CONNECTEDNESS IN SOFT m-STRUCTURE *

Samajh Singh Thakur and Alpa Singh Rajput

Abstract. In the present paper, we introduce the concept of soft connectedness in a soft m-structure and study some of its properties and characterizations.

Keywords: Soft m-structure, Soft m-connectedness and Soft m-connectedness between soft sets.

1. Introduction

The concept of soft set is fundamentally important in almost every scientific field. Soft set theory is a new mathematical tool dealing with uncertainty and has been applied in several directions since its introduction by Molodtsov [19] in 1999. The operations on soft sets and soft structures have been studied in [1, 16, 23]. Maji et. al [15] gave the first practical application of soft sets in decision theory. In 2011 Shabir and Naz [22] initiated a study of soft topological spaces. In recent years, many soft topological concepts such as soft connectedness and their strong forms [8, 11, 17, 20, 24], soft separation axioms [14, 20, 22], weak and strong forms of soft open sets and soft continuity [17, 2, 3, 4, 5, 6, 9, 10, 12, 13, 25] have been introduced and studied. Recently, the authors of this paper [21] initiated a study of soft m-structures. In the present paper we introduce the concept of soft connectedness in soft m-structures and we study some of its properties and characterizations.

2. Preliminaries

Let U be an initial universe set, E be a set of parameters, P(U) denote the power set of U and A ⊆ E.

Definition 2.1. [19] A pair (F, A) is called a soft set over U, where F is a mapping given by F: A → P(U). In other words, a soft set over U is a parameterized family of subsets of the universe U. For all e ∈ A, F(e) may be considered a set of e-approximate elements of the soft set (F, A).

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Definition 2.2. [16] For two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), we say that \((F, A)\) is a soft subset of \((G, B)\), denoted by \((F, A) \subseteq (G, B)\), if
(a) \(A \subseteq B\) and
(b) \(F(e) \subseteq G(e)\) for all \(e \in E\).

Definition 2.3. [16] Two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\) are said to be soft equal denoted by \((F, A) = (G, B)\) if \((F, A) \subseteq (G, B)\) and \((G, B) \subseteq (F, A)\).

Definition 2.4. [7] The complement of a soft set \((F, A)\), denoted by \((F, A)^c\), is defined by \((F, A)^c = (F^c, A)\), where \(F^c : A \rightarrow \mathcal{P}(U)\) is a mapping given by \(F^c(e) = U - F(e)\), for all \(e \in E\).

Definition 2.5. [16] Let a soft set \((F, A)\) over \(U\).
(a) A null soft set denoted by \(\phi\) if for all \(e \in A\), \(F(e) = \phi\).
(b) An absolute soft set denoted by \(\tilde{U}\), if for each \(e \in A\), \(F(e) = U\).

Clearly, \(\tilde{U}^c = \phi\) and \(\phi^c = \tilde{U}\).

Definition 2.6. [7] The union of two sets \((F, A)\) and \((G, B)\) over a common universe \(U\) is a soft set \((H, C)\), where \(C = A \cup B\) and for all \(e \in C\),
\[
H(e) = \begin{cases} 
F(e), & \text{if } e \in A - B \\
G(e), & \text{if } e \in B - A \\
F(e) \cup G(e), & \text{if } e \in A \cap B 
\end{cases}
\]

Definition 2.7. [7] The intersection of two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\) is a soft set \((H, C)\) where \(C = A \cap B\) and \(H(e) = F(e) \cap G(e)\) for each \(e \in E\).

Let \(X\) and \(Y\) be initial universe sets and \(E\) and \(K\) be non-empty sets of the parameters, \(S(X, E)\) denotes the family of all soft sets over \(X\), and \(S(Y, K)\) denotes the family of all soft sets over \(Y\).

Definition 2.8. [12] Let \(S(X,E)\) and \(S(Y,K)\) be families of soft sets. Let \(u: X \rightarrow Y\) and \(p: E \rightarrow K\) be mappings. Then a mapping \(f_{pu}: S(X, E) \rightarrow S(Y, K)\) is defined as:

(i) Let \((F, A)\) be a soft set in \(S(X, E)\). The image of \((F, A)\) under \(f_{pu}\), written as \(f_{pu}(F, A) = \{ f_{pu}(F), p(A) \}\), is a soft set in \(S(Y, K)\) such that
\[
f_{pu}(F)(k) = \begin{cases} 
\bigcup_{e \in p^{-1}(k)} \bigcap_{A} u(F(e)), & p^{-1}(k) \cap A \neq \phi \\
\phi, & p^{-1}(k) \cap A = \phi
\end{cases}
\]
For all $k \in K$.

(ii) Let $(G, B)$ be a soft set in $S(Y, K)$. The inverse image of $(G, B)$ under $f_{pu}$, written as $f_{pu}^{-1}(G,B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $S(X, E)$ such that

$$f_{pu}^{-1}(G)(e) = \begin{cases} u^{-1}G(p(e)), & p(e) \in B \\ \phi, & p(e) \notin B \end{cases}$$

For all $e \in E$.

**Definition 2.9.** [25] Let $f_{pu} : S(X, E) \rightarrow S(Y, K)$ be a mapping and $u : X \rightarrow Y$ and $p : E \rightarrow K$ be mappings. Then $f_{pu}$ is soft onto, if $u : X \rightarrow Y$ and $p : E \rightarrow K$ are onto and $f_{pu}$ is soft one-one, if $u : X \rightarrow Y$ and $p : E \rightarrow K$ are one-one.

**Definition 2.10.** [22] A subfamily $\tau$ of $S(X, E)$ is called a soft topology over $X$ if:

1. $\tilde{\phi}, \tilde{X}$ belong to $\tau$.
2. The union of any number of soft sets in $\tau$ belongs to $\tau$.
3. The intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called a soft topological space over $X$. The members of $\tau$ are called soft open sets in $X$ and their complements are called soft closed sets in $X$.

**Definition 2.11.** If $(X, \tau, E)$ is a soft topological space and a soft set $(F, E)$ over $X$.

(a) The soft closure of $(F, E)$ is denoted by $\text{Cl}(F, E)$, and defined as the intersection of all soft closed super sets of $(F, E)$ [22].

