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EXISTENCE AND STABILITY RESULTS FOR COINCIDENCE POINTS OF NONLINEAR CONTRACTIONS

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Abstract. In this paper we define α - admissibility of multi-valued mapping with respect to a single-valued mapping and use this concept to establish a coincidence point theorem for pairs of nonlinear multi-valued and single-valued mappings under the assumption of an inequality with rational terms. We illustrate the result with an example. In the second part of the paper we prove the stability of the coincidence point sets associated with the pairs of mappings in our coincidence point theorem. For that purpose we define the corresponding stability concepts of coincidence points. The results are primarily in the domain of nonlinear set-valued analysis.
Keywords: Hausdorff metric, α-admissible mappings, coincidence point, stability

1. Introduction and Mathematical Background

Our aim in this paper is to establish the existence of coincidence points of two nonlinear single-valued and multi-valued contractions in complete metric spaces. We also discuss stability of the sets of such coincidence points. The study is a part of set-valued analysis.

The famous Banach's contraction mapping principle [2] was extended to the case of set-valued mappings by Nadler in 1969 [17]. The work is recognized as the origin of metric fixed point theory of multi-valued functions. Following the work of Nadler, a large number of works on multi-valued fixed point theory have appeared in literatures. We first describe below some essential concepts for our discussion in this paper. Let $(X,\ d)$ be a metric space. Then

$N(X) = \{A : A \text{ is a non-empty subset of } X\}$,

$B(X) = \{A : A \text{ is a non-empty bounded subset of } X\}$,

$CB(X) = \{A : A \text{ is a non-empty closed and bounded subset of } X\}$ and

$C(X) = \{A : A \text{ is a non-empty compact subset of } X\}$.

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For $x \in X$ and $B \in N(X)$, the function $D(x, B)$, and for $A, B \in CB(X)$, the function $H(A, B)$ are defined as follows:

$$D(x, B) = \inf\{d(x, y) : y \in B\} \text{ and } H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}.$$

H is known as the Hausdorff metric induced by the metric d on $CB(X)$ [17]. Further, if (X, d) is complete then $(CB(X), H)$ is also complete.

The following is the multi-valued contraction mapping theorem due to Nadler [17].

Theorem 1.1. *Let (X, d) be a complete metric space, and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping. Assume that there exists $k \in (0, 1)$ such that*

$$H(Tx, Ty) \leq k\, d(x, y), \quad \textit{for all } x, y \in X.$$

Then T has a fixed point, that is, there exists $z \in X$ such that $z \in Tz$.

The result of the following lemma was proved by Nadler for establishing the above theorem.

Lemma 1.1. [17] Let (X, d) be a metric space and $A, B \in CB(X)$. Let $q > 1$. Then for every $x \in A$, there exists $y \in B$ such that $d(x, y) \leq q\, H(A, B)$.

The above lemma is also valid for $q \geq 1$. We give the result in the following lemma.

Lemma 1.2. Let (X, d) be a metric space and $A, B \in C(X)$. Let $q \geq 1$. Then for every $x \in A$, there exists $y \in B$ such that $d(x, y) \leq q\, H(A, B)$.

Proof. Let $A, B \in C(X)$ and $x \in A$. Since $A, B \in C(X)$ implies $A, B \in CB(X)$, by Lemma 1.1 the result is true if $q > 1$. So, we shall prove the result for $q = 1$. Now, we know that

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}.$$

From the definition, $p = D(x, B) = \inf\{d(x, b) : b \in B\} \leq H(A, B)$. Then there exists a sequence $\{y_n\}$ in B such that $d(x, y_n) \to p$ as $n \to \infty$. Since B is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n(k)}\}$. Hence there exists $y \in X$ such that $y_{n(k)} \to y$ as $k \to \infty$. As B is compact, it is closed and $y \in B$. Now, $\lim_{n\to\infty} d(x, y_n) = p$ implies that $\lim_{k\to\infty} d(x, y_{n(k)}) = p$, that is, $d(x, y) = p = D(x, B) \leq H(A, B)$. Hence the proof is completed.

The following is a consequence of Lemma 1.2.

