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ON SOME GENERALIZED DIFFERENCE SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY A SEQUENCE OF MODULI

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Abstract. We introduce the paranormed sequence spaces $c^F(\mathbf{f}, \Lambda, \Delta_m, p)$, $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ of fuzzy numbers associated with the multiplier sequence $\Lambda = (\lambda_k)$ determined by a sequence of moduli $\mathbf{f} = (f_k)$. Some of their properties like solidity, symmetricity, completeness etc. and inclusion relations are studied.

Keywords. fuzzy numbers, paranorm, modulus function, difference sequence

AMS subject classifications . 40A05; 40B05, 46S40.

1. Introduction and Preliminaries :

The idea of difference sequence spaces was first introduced by Kizmaz [10] and this concept was generalized by Et and Colak [7], Tripathy and Chandra [16] and many others. The concept of fuzzy set was introduced by L A. Zadeh [19] in the year 1965. Based on this, sequences of fuzzy numbers have been introduced by different authors and many important properties have been investigated. Applying the notion of fuzzy real numbers, different classes of fuzzy real-valued sequences have been introduced and investigated by Tripathy and Baruah [15],Tripathy and Debnath [18] and many researchers [2, 3, 4, 5] on sequence space.

A fuzzy real number X is a fuzzy set on R i.e. a mapping $X : R \to I = [0, 1]$ associating each real number t, with its grade of membership *X(t)*. The α-level set of a fuzzy real number X is denoted by $[X]_\alpha, 0 < \alpha \leq 1$, where $[X]_\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level set is the closure of the strong 0-cut i.e. $[X]_0 = cl(\{t \in R : X(t) > 0\})$.

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A fuzzy real number X is said to be upper semi continuous if for each $\epsilon > 0, X^{-1}([0, a+\epsilon))$, for all $a \in [0,1)$ is open in the usual topology of R. If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called normal. A fuzzy real number X is said to be convex, if $X(t) \geq X(s) \bigwedge X(r) = \min(X(s), X(r))$, where $s < t < r$. The class of all upper semi continuous, normal and convex fuzzy real numbers is denoted by $R(I)$.

Let $X, Y \in R(I)$ and the $\alpha-$level sets be $[X]_\alpha = [a_1^\alpha, a_2^\alpha], [Y]_\alpha = [b_1^\alpha, b_2^\alpha], \alpha \in [0,1]$. The absolute value of $X \in R(I)$ is defined by

$$|X|(t)=\begin{cases} \max\{X(t), X(-t)\}, & \text{for t} > 0 \\ 0, & \text{for t} \leq 0 \end{cases}$$

The additive identity and multiplicative identity of $R(I)$ are $\bar{0}$ and $\bar{1}$, respectively.

Let D be the set of all closed and bounded intervals $X = [X^L, X^R]$ on the real line R. For $X, Y \in D$ we define $X \leq Y$ iff $X^L \leq Y^L$ and $X^R \leq Y^R$. and $d(X, Y)=\max\{|X^L - Y^L|, |X^R - Y^R|\}$ where $X = [X^L, X^R]$ and $Y = [Y^L, Y^R]$. Then it can be shown that (D, d) is a complete metric space.

Let us define a mapping $\overline{d} : R(I) \times R(I) \longrightarrow R$ by $\overline{d}(X, Y)=\sup_\alpha d([X]^\alpha, [Y]^\alpha)$, for $X, Y \in R(I)$.

Throughout the paper $p = (p_k)$ is a sequence of strictly positive real numbers. The notion of paranormed sequences was first introduced by S. Simons [14]. It was further investigated by Maddox [11], Tripathy and Chandra [16] and many others. The notion of multiplier sequences $\Lambda = (\lambda_k)$ was studied by Goes and Goes [8] at the initial stage. It was further investigated by Kamthan [9], Tripathy and Mahanta [18] and many others. The notion of modulus function was introduced by Nakano [12]. It was further investigated with applications to sequences by Tripathy and Chandra [16] and many others.

Definition 1.1 A modulus function f is a mapping from $[0, \infty)$ into $[0, \infty)$ such that
(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f(x + y) \leq f(x) + f(y)$;
(iii) f is increasing;
(iv) f is continuous from the right at 0. Hence f is continuous everywhere in $[0, \infty)$, by (ii) and (iv).