(b) The soft interior of $(F, E)$ is denoted by $\text{Int}(F, E)$, and defined as the soft union of all soft open subsets of $(F, E)$ [25].

**Definition 2.12.** [25] The soft set $(F,E) \in S(X,E)$ is called a soft point if there exist $x \in X$ and $e \in E$ such that $F(e) = \{x\}$ and $F(e') = \phi$ for each $e' \in E - \{e\}$, and the soft point $(F,E)$ is denoted by $x_e$.

**Definition 2.13.** A soft set $(A, E)$ of a soft topological space $(X, \tau, E)$ is called:

(a) Soft regular open $(A, E) = \text{Int}(\text{Cl}(A, E))$ [6];

(b) Soft $\alpha$-open if $(A, E) \subset \text{Int}(\text{Cl}(\text{Int}(A, E)))$ [3];

(c) Soft semi-open if $(A, E) \subset \text{Cl}(\text{Int}(A, E))$ [17];

(d) Soft preopen if $(A, E) \subset \text{Int}(\text{Cl}(A, E))$ [2];

(e) Soft $b$-open if $(A, E) \subset \text{Int}(\text{Cl}(A, E)) \cup \text{Cl}(\text{Int}(A, E))$ [5].
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(f) Soft $\beta$-open if $(A, E) \subset \text{Cl}(\text{Int}(\text{Cl}(A, E)))$ [4]

The family of all soft regular open (resp. soft $\alpha$-open, soft semi-open, soft preopen, soft $\beta$-open, soft $b$-open) sets of $X$ will be denoted by $\text{SRO}(X, E)$ (resp. $\text{SaO}(X, E)$, $\text{SSO}(X, E)$, $\text{SPO}(X, E)$, $\text{S}\beta\text{O}(X, E)$, $\text{SbO}(X, E)$).

**Definition 2.14.** Let $(A, E)$ be a soft subset of a soft topological space $(X, \tau, E)$. Then:

(a) The intersection of all soft semi-open sets containing $(A, E)$ is called semi-closure of $(A, E)$. It is denoted by $\text{sCl}(A, E)$ [17].

(b) The intersection of all soft preopen sets containing $(A, E)$ is called preclosure of $(A, E)$. It is denoted by $\text{pCl}(A, E)$ [2].

(c) The intersection of all soft $\alpha$ open sets containing $(A, E)$ is called $\alpha$-closure of $(A, E)$. It is denoted by $\alpha\text{Cl}(A, E)$ [3].

(d) The intersection of all soft $b$-open sets containing $(A, E)$ is called $b$-closure of $(A, E)$. It is denoted by $\beta\text{Cl}(A, E)$ [5].

**Definition 2.16.** A soft mapping $f_{pu} : (X, \tau, E) \rightarrow (X, \sigma, K)$ is said to be:

(a) Soft continuous if $f_{pu}^{-1}(U, K) \in \tau$ for every soft set $(U, K) \in \sigma$ [25].

(b) Soft $\alpha$-continuous if $f_{pu}^{-1}(U, K) \in \text{SaO}(X, E)$ for every soft set $(U, K) \in \sigma$ [3].

(c) Soft semi-continuous if $f_{pu}^{-1}(U, K) \in \text{SSO}(X, E)$ for every soft set $(U, K) \in \sigma$ [17].

(d) Soft precontinuous if $f_{pu}^{-1}(U, K) \in \text{SPO}(X, E)$ for every soft set $(U, K) \in \sigma$ [2].

(e) Soft $b$-continuous if $f_{pu}^{-1}(U, K) \in \text{SbO}(X, E)$ for every soft set $(U, K) \in \sigma$ [5].

(f) Soft $\beta$-continuous if $f_{pu}^{-1}(U, K) \in \text{S}\beta\text{O}(X, E)$ for every soft set $(U, K) \in \sigma$ [4].
(e) Soft b-open if \( f_{pu}(U, E) \in SbO(Y, K) \) for every soft set \((U, E) \in \tau \) [5].

(f) Soft \( \beta \)-open if \( f_{pu}(U, E) \in S\beta O(Y, K) \) for every soft set \((U, E) \in \tau \) [4].

**Definition 2.17.** [14] Let \((X, \tau, E)\) be a soft topological space, and \((A, E), (B, E)\) be two soft sets over \(X\). The soft sets \((A, E)\) and \((B, E)\) are said to be soft-separated, if \((A, E) \cap \text{Cl}(B, E) = \phi\) and \(\text{Cl}(A, E) \cap (B, E) = \phi\).

**Definition 2.18.** [14] Let \((X, \tau, E)\) be a soft topological space and if there exist two non-empty soft separated sets \((A, E), (B, E)\) such that \((A, E) \cup (B, E) = \tilde{X}\), then \((A, E)\) and \((B, E)\) are said to be a soft disconnection for a soft topological space \((X, \tau, E)\). \((X, \tau, E)\) is said to be soft-disconnected if \((X, \tau, E)\) has a soft disconnection. Otherwise, \((X, \tau, E)\) is said to be soft-connected.

**Definition 2.19.** [17] Let \((X, \tau, E)\) be a soft topological space. The nonempty soft sets \((F, A)\) and \((F, B)\) in \(S(X, E)\) are called soft semi-separated iff \(s\text{Cl}(F, A) \cap (F, B) = (F, A) \cap s\text{Cl}(F, B) = \phi\).

**Definition 2.20.** [17] Let \((X, \tau, E)\) be a soft topological space. If there does not exist a soft semi-separation of \(X\), then it is said to be soft s-connected.

**Definition 2.21.** [24] Let \((X, \tau, E)\) be a soft topological space. The nonempty soft sets \((F, A)\) and \((F, B)\) in \(S(X, E)\) are called soft pre-separated iff \(p\text{Cl}(F, A) \cap (F, B) = (F, A) \cap p\text{Cl}(F, B) = \phi\).

**Definition 2.22.** [24] Let \((X, \tau, E)\) be a soft topological space. If there does not exist a soft pre-separation of \(X\), then it is said to be soft P-connected.

