,q}}{[n+1]_{p,q}}q^{n+1-k} - \lambda_{n+1,k}^1(g)|
\]
\[
\leq \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}p^k|\lambda_{p,q}^1(g) - \lambda_{n+1,k}^1(g)| + \frac{[k]_{p,q}}{[n+1]_{p,q}}q^{n+1-k}|\lambda_{p,q}^1(g) - \lambda_{n+1,k}^1(g)|
\]
\[
\leq 1 + \frac{1}{1-q} + \frac{q}{p}\|g'\| \left(\frac{q}{p}\right)^{3/2} + (1 + p)\|g'\| \left(\frac{q}{p}\right)^{k/2}
\]
\[
\leq 1 + \frac{1}{1-q} + \frac{q}{p}\|g'\| \left(\frac{q}{p}\right)^{3/2} + (1 + p)\|g'\| \left(\frac{q}{p}\right)^{(n-k+2)/2}
\]
\[
\leq \|g'\| \left(\frac{q}{p}\right)^{n/2} \left\{\frac{1}{1-q} + \frac{q}{p}\right\} + \sqrt{\frac{q}{p}}(1 + p)
\].

This means that we may choose \(\beta_n = (q/p)^{n/2}, n \geq 1\) (see Theorem 1.1). Then for all \(n, m \geq 1\), we have
\[
\beta_n + \beta_{n+1} + \ldots + \beta_{n+m-1} = \left(\frac{q}{p}\right)^{n/2} + \left(\frac{q}{p}\right)^{(n+1)/2} + \ldots + \left(\frac{q}{p}\right)^{(n+m-1)/2}
\].
Further, by Lemma 2.3 and (1.1),

\[
\begin{align*}
\frac{2}{p} & = \left(\frac{q}{p}\right)^{n/2} \frac{1 - \left(\frac{q}{p}\right)^{m/2}}{1 - \left(\frac{q}{p}\right)} < \frac{\sqrt{p}}{\sqrt{p} - \sqrt{q}} \left(\frac{q}{p}\right)^{n/2}. \\
\end{align*}
\]

Thus we may choose \(\alpha_n = (q/p)^{n/2}, \ n \geq 1\). Applying Theorem 1.1, we get the statement of our theorem.

In the next theorem we shall estimate the error \(|\tilde{D}_{\infty}^{p,q}(f; x) - f(x)|\) with the aid of the modulus of continuity (1.4).

**Theorem 2.2.** For the limit \((p, q)\)-Bernstein-Durrmeyer operator \(\tilde{D}_{\infty}^{p,q}\), we have

\[
|\tilde{D}_{\infty}^{p,q}(f; x) - f(x)| \leq 2\omega \left(f, \sqrt{\delta_{p,q}(x)}\right)
\]

for all \(f \in C[0, 1]\) and \(x \in [0, 1]\), where

\[
\delta_{p,q}(x) = \frac{1}{p^2} (p - q) \{2p^2 + (3p + 1)x + (p^3 - 1)x^2\}.
\]

**Proof.** In view of Lemma 2.3, we have

\[
\tilde{D}_{\infty}^{p,q}(1, x) = 1.
\]

Further, by Lemma 2.3 and (1.1),

\[
\tilde{D}_{\infty}^{p,q}(t; x) = \frac{p^{n+1} + p^n q[n/q]_{x/p}}{p^{n+1}[n + 2]_{q/p}} = \frac{p + q[n/q]_{x/p}}{p[n + 2]_{q/p}} \to \frac{p + q}{p} =: \tilde{D}_{\infty}^{p,q}(t; x)
\]

as \(n \to \infty\), and analogously, we have

\[
\tilde{D}_{n}^{p,q}(t^2; x) = \frac{p^{2n+2}[q]_{p,q}}{[n + 2]_{p,q}[n + 3]_{p,q}} + \frac{(2q^2 + p)q^n[2]_{p,q}}{[n + 2]_{p,q}[n + 3]_{p,q}} + \frac{q^3[n]_{p,q}[p^2]_{p,q}x^2}{[n + 2]_{p,q}[n + 3]_{p,q}} + \frac{p^n+1x(1 - x)}{p^n+1} + \frac{q^3}{p^3} \left\{x^2 + \frac{p - q}{p}x(1 - x)\right\}
\]

as \(n \to \infty\). Then

\[
\tilde{D}_{\infty}^{p,q}((t - x)^2; x)
\]

\[
= \frac{1}{p^3}(p + q)(p - q)^2 + \frac{q^3}{p^3}(p - q)x(1 - x) + \frac{1}{p^3}(pq + 2q^2 - 2p^2)(p - q)x
\]

\[
+ \frac{1}{p^3}(p^2 - pq - q^2)(p - q)x^2
\]

\[
\leq \frac{2}{p^3}(p - q) + \frac{1}{p^3}(p - q)x(1 - x) + \frac{3}{p^3}(p - q)x + \frac{1}{p}(p - q)x^2
\]

(2.16) \(= \delta_{p,q}(x)\).
For the modulus of continuity (1.4) we have $\omega(f, \lambda \delta) \leq (1 + \lambda) \omega(f, \delta), \lambda \geq 0$. Then

$$|f(t) - f(x)| \leq \omega(f, |t - x|) \leq (1 + \delta^{-1}|t - x|) \omega(f, \delta)$$

for $t, x \in [0, 1]$. Hence, by (2.15), Hölder’s inequality and (2.16), we obtain

$$|\tilde{D}^{\infty}_{p,q}(f; x) - f(x)|$$

$$\leq \tilde{D}^{\infty}_{p,q}(|f(t) - f(x)|; x) \leq \omega(f, \delta) \left(1 + \delta^{-1}\tilde{D}^{\infty}_{p,q}(|t - x|; x)\right)$$

$$\leq \omega(f, \delta) \left(1 + \delta^{-1}\tilde{D}^{\infty}_{p,q}(|t - x|^2; x)\right)^{1/2} \leq \omega(f, \delta) \left(1 + \delta^{-1}\sqrt{\delta_{p,q}(x)}\right).$$

Choosing $\delta = \sqrt{\delta_{p,q}(x)}$, we get the assertion of the theorem. □

**Remark 2.2.** If $p = p(q)$ and $q \to 1$, then Theorem 2.2 implies that $\tilde{D}^{\infty}_{p,q}(f; x)$ converges uniformly to $f(x)$ for $x \in [0, 1]$.

**Acknowledgements.** The authors are thankful to the reviewers for valuable suggestions leading to overall improvements in the paper.

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