The idea of difference sequences for real numbers was first introduced by Kizmaz [10] and it was further generalized by Tripathy and Esi as follows: $Z(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in Z\}$, for $Z = c, c_0, l_\infty$, where $\Delta_m x_k = x_k - x_{k+m}$, for all $k \in N$. Let w^F be the linear space of all sequences of fuzzy numbers. Any linear subspace E^F of w^F is called a fuzzy sequence space. Throughout the paper w^F, c^F, c_0^F and

l_∞^F denote the classes of all, convergent, null, and bounded sequences of fuzzy real numbers respectively.

Definition 1.2 A sequence (X_n) of fuzzy real numbers is said to be convergent to a fuzzy number X_0 if for each $\epsilon > 0$ there exists a positive integer n_0 such that $\overline{d}(X_k, X_0) < \epsilon$ for all $n > n_0$.

Definition 1.3 A sequence (X_n) of fuzzy real numbers is said to be bounded if $\sup_n \overline{d}(X_n, \overline{0}) < \infty$.

Definition 1.4: A fuzzy real valued sequence space E^F is said to be solid (or normal) if for the scalar $|\alpha_n| \le 1$ for all $n \in N$ and $(X_n) \in E^F$ implies $(\alpha_n X_n) \in E^F$.

Definition 1.5: A fuzzy real valued sequence space E^F is said to be monotone if E^F contains the canonical pre image of all its step spaces.

Remark 1.1:If a class of sequences of fuzzy numbers is solid, then it is monotone.

Definition 1.6:A fuzzy real valued sequence space E^F is said to be symmetric if $(X_n) \in E^F \Rightarrow (X_{\pi(n)}) \in E^F$ where π is a permutation of N.

Definition 1.7: A fuzzy real valued sequence space E^F is said to be convergence free if $(X_n) \in E^F$ whenever $(Y_n) \in E^F$ and $Y_n = \overline{0}$ implies $X_n = \overline{0}$.

If (a_k) and (b_k) are two sequences of complex terms and $p = (p_k) \in l_\infty$, we have the following known inequality:

$$|a_k + b_k|^{p_k} \le D(|a_k|^{p_k} + |b_k|^{p_k})$$

where $H = \sup_k p_k$ and $D = \max\{1, 2^{H-1}\}$.

Recently the paranormed sequence spaces $c(f, \Lambda, \Delta_m, p)$,$c_0(f, \Lambda, \Delta_m, p)$ and $l_\infty(f, \Lambda, \Delta_m, p)$ was introduced by Tripathy and Chandra [16] and the fuzzy analogues of the same was introduced by Tripathy and Debnath [18]. We now give the generalization of the existing spaces in fuzzy setting are as follows.

Definition 1.8: Let $\mathbf{f} = (f_k)$ be a sequence of modulli, then for a given multiplier sequence $\Lambda = (\lambda_k)$, we introduce the following fuzzy real valued sequence spaces:

$c^F(\mathbf{f}, \Lambda, \Delta_m, p)$=$\{(X_k) \in w^F : (f_k(\overline{d}(\Delta_m(\lambda_k X_k), L)))^{p_k} \to 0$ as $k \to \infty$, for some $L \in R(I)\}$

$c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$=$\{(X_k) \in w^F : (f_k(\overline{d}(\Delta_m(\lambda_k X_k), \overline{0})))^{p_k} \to 0$ as $k \to \infty\}$,

$l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$=$\{(X_k) \in w^F$:$\sup_k(f_k(\overline{d}(\Delta_m(\lambda_k X_k), L)))^{p_k} < \infty\}$.

When$f_k(x) = f(x)$, for all $k \in N$,the above sequence spaces are denoted by $c^F(f, \Lambda, \Delta_m, p)$, $c_0^F(f, \Lambda, \Delta_m, p)$, $l_\infty^F(f, \Lambda, \Delta_m, p)$, respectively, which was introduced by Tripathy and Debnath [14]. When $f_k(x) = f(x) = x$ for all $x \in [0, \infty)$, the above sequence spaces are denoted as $c^F(\Lambda, \Delta_m, p), c_0^F(\Lambda, \Delta_m, p), l_\infty^F(\Lambda, \Delta_m, p)$,respectively. When $\lambda_k = 1$, for all $k \in N$, the above sequence spaces are denoted as

$$c^F(f, \Delta_m, p), c_0^F(f, \Delta_m, p), l_\infty^F(f, \Delta_m, p),$$

respectively.

Taking $f(x) = x$, for all $x \in [0, \infty))$, and $\lambda_k = 1, p_k = 1$ for all $k \in N$, the above spaces are denoted as $c^F(\Delta_m), c_0^F(\Delta_m)$ and $l_\infty^F(\Delta_m)$, respectively. Further taking m = 1, we get the spaces $c^F(\Delta), c_0^F(\Delta)$ and $l_\infty^F(\Delta)$, respectively.