**Definition 2.23.** [21] A subfamily \(m_{(X, E)}\) of \(S(X, E)\) is called a soft minimal structure (briefly soft m-structure) over \(X\) if \(\phi \in m_{(X, E)}\) and \(\tilde{X} \in m_{(X, E)}\).

\((X, m_{(X, E)})\) is called a soft space with a soft minimal structure \(m_{(X, E)}\) or simply a soft m-space. Each member of \(m_{(X, E)}\) is called a soft m-open set and the complement of a soft m-open set is called a soft m-closed set.

**Remark 2.1.** [21] Let \((X, \tau, E)\) be a soft topological space. Then the families \(\tau, SSO(X, E), SPO(X, E), S\alpha O(X, E), S\beta O(X, E), SRO(X, E)\) are all soft m-structures over \(X\).

**Definition 2.24.** [21] Let \(X\) be a nonempty set, \(E\) be a set of parameters and \(m_{(X, E)}\) be a soft m-structure over \(X\). The soft \(m_{(X, E)}\)-closure and the soft \(m_{(X, E)}\)-interior of the soft set \((A, E)\) over \(X\) are defined as follows:

1. \(m_{(X, E)}\)-\text{Cl}(A, E) = \cap \{(F, E) : (F, E) \subset (A, E), (F, E)^c \in m_{(X, E)}\}\).
2. \(m_{(X, E)}\)-\text{Int}(A, E) = \cup \{(F, E) : (F, E) \subset (A, E), (F, E) \in m_{(X, E)}\}\).
Remark 2.2. [21] Let $(X,\tau,E)$ be a soft topological space and $(A,E)$ be a soft set over $X$. If $m_{(X,E)} = \tau$ (respectively $SO(X,E)$, $S\alpha O(X,E)$, $S\beta O(X,E)$, $S\delta O(X,E)$), then we have:


Theorem 2.1. [21] Let $S(X,E)$ be a family of soft sets and $m_{(X,E)}$ a soft minimal structure over $X$. For soft sets $(A,E)$ and $(B,E)$ of $X$, the following holds:

(a) (i): $m_{(X,E)}\cdot Int(A,E)^c = (m_{(X,E)} - Cl(A,E))^c$ and (ii): $m_{(X,E)}\cdot Cl(A,E)^c = (m_{(X,E)} - Int(A,E))^c$.

(b) If $(A,E)^c \in m_{(X,E)}$, then $m_{(X,E)}\cdot Cl(A,E) = (A,E)$ and if $(A,E) \in m_{(X,E)}$, then $m_{(X,E)}\cdot Int(A,E) = (A,E)$.

(c) $m_{(X,E)}\cdot Cl(\phi) = \phi$, $m_{(X,E)}\cdot Cl(\tilde{X}) = \tilde{X}$, $m_{(X,E)}\cdot Int(\phi) = \phi$, $m_{(X,E)}\cdot Int(\tilde{X}) = \tilde{X}$.

(d) If $(A,E) \subset (B,E)$, then $m_{(X,E)}\cdot Cl(A,E) \subset m_{(X,E)}\cdot Cl(B,E)$, $m_{(X,E)}\cdot Int(A,E) \subset m_{(X,E)}\cdot Int(B,E)$.

(e) $(A,E) \subset m_{(X,E)}\cdot Cl(A,E)$ and $m_{(X,E)}\cdot Int(A,E) \subset (A,E)$.

(f) $m_{(X,E)}\cdot Cl(m_{(X,E)}\cdot Cl(A,E)) = m_{(X,E)}\cdot Cl(A,E)$ and $m_{(X,E)}\cdot Int(m_{(X,E)}\cdot Int(A,E)) = m_{(X,E)}\cdot Int(A,E)$.

Definition 2.25. [21] A soft mapping $f_{pu} : (X,m_{(X,E)}) \rightarrow (Y,m_{(Y,K)})$, where the minimal soft structure $m_{(X,E)}$ and $m_{(Y,K)}$ over $X$ and $Y$, respectively, is said to be soft M-continuous if for each $x_e \in S(X,E)$ and each $(V,K) \in m_{(Y,K)}$ containing $f_{pu}(x_e)$, there exists $(U,E) \in m_{(X,E)}$ containing $x_e$ such that $f_{pu}(U,E) \subset (V,K)$.

Throughout this paper soft clopen means soft closed and open.

3. Connectedness in soft m-structure

Definition 3.1. [21] A soft minimal structure $m_{(X,E)}$ over $X$ is said to have the property B if the union of any family of subsets belongs to $m_{(X,E)}$ belongs to $m_{(X,E)}$.

Definition 3.2. Let $X$ be a nonempty set, $E$ be a set of parameters and $m_{(X,E)}$ be a soft m-structure over $X$ with property B. In $(X,m_{(X,E)})$ two nonempty soft sets $(A,E)$ and $(B,E)$ over $X$ are called soft m-separated if $m_{(X,E)}\cdot Cl(A,E) \cap (B,E) = (A,E) \cap m_{(X,E)}\cdot Cl(B,E) = \phi$. 
Remark 3.1. Let \((X,\tau,E)\) be a soft topological space over \(X\). If \(m_{(X,E)} = \tau\) (resp. \(SSO(X,E),SPO(X,E),ShO(X,E)\)) and \(m_{(X,E)} - \text{Cl}(A,E) = \text{Cl}(A,E)\) (resp. \(s\text{Cl}(A,E), \text{pCl}(A,E), b\text{Cl}(A,E)\)) we get definitions of soft separated (resp. soft semi-separated, soft preseparated, soft b-separated) sets.

Definition 3.3. Let \(m_{(X,E)}\) be a soft m-structure over \(X\) with the property \(B\). Then \((X,m_{(X,E)})\) is said to be soft m-connected if there does not exist two nonempty soft m-separated sets \((A,E)\) and \((B,E)\) over \(X\), such that \((A,E) \cup (B,E) = \bar{X}\). Otherwise it is soft m-disconnected. In this case, the pair \((A,E)\) and \((B,E)\) is called soft m-disconnection over \(X\).

Remark 3.2. Let \((X,\tau,E)\) be a soft topological space over \(X\). If we replace soft m-separated by soft separated (resp. soft semi-separated, soft preseparated, soft b-separated) sets we get a definition for soft connectedness (resp. soft semi-connectedness, soft preconnectedness, soft b-connectedness).

Theorem 3.1. Let \((X,m_{(X,E)})\) be a soft m-space with the property \(B\). Then the following conditions are equivalent:

1. \((X,m_{(X,E)})\) has a soft m-disconnection.
2. There exist two disjoint soft m-closed sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \bar{X}\).
3. There exist two disjoint soft m-open sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \bar{X}\).
4. \((X,m_{(X,E)})\) has a proper soft m-open and soft m-closed set over \(X\).

Proof: (1) \(\Rightarrow\) (2): Let \((X,m_{(X,E)})\) have a soft m-disconnection \((A,E)\) and \((B,E)\). Then \((A,E) \cap (B,E) = \emptyset\) and
\[
m_{(X,E)} - \text{Cl}(A,E) = m_{(X,E)} - \text{Cl}(A,E) \cap ((A,E) \cup (B,E)) = (m_{(X,E)} - \text{Cl}(A,E) \cap (A,E)) \cup (m_{(X,E)} - \text{Cl}(A,E) \cap (B,E)) = (A,E).
\]
Therefore, \((A,E)\) is a soft m-closed set over \(X\). Similarly, we can see that \((B,E)\) is also a soft m-closed set over \(X\).

(2) \(\Rightarrow\) (3): Let \((X,m_{(X,E)})\) has a soft m-disconnection \((A,E)\) and \((B,E)\) such that \((A,E)\) and \((B,E)\) are soft m-closed. Then \((A,E)^c\) and \((B,E)^c\) are soft m-open sets in \(m_{(X,E)}\). Then it is easy to see \((A,E)^c \cap (B,E)^c = \emptyset\) and \((A,E)^c \cup (B,E)^c = \bar{X}\).

(3) \(\Rightarrow\) (4): Let \((X,m_{(X,E)})\) have a soft m-disconnection \((A,E)\) and \((B,E)\) such that \((A,E)\) and \((B,E)\) are soft m-open over \(X\). Then \((A,E)\) and \((B,E)\) are also soft closed in \((X,m_{(X,E)})\).

(4) \(\Rightarrow\) (1): Let \((X,m_{(X,E)})\) has a proper soft m-open and soft m-closed set \((F,E)\) over \(X\). Put \((H,E) = (F,E)^c\). Then \((H,E)\) and \((F,E)\) are non-empty soft m-closed sets in \((X,m_{(X,E)})\). \((H,E) \cap (F,E) = \emptyset\) and \((H,E) \cup (F,E) = \bar{X}\). Therefore, \((H,E)\) and \((F,E)\) is a soft m-disconnection of \((X,m_{(X,E)})\).
Remark 3.3. Let \((X,\tau,E)\) be a soft topological space over \(X\), if \(m_{(X,E)} = \tau\) (resp. \(SSO(X,E), SPO(X,E), SbO(X,E)\)) Then the following conditions are equivalent:

1. \((X,\tau,E)\) has a soft disconnection (resp. soft semi-disconnection, soft pre disconnection, soft b-disconnection).

2. There exist two disjoint soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \tilde{X}\).

3. There exist two disjoint soft open (resp. soft semi-open, soft pre-open, soft b-open) sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \tilde{X}\).

4. \((X,\tau,E)\) has a proper soft open (resp. soft semi-open, soft pre-open, soft b-open) and soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) set over \(X\).

Corollary 3.1. Let \((X,m_{(X,E)})\) be a soft m-space with the property \(B\). Then the following conditions are equivalent:

1. \((X,m_{(X,E)})\) is a soft m-connected.

2. There does not exist two disjoint soft m-closed sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \tilde{X}\).

3. There does not exist two disjoint soft m-open sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \tilde{X}\).

4. \((X,m_{(X,E)})\) at most has two soft m-closed and soft m-open sets over \(X\), that is, \(\phi\) and \(\tilde{X}\).

Remark 3.4. Let \((X,\tau,E)\) be a soft topological space over \(X\), if \(m_{(X,E)} = \tau\) (resp. \(SSO(X,E), SPO(X,E), SbO(X,E)\)). Then the following conditions are equivalent:

1. \((X,\tau,E)\) is a soft connected (resp. soft semi-connected, soft preconnected, soft b-connected).

2. There does not exist two disjoint soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \tilde{X}\).

3. There does not exist two disjoint soft open (resp. soft semi-open, soft pre-open, soft b-open) sets \((A,E),(B,E)\) such that \((A,E) \cup (B,E) = \tilde{X}\).

4. \((X,\tau,E)\) has a proper soft open (resp. soft semi-open, soft pre-open, soft b-open) and soft closed (resp. soft semi-closed, soft pre-closed, soft b-closed) set over \(X\).

Definition 3.4. Let \((X,m_{(X,E)})\) be a soft m-space with the property \(B\), \(Y \subset X\) in \((X,m_{(X,E)})\). The soft space \((Y,m_{(Y,E)})\) is called a soft m-subspace of \((X,m_{(X,E)})\) if \(m_{(Y,E)} = \{((A,E) \cap \tilde{Y} : (A,E) \in m_{(X,E)})\} \cap \tilde{Y}\).

Lemma 3.1. Let \((X,m_{(X,E)})\) be a soft m-space with the property \(B\), \((Y,m_{(Y,E)})\) be a soft m-subspace of \((X,m_{(X,E)})\). If \((A,E) \subset \tilde{Y} \subset \tilde{X}\). Then \(m_{(Y,E)} \cap \text{Cl}(A,E) = m_{(X,E)} \cap \text{Cl}(A,E) \cap \tilde{Y}\).