Similarly taking different combinations and restrictions, we will get different paranormed sequence spaces.

2. Main Result

Theorem 2.1: The classes of sequences $c^F(\mathbf{f}, \Lambda, \Delta_m, p), c_0^F(\mathbf{f}, \Lambda, \Delta_m, p), l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ are closed with respect to addition and scalar multilpication.

Proof: We prove the theorem for the class of sequences $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$. The other classes can be proved similarly.
Suppose $(X_k), (Y_k) \in c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$. Then we have
$(f_k(\overline{d}(\Delta_m(\lambda_k X_k), \overline{0})))^{p_k} \to 0$ as $k \mapsto \infty$,
and $(f_k(\overline{d}(\Delta_m(\lambda_k Y_k), \overline{0})))^{p_k} \to 0$ as $k \mapsto \infty$,

Now for a,b $\in R$ we have

$$(f_k(\overline{d}(\Delta_m(\lambda_k(aX_k+bY_k)), \overline{0})))^{p_k} \leq (f_k(\overline{d}(\Delta_m(a\lambda_k X_k), \overline{0}))+(\overline{d}(\Delta_m(b\lambda_k Y_k), \overline{0})))^{p_k}$$

(by Minkowski inequality and f is continuous)

$$\leq (f_k(\overline{d}(\Delta_m(a\lambda_k X_k), \overline{0})) + f_k(\overline{d}(\Delta_m(b\lambda_k Y_k), \overline{0})))^{p_k}$$

$$\leq D[a(f_k(\overline{d}(\Delta_m(\lambda_k X_k), \overline{0}))^{p_k} + b(f_k(\overline{d}(\Delta_m(b\lambda_k Y_k), \overline{0})))^{p_k}$$

Hence $(aX_k + bY_k) \in c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$

Theorem 2.2: Let $p = (p_k) \in l_\infty$. Then the class of sequences $c^F(\mathbf{f}, \Lambda, \Delta_m, p)$, $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$, $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ are paranormed spaces paranormed by g, define by

$$g(X) = sup_k(f_k(\overline{d}(\Delta_m(\lambda_k X_k), \overline{0})))^{p_k/M}$$

where $M = \max(1, \sup p_k)$ and $X = (X_k)$.

Proof: Clearly, $g(X) \geq 0, g(-X) = g(X)$ and $g(X + Y) \leq g(X) + g(Y)$. Next we show the continuity of the product. Let a be fixed and $g(X) \to 0$.Then it is obvious that $g(aX) \to 0$. Let $a \to 0$ and X be fixed. Since each f_k is continuous, we have

$(f_k(|a|\overline{d}(\Delta_m(\lambda_k X_k), \overline{0})))^{p_k/M} \to 0$ as $a \to 0$.

Thus, $\sup_k (f_k(|a|\overline{d}(\Delta_m(\lambda_k X_k), \overline{0})))^{p_k/M} \to 0$ as $a \to 0$
Hence $g(aX) \to 0$ as $a \to 0$ Therefore g is paranorm.

Proposition 2.1: $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p) \subset c^F(\mathbf{f}, \Lambda, \Delta_m, p) \subset l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and the inclusions are proper.

Example 2.1: Let $f_k(x) = f(x) = x$, for all $x \in [0, \infty); m = 1, \lambda_k = 1, p_k = 1$ $k \in N$

$$X_k(t) = \begin{cases} \frac{t}{k} + 1, & -k \le t \le 0 \\ 1 - \frac{t}{k}, & 0 \le t \le k \\ 0, & \text{otherwise} \end{cases} .$$

Then $(X_k) \in c^F(f, \Lambda, \Delta_1, p)$ but $(X_k) \notin c_0^F(f, \Lambda, \Delta_1, p)$.

Hence the inclusions are proper.

Theorem 2.3: The classes of sequences $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p), c^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ are neither solid nor monotone in general, but the spaces $c_0^F(\mathbf{f}, \Lambda, p), c^F(\mathbf{f}, \Lambda, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, p)$ are solid and as such are monotone.