Proof: We have \(m_{(Y,E)} \cap \text{Cl}(A,E) = \cap \{(F,E) : (A,E) \subset (F,E), \tilde{Y} \subset (F,E) \in m_{(Y,E)}\}\) = \(\cap \{(F,E) \cap \tilde{Y} : (A,E) \subset (F,E) \cap \tilde{Y}, \tilde{X} \subset (F,E) \in m_{(X,E)}\}\) = \(\cap \{ (F,E) \cap \tilde{Y} : (A,E) \subset (F,E), \tilde{X} \subset (F,E) \in m_{(X,E)}\}\) = \(\cap \{ (F,E) : (A,E) \subset (F,E), \tilde{X} \subset (F,E) \in m_{(X,E)}\}\) \(\cap \tilde{Y} = m_{(X,E)} \cap \text{Cl}(A,E) \cap \tilde{Y}\).

Therefore, the lemma holds.
Lemma 3.2. Let \((X, m_{(X,E)})\) be a soft m-space with the property \(B\), \((Y, m_{(Y,E)})\) be a soft m-subspace of \((X, m_{(X,E)})\). If \((A,E)\) and \((B,E)\) are soft sets in \((Y, m_{(Y,E)})\), then \((A,E)\) and \((B,E)\) are soft m-separated in \((Y, m_{(Y,E)})\) if and only if \((A,E)\) and \((B,E)\) are soft m-separated in \((X, m_{(X,E)})\).

Proof: We have \(m_{(Y,E)} \cdot \text{Cl}(A,E) \cap (B,E) = (m_{(X,E)} \cdot \text{Cl}(A,E) \cap \tilde{Y}) \cap (B,E) = m_{(X,E)} \cdot \text{Cl}(A,E) \cap (B,E)\) by lemma 3.1.

Similarly, we have

\[m_{(Y,E)} \cdot \text{Cl}(B,E) \cap (A,E) = m_{(X,E)} \cdot \text{Cl}(B,E) \cap (A,E).\]

Therefore, the lemma holds.

Lemma 3.3. Let \((X, m_{(X,E)})\) be a soft m-space with the property \(B\), \(\tilde{Y} \subset \tilde{X}\).

\((Y, m_{(Y,E)})\) be a soft m-subspace of \((X, m_{(X,E)})\). \((Y, m_{(Y,E)})\) is soft m-connected. If \((A,E)\) and \((B,E)\) are soft m-separated in \((X, m_{(X,E)})\), such that \(\tilde{Y} \subset (A,E) \cup (B,E)\), then \(\tilde{Y} \subset (A,E)\) or \(\tilde{Y} \subset (B,E)\).

Proof: We have \(\tilde{Y} \subset (A,E) \cup (B,E)\), we have \(\tilde{Y} = (\tilde{Y} \cap (A,E) \cup (\tilde{Y} \cap (B,E))\).

By lemma 3.2, \(\tilde{Y} \cap (A,E)\) and \(\tilde{Y} \cap (B,E)\) are soft m-separated in \((Y, m_{(Y,E)})\). Since \((Y, m_{(Y,E)})\) is soft m-connected, we have \(\tilde{Y} \cap (A,E) = \phi\) or \(\tilde{Y} \cap (B,E) = \phi\). Therefore, \(\tilde{Y} \subset (A,E)\) or \(\tilde{Y} \subset (B,E)\).

Lemma 3.4. Let \(\{(X_\alpha, m_{(X_\alpha,E)}): \alpha \in J\}\) be a soft family non-empty soft m-connected subspaces of \((X, m_{(X,E)})\). If \(\bigcap_{\alpha \in J} X_\alpha \neq \phi\), then \((\bigcup_{\alpha \in J} X_\alpha, m_{\bigcup_{\alpha \in J} m_{(X_\alpha,E)}})\) is a soft m-connected subspace of \((X, m_{(X,E)})\).

Proof: Let \(Y = \bigcup_{\alpha \in J} X_\alpha\). Choose a soft point \(x_\alpha \in \tilde{Y}\). Let \((C,E)\) and \((D,E)\) be a soft m-disconnection of \((\bigcup_{\alpha \in J} X_\alpha, m_{\bigcup_{\alpha \in J} m_{(X_\alpha,E)}})\). Then, \(x_\alpha \in (C,E)\) and \(x_\alpha \in (D,E)\), we assume that \(x_\alpha \in (C,E)\). For each \(\alpha \in J\), since \(\{(X_\alpha, m_{(X_\alpha,E)}): \alpha \in J\}\) is soft m-connected, it follows from lemma 3.3 that \(\tilde{X_\alpha} \subset (C,E)\) or \(\tilde{X_\alpha} \subset (D,E)\). Therefore, we have \(\tilde{Y} \subset (C,E)\) since \(x_\alpha \in (C,E)\) and then \((D,E) = \phi\), which is a contradiction. Thus \((\bigcup_{\alpha \in J} X_\alpha, m_{\bigcup_{\alpha \in J} m_{(X_\alpha,E)}})\) is a soft m-connected subspace of \((X, m_{(X,E)})\).

Theorem 3.2. Let \(\{(X_\alpha, m_{(X_\alpha,E)}): \alpha \in J\}\) be a soft family non-empty soft m-connected subspaces of \((X, m_{(X,E)})\). If \(X_\alpha \cap X_\beta \neq \phi\) for \(\alpha, \beta \in J\), then \((\bigcup_{\alpha \in J} X_\alpha, m_{\bigcup_{\alpha \in J} m_{(X_\alpha,E)}})\) is a soft m-connected subspace of \((X, m_{(X,E)})\).

Proof: Let \(\alpha_\alpha \in J\). For \(\beta \in J\), Put \(A_\beta = X_{\alpha_\beta} \cup X_\beta\). By lemma 3.4, \(\{(A_\beta, m_{(X_\alpha,E)}): \beta \in J\}\) is soft m-connected. Then, \(\{(A_\beta, m_{(X_\alpha,E)}): \beta \in J\}\) is a family soft m-connected subspace of \((X, m_{(X,E)})\) and \(\bigcap_{\beta \in J} A_\beta = X_{\alpha_\beta} \neq \phi\). Obviously, \((\bigcup_{\alpha \in J} X_\alpha = (\bigcup_{\beta \in J} A_\beta)\). It follows from lemma 3.4 that \((\bigcup_{\alpha \in J} X_\alpha, m_{\bigcup_{\alpha \in J} m_{(X_\alpha,E)}})\) is a soft m-connected subspace of \((X, m_{(X,E)})\).

Theorem 3.3. Let \((X, m_{(X,E)})\) be a soft m-space with the property \(B\), \(\tilde{Y} \subset \tilde{X}\).

\((Y, m_{(Y,E)})\) be a soft m-subspace of \((X, m_{(X,E)})\). If \(\tilde{Y} \subset \tilde{A} \subset m_{(X,E)} \cdot \text{Cl}(F,E)\), then
Let \((A, m_{(A,E)})\) be a soft connected \(m\)-subspace of \((X, m_{(X,E)})\). In particular, \(m_{(X,E)}^{-}\text{Cl}(F,E)\) is a soft connected \(m\)-subspace of \((X, m_{(X,E)})\).

Proof: Let \((C,E)\) and \((D,E)\) be a soft \(m\)-disconnection of \((A, m_{(A,E)})\). By lemma 3.3, we have \(\tilde{A} \subset (C,E)\) or \(\tilde{A} \subset (D,E)\). We assume that \(\tilde{A} \subset (C,E)\). By lemma 3.2, we have \(m_{(X,E)}^{-}\text{Cl}(C,E) \cap (D,E) = \emptyset\) and, hence, \(\tilde{A} \cap (D,E) = \emptyset\), which is a contradiction.

Theorem 3.4. Let \(f_{pu} : (X, m_{(X,E)}) \rightarrow (Y, m_{(Y,K)})\) be a soft \(M\)-continuous mapping, where \(m_{(X,E)}\) and \(m_{(Y,K)}\) are soft minimal structures over \(X\) and \(Y\), respectively. If \((X, m_{(X,E)})\) is soft \(m\)-connected, then the soft image of \((X, m_{(X,E)})\) is also soft \(m\)-connected.

Proof: Let \(f_{pu} : (X, m_{(X,E)}) \rightarrow (Y, m_{(Y,K)})\) be a soft continuous mapping. Conversely, suppose that \((Y, m_{(Y,K)})\) is soft \(m\)-disconnected and the pair \((A,K)\) and \((B,K)\) is a soft \(m\)-disconnection of \((Y, m_{(Y,K)})\). Since \(f_{pu} : (X, m_{(X,E)}) \rightarrow (Y, m_{(Y,K)})\) is soft continuous, then \(f_{pu}^{-1}(A,K) \in m_{(X,E)}\), \(f_{pu}^{-1}(B,K) \in m_{(X,E)}\). Clearly, the pair \(f_{pu}^{-1}(A,K)\) and \(f_{pu}^{-1}(B,K)\) is a soft \(m\)-disconnection of \((X, m_{(X,E)})\), which is a contradiction. Hence, \((Y, m_{(Y,K)})\) is soft \(m\)-connected. This completes the proof.

Remark 3.5. Let \((X, \tau, E)\) and \((Y, \vartheta, K)\) be two soft topological spaces over \(X\) and \(Y\), respectively. If \(m_{(X,E)} = \tau\), \(m_{(Y,K)} = \vartheta\), \(f_{pu} : (X, \tau, E) \rightarrow (Y, \vartheta, K)\) is a soft continuous mapping. If \((X, \tau, E)\) is soft connected (resp. soft semi-connected, soft preconnected, soft \(b\)-connected) then the soft image of \((X, \tau, E)\) is also soft connected (resp. soft semi-connected, soft preconnected, soft \(b\)-connected).

Definition 3.5. Let \(m_{(X,E)}\) be a soft \(m\)-structure over \(X\). A soft set \((F,E)\) in \((X, m_{(X,E)})\) is soft \(m\)-connected if it is soft \(m\)-connected as a soft \(m\)-subspace.

Remark 3.6. Let \((X, \tau, E)\) be a soft topological space over \(X\). A soft set \((F,E)\) in \((X, \tau, E)\) is soft connected (resp. soft semi-connected, soft preconnected and soft \(b\)-connected) if it is soft connected (resp. soft semi-connected, soft preconnected and soft \(b\)-connected) as a soft subspace.

Theorem 3.5. Let \(m_{(X,E)}\) be a soft \(m\)-structure over \(X\), \((G,E)\) be a soft \(m\)-connected set in \((X, m_{(X,E)})\) and \((F,E)\) be a soft set over \(X\) such that \((G,E) \subset (F,E) \subset m_{(X,E)}^{-}\text{Cl}(G,E)\). Then \((F,E)\) is soft \(m\)-connected.

Proof: It is sufficient that \(m_{(X,E)}^{-}\text{Cl}(G,E)\) is soft \(m\)-connected. On the contrary, suppose that \(m_{(X,E)}^{-}\text{Cl}(G,E)\) is soft \(m\)-disconnected. Then there exists a soft \(m\)-disconnection \(((H,E),(K,E))\) of \(m_{(X,E)}^{-}\text{Cl}(G,E)\). That is, there are \(((H,E) \cap (G,E)),((K,E) \cap (G,E))\) soft sets in \((G,E)\) such that \(((H,E) \cap (G,E)) \cap ((K,E) \cap (G,E))) = ((H,E) \cap (K,E)) \cap (G,E) = \emptyset\), and \(((H,E) \cap (G,E)) \cup ((K,E) \cap (G,E)) = (H,E) \cup (K,E)) \cap (G,E) = (G,E). This yields that the pair \(((H,E) \cap (G,E))\) and \(((K,E) \cap (G,E))\) is a soft \(m\)-disconnection of \((G,E)\), which is a contradiction. This proves that \(m_{(X,E)}^{-}\text{Cl}(G,E)\) is soft \(m\)-connected. Hence, the proof is complete.
Lemma 3.5. Let $m_{(X,E)}$ be a soft m-structure over $X$ with the property $B$, and let $(A,E)$ and $(B,E)$ be two soft sets over $X$. In $(X,m_{(X,E)})$ the following statements are equivalent:

1. $\phi$, $\tilde{X}$ are only soft m-open and soft m-closed set in $m_{(X,E)}$.
2. $(X,m_{(X,E)})$ is not a soft union of two disjoint soft sets $(A,E)$ and $(B,E) \in m_{(X,E)}$.
3. $(X,m_{(X,E)})$ is not a soft union of two disjoint soft sets $(A,E)^c$ and $(B,E)^c \in m_{(X,E)}$.
4. $(X,m_{(X,E)})$ is not a soft union of two nonempty soft m-separated sets.