Proof: Let (X_k) be a given sequence and (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$, for all $k \in N$. Then we have, $(f_k(\overline{d}(\alpha_k(\lambda_k X_k), \overline{0})))^{p_k} \le (f_k(\overline{d}(\lambda_k X_k), \overline{0}))^{p_k}$, for all $k \in N$. The solidness of $c_0^F(\mathbf{f}, \Lambda, p), c^F(\mathbf{f}, \Lambda, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, p)$ follows from the above inequality. The monotonicity of these two classes of sequences follows by remark 2.1. The first part of the proof follows from the following examples.

Example 2.2: Let $f_k(x) = f(x) = x$, for all $x \in [0, \infty); m = 1, \lambda_k = 1, p_k = 1$ for odd k and $p_k = 2$ for even k, $k \in N$

$$X_k(t) = \begin{cases} 1, & \text{for } t = k \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X_k) \in c^F(f, \Lambda, \Delta_1, p)$ and $l_\infty^F(f, \Lambda, \Delta_1, p)$. For a sequence space E, consider its step space E_j defined by

$$Y_k = \begin{cases} X_k, & \text{for k even} \\ 0, & \text{for k odd.} \end{cases}$$

Then (Y_k) neither belongs to $c^F(f, \Lambda, \Delta_1, p)_J$ nor to $l_\infty^F(f, \Lambda, \Delta_1, p)_J$. Hence the spaces are not monotone. So, are not solid.

Example 2.3: Let $f_k(x) = f(x) = x$, for all $x \in [0, \infty); m = 1, \lambda_k = 2 + k^{-1}, p_k = 1$ for odd k and $p_k = 2$ for even k, $k \in N$

$$X_k(t) = \begin{cases} 1, & \text{for t} = k \\ 0, & \text{0therwise} \end{cases} .$$

Then $(X_k) \in c_0^F(f, \Lambda, \Delta_1, p)$. Now consider the step as defined in the earlier example. Then $(Y_k) \notin c_0^F(f, \Lambda, \Delta_1, p)$. Hence $c_0^F(f, \Lambda, \Delta_1, p)$ is not monotone, as such $c_0^F(f, \Lambda, \Delta_1, p)$ is not solid.

Theorem 2.4: The classes of sequences $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p), c^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(f, \Lambda, \Delta_m, p)$ are not convergence free in general.

Proof: The result follows from the following example.

Example 2.3: Let $f_k(x) = f(x) = x$, for all $x \in [0, \infty); m = 1, \lambda_k = 1, p_k = 1$ for odd k and $p_k = 2$ for even k, $k \in N$. Consider the sequence (X_k) defined in Example 3.1. Clearly $(X_k) \in c^F(\mathbf{f}, \Lambda, \Delta_1, p)$. Consider the sequence (Y_k) defined by

$$Y_k = \begin{cases} X_k, & \text{for } k = 3i, i \in N \\ 0, & \text{otherwise.} \end{cases}$$

Then $(Y_k) \notin c^F(\mathbf{f}, \Lambda, \Delta_1, p)$. Hence the class of sequences $c^F(\mathbf{f}, \Lambda, \Delta_m, p)$ is not convergence free. Similar examples can be constructed to show that the classes of sequences $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ are not convergence free.

Theorem 2.5: The classes of sequences
$c_0^F(\mathbf{f}, \Lambda, \Delta_m, p), c^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$
are not symmetric in general.

Proof: The result follows from the following example.

Example 2.4: Let $f_k(x) = f(x) = x, m = 1$, $\lambda_k = 1, p_k = 1$ for odd k and $p_k = 2$ for even $k, k \in N$. Consider the sequence (X_k) defined in Example 3.1. Clearly $(X_k) \in c^F(\mathbf{f}, \Lambda, \Delta_m, p)$

Consider its rearrangement (Y_k) as follows
$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, X_7, X_{36}, X_8, \ldots)$.
Then $(Y_k) \notin c^F(\mathbf{f}, \Lambda, \Delta_m, p)$. Hence the class of sequences $c^F(\mathbf{f}, \Lambda, \Delta_m, p)$ is not symmetric. Similar examples can be constructed to show that the classes of sequences $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ are not symmetric.

Conclusion: In this paper we have introduced the paranormed sequence spaces $c^F(\mathbf{f}, \Lambda, \Delta_m, p)$, $c_0^F(\mathbf{f}, \Lambda, \Delta_m, p)$ and $l_\infty^F(\mathbf{f}, \Lambda, \Delta_m, p)$ of fuzzy numbers associated with the multiplier sequence $\Lambda = (\lambda_k)$ determined by a sequence of moduli $\mathbf{f} = (f_k)$. Which are the generalizations of the existing sequence spaces of fuzzy numbers. More generalizations may be done based on this paper.

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