Remark 3.7. Let $(X,\tau,E)$ be a soft topological space over $X$, so we put $m_{(X,E)} = \tau$ (resp. $SSO(X,E),SPO(X,E),SbO(X,E)$). Also, let $(A,E)$ and $(B,E)$ be two soft sets over $X$. In $(X,\tau,E)$ the following statements are equivalent:

1. $\phi$ and $\tilde{X}$ are only soft clopen (resp. soft semi-clopen, soft preclopen, soft b-clopen) sets in $(X,\tau,E)$.
2. $(X,\tau,E)$ is not a soft union of two soft disjoint soft open (resp. soft semi-open, soft pre open, soft b-open) sets.
3. $(X,\tau,E)$ is not a soft union of two soft disjoint soft closed (resp. soft semi-closed, soft preclosed, soft b-closed) sets.
4. $(X,\tau,E)$ is not a soft union of two nonempty soft separated (soft semi separated, soft preseparated, soft b-separated) sets.

Theorem 3.6. Let $m_{(X,E)}$ be a soft m-structure over $X$ with the property $B$. In $(X,m_{(X,E)})$ the following statements are equivalent:

1. $(X,m_{(X,E)})$ is a soft m-connected space.
2. $(X,m_{(X,E)})$ is not a soft union of any two soft m-separated sets.

Proof : (1) $\rightarrow$ (2) : Assume (1). Suppose (2) is false, then let $(A,E)$ and $(B,E)$ be two soft m-separated sets such that $\tilde{X} = (A,E) \cup (B,E)$. Since $(X,m_{(X,E)})$ is soft m-connected $m_{(X,E)}\text{-Cl}(A,E) \cap (B,E) = (A,E) \cap m_{(X,E)}\text{-Cl}(B,E) = \phi$. Since $(A,E) \subset m_{(X,E)}\text{-Cl}(A,E)$ and $(B,E) \subset m_{(X,E)}\text{-Cl}(B,E)$, then $(A,E) \cup (B,E) = \phi$. Now $m_{(X,E)}\text{-Cl}(A,E) \subset (B,E)^c = (A,E)$. Hence, $m_{(X,E)}\text{-Cl}(A,E) = (A,E)$. Therefore, $(A,E)^c \in m_{(X,E)}$. By the same way we show that $(B,E)^c \in m_{(X,E)}$, which is a contradiction with remark 3.5. This shows that (2) is true. Therefore (1) $\rightarrow$ (2).

(2) $\rightarrow$ (1) : Assume that (2) is not true. Let $(A,E)^c$ and $(B,E)^c$ be two soft m-disjoint nonempty and $(A,E)^c$ and $(B,E)^c \in m_{(X,E)}$ such that $\tilde{X} = (A,E)^c \cup (B,E)^c$. Then, $m_{(X,E)}\text{-Cl}(A,E)^c \cap (B,E)^c = (A,E) \cap m_{(X,E)}\text{-Cl}(B,E)^c = (A,E)^c \cap (B,E)^c = \phi$. This contradicts the hypothesis in (2). This show that (1) is true. Therefore, (2) $\rightarrow$ (1).

Remark 3.8. Let $(X,\tau,E)$ be a soft topological space over $X$, so we put $m_{(X,E)} = \tau$. Then, the following statements are equivalent:

1. $(X,\tau,E)$ is a soft connected (soft semi-connected, soft preconnected, soft b-connected) space.
(2) \((X, \tau, E)\) is not the soft union of any two soft separated (soft semi separated, soft preseparated, soft b-separated) sets.

**Remark 3.9.**  (1) Let \(m_{(X,E)}\) be a soft m-structure over \(X\) with the property \(B\), and let \((A,E)\) be a soft set over \(X\). If \(\phi \neq (A,E) \subset (X,m_{(X,E)})\) then \((A,E)\) is a soft m-connected set in \(m_{(X,E)}\) whenever \((X,m_{(X,E)})\) is a soft m-connected space.

(2) Let \((X, \tau, E)\) be a soft topological space over \(X\), so we put \(m_{(X,E)} = \tau\). If \(\phi \neq (A,E) \subset (X, \tau, E)\) then \((A,E)\) is a soft connected (soft semi-connected, soft preconnected, soft b-connected) set over \(X\) whenever \((X, \tau, E)\) is a soft connected (soft semi-connected, soft preconnected, soft b-connected) space.

**Theorem 3.7.** Let \(m_{(X,E)}\) be a soft m-structure over \(X\) with the property \(B\). In \((X,m_{(X,E)})\), let the soft set \((A,E)\) be a soft m-connected set. Let \((B,E)\) and \((C,E)\) be soft m-separated sets. If \((A,E) \subset (B,E) \cup (C,E)\). Then, either \((A,E) \subset (B,E)\) or \((A,E) \subset (C,E)\).

Proof: Suppose \((A,E)\) is a soft m-connected set and \((B,E),(C,E)\) are soft m-separated sets such that \((A,E) \subset (B,E) \cup (C,E)\). Let \((A,E)\) not subset \((B,E)\) and \((A,E)\) is not a subset of \((C,E)\). Suppose \((A_1,E) = (B,E) \cap (A,E) \neq \phi\) and \((A_2,E) = (C,E) \cap (A,E) \neq \phi\). Then, \((A,E) = (A_1,E) \cup (A_2,E)\). Since \((A_1,E) \subset (B,E)\), \((A_2,E) \subset (C,E)\). Hence, \(m_{(X,E)}-Cl(A_1,E) \subset m_{(X,E)}-Cl(B,E)\). Since \(m_{(X,E)}-Cl(B,E) \cap (C,E) = \phi\) then \(m_{(X,E)}-Cl(A_1,E) \cap (A_2,E) = \phi\). Since \((A_2,E) \subset (C,E)\), \(m_{(X,E)}-Cl(A_2,E) \subset m_{(X,E)}-Cl(C,E)\). Since \(m_{(X,E)}-Cl(C,E) \cap (B,E) = \phi\). Then \(m_{(X,E)}-Cl(A_2,E) \cap (A_1,E) = \phi\). But \((A,E) = (A_1,E) \cup (A_2,E)\). Therefore, \((A,E)\) is not a soft m-connected space. This is a contradiction. Then either \((A,E) \subset (B,E)\) or \((A,E) \subset (C,E)\).

**Remark 3.10.** Let \((X, \tau, E)\) be a soft topological space over \(X\), so we put \(m_{(X,E)} = \tau\). Also, let \((A,E)\) be a soft connected (resp. soft semi-connected, soft preconnected, soft b-connected) set. Let \((B,E)\) and \((C,E)\) be soft separated (resp. soft semi-separated, soft preseparated, soft b-separated) sets. If \((A,E) \subset (B,E) \cup (C,E)\). Then, either \((A,E) \subset (B,E)\) or \((A,E) \subset (C,E)\).

Let \(m_{(X,E)}\) be a soft m-structure over \(X\) with the property \(B\). In \((X,m_{(X,E)})\), let the soft set \((A,E)\) be a soft m-connected set, then \(m_{(X,E)}-Cl(A,E)\) is soft m-connected.

Proof: Suppose the soft set \((A,E)\) is a soft m-connected set and \(m_{(X,E)}-Cl(A,E)\) is not. Then there exist two soft m-separated sets \((B,E)\) and \((C,E)\) such that \(m_{(X,E)}-Cl(A,E) = (B,E) \cup (C,E)\). But \((A,E) \subset m_{(X,E)}-Cl(A,E)\) then \((A,E) \subset (B,E) \cup (C,E)\) and since \((A,E)\) is a soft m-connected set, then by Theorem 3.7 either \((A,E) \subset (B,E)\) or \((A,E) \subset (C,E)\).

(i) If \((A,E) \subset (B,E)\) then \(m_{(X,E)}-Cl(A,E) \subset m_{(X,E)}-Cl(B,E)\). But \(m_{(X,E)}-Cl(B,E) \cap (C,E) = \phi\). Hence, \(m_{(X,E)}-Cl(A,E) \cap (C,E) = \phi\). Since \((C,E) \subset m_{(X,E)}-Cl(A,E)\), then \((C,E) = \phi\) this is a contradiction.

(ii) If \((A,E) \subset (C,E)\) then in the same way we can prove that \((B,E) = \phi\), which is a contradiction. Therefore, \(m_{(X,E)}-Cl(A,E)\) is soft m-connected.
Remark 3.11. Let \((X, \tau, E)\) be a soft topological space over \(X\), we put \(m_{(X,E)} = \tau\) let soft set \((A, E)\) be a soft connected (resp. soft semi connected, soft pre connected, soft \(b\)-connected) set then \(m_{(X,E)} \cdot \text{Cl}(A,E)\) is soft connected (resp. soft semi connected, soft pre connected, soft \(b\)-connected).

Theorem 3.8. Let \(m_{(X,E)}\) be a soft \(m\)-structure over \(X\) with the property \(B\). In \((X, m_{(X,E)})\), let the soft set \((A,E)\) be a soft \(m\)-connected set and \((A,E) \subset (B,E) \subset m_{(X,E)} \cdot \text{Cl}(A,E)\) then \((B,E)\) is soft \(m\)-connected.

Proof: If \((B,E)\) is not soft \(m\)-connected, then there exist two soft sets \((C,E)\) and \((D,E)\) such that \(m_{(X,E)} \cdot \text{Cl}(C,E) \cap (D,E) = (C,E) \cap m_{(X,E)} \cdot \text{Cl}(D,E) = \emptyset\) and \((B,E) = (C,E) \cup (D,E)\). Since \((A,E) \subset (B,E)\), thus either \((A,E) \subset (C,E)\) or \((A,E) \subset (D,E)\). Suppose \((A,E) \subset (C,E)\) then \(m_{(X,E)} \cdot \text{Cl}(A,E) \subset m_{(X,E)} \cdot \text{Cl}(C,E)\), thus \(m_{(X,E)} \cdot \text{Cl}(A,E) \subset (D,E) = m_{(X,E)} \cdot \text{Cl}(C,E) \subset (D,E) = \emptyset\). But \((D,E) \subset (B,E) \subset m_{(X,E)} \cdot \text{Cl}(A,E)\), thus \(m_{(X,E)} \cdot \text{Cl}(A,E) \cap (D,E) = (D,E)\). Therefore, \((D,E) = \emptyset\) which is a contradiction. Thus, \((B,E)\) is a soft \(m\)-connected set.

If \((A,E) \subset (B,E)\), then we can prove that \((C,E) = \emptyset\). This is a contradiction. Then \((B,E)\) is soft \(m\)-connected.

Remark 3.12. Let \((X, \tau, E)\) be a soft topological space over \(X\), so we put \(m_{(X,E)} = \tau\). Also, let the soft set \((A,E)\) be a soft connected (resp. soft semi-connected, soft preconnected, soft \(b\)-connected) set and \((A,E) \subset (B,E) \subset m_{(X,E)} \cdot \text{Cl}(A,E)\), then \((B,E)\) is soft connected (resp. soft semi-connected, soft preconnected, soft \(b\)-connected).

Remark 3.13. Let \((X, \tau, E)\) be a soft topological space over \(X\), and \((F,E)\) be a soft set over \(X\). \((X, \tau, E)\) is soft connected (soft semi-connected, soft preconnected, soft \(b\)-connected) if and only if there does not exist nonempty soft set \((F,E)\) over \(X\) which is both soft open (resp. soft semi-open, soft preopen, soft \(b\)-open) and soft closed (resp. soft semi-closed, soft pre-closed, soft \(b\)-closed) set over \(X\).

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S.S. thakur  
Faculty of Science  
Department of Applied Mathematics  
P. O. Box 60  
Jabalpur Engineering College, Jabalpur (M. P.) 482011 India  
samajh_singh@rediffmail.com

Alpa Singh Rajput  
Department of Mathematics  
P.O. Box 73  
Jabalpur Engineering College, Jabalpur (M. P.) 482011 India  
alpasinghrajput09@gmail.com