Lemma 1.3. Let A and B be two nonempty compact subsets of a metric space (X, d) and $T : A \to C(B)$ be a multi-valued mapping. Let $q \geq 1$. Then for $a, b \in A$ and $x \in Ta$, there exists $y \in Tb$ such that $d(x, y) \leq q\, H(Ta, Tb)$.

Definition 1.1. Let $T : X \to CB(Y)$ be a multi-valued mapping, where $(X,\ \rho)$, $(Y,\ d)$ are two metric spaces and H is the Hausdorff metric on $CB(Y)$. The mapping T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X, $H(Tx,\ Tx_n) \to 0$ whenever $\rho(x,\ x_n) \to 0$ as $n \to \infty$.

A new contraction mapping was introduced by Samet et al [19] which is known as $(\alpha - \psi)$ - contraction for which a fixed point result was established in the same work. Subsequently there have been generalizations and extensions of this result through a good number of works like [1, 8, 10, 13, 20] through which the $(\alpha - \psi)$ - type contractions have emerged as an important area of study in recent literatures of fixed point theory. The following definition is associated with this area of study.

Definition 1.2. [19] Let X be a nonempty set, $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. The mapping T is α-admissible if for $x,\ y \in X$,

$$\alpha(x,\ y) \geq 1 \Longrightarrow \alpha(Tx,\ Ty) \geq 1.$$

Following the above definition we define multi-valued α-admissible mapping with respect to a single-valued mapping.

Definition 1.3. Let X be a nonempty set, $T : X \longrightarrow N(X)$ a multi-valued mapping, $g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. The mapping T is called multi-valued α-admissible with respect to the mapping g if for $x_0,\ y_0 \in X$,

$$\alpha(gx_0,\ gy_0) \geq 1 \Longrightarrow \alpha(x_1,\ y_1) \geq 1,\ \text{where } x_1 \in Tx_0 \text{ and } y_1 \in Ty_0.$$

We use the concept of compatibility between a single-valued and a multi-valued mapping.

Definition 1.4. [21] Let $(X,\ d)$ be a metric space, $T : X \longrightarrow CB(X)$ a multi-valued mapping and $g : X \longrightarrow X$. The pair of mappings $(g,\ T)$ is said to compatible if $\lim\limits_{n\longrightarrow\infty} D(gy_{n+1}, Tgx_n) = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim\limits_{n\longrightarrow\infty} gx_n = y_n = t$ for some t in X, where $y_{n+1} \in Tx_n$ for $n = 1,\ 2,\ 3, ...$

The compatibility concept between two mappings has an extensive role in the theory of common fixed points and coincidence points. Originally introduced by Jungck [11], this concept has been variously generalized and used in fixed point related problems. Some of the works in this area are noted in [3, 4, 6, 7, 12, 18].

Definition 1.5. Let X be a nonempty set, $g : X \longrightarrow X$ a single-valued mapping and $T : X \longrightarrow N(X)$ a multivalued mapping. A point $x \in X$ is a coincidence point of g and T if $gx \in Tx$.

The set of all coincidence points of g and T is denoted by $C(g,\ T)$, that is, $C(g,\ T) = \{x \in X : gx \in Tx\}$.

The second part of the paper deals with the stability of coincidence point sets of the single-valued and multi-valued contractions we consider here. Such a concept is a generalization of the stability of fixed point sets which have been considered in several papers like [5, 8, 9, 14, 15, 16] in recent times. The following is the essential background for this study.

Let $(X,\ d)$ be a metric space and $\Big\{T_n : X \longrightarrow X : n \in \mathbb{N}\Big\}$ be a sequence of mappings. We recall that the fixed point sets $F(T_n)$ of a sequence $\{T_n\}$ are stable if $H\Big(F(T_n),\ F(T)\Big) \longrightarrow 0$ as $n \longrightarrow \infty$ where $T = \lim\limits_{n\to\infty} T_n$ and H is the Hausdorff metric. We propose the following two definitions of stability for coincidence point sets. The first definition is the stability of coincidence point sets of a sequence of pair of mappings $\Big\{(g,\ T_n) : n \in \mathbb{N}, T_n : X \longrightarrow X \text{ and } g : X \longrightarrow X\Big\}$ while the other is the stability of coincidence point sets of a sequence of pair of mappings $\Big\{(g,\ T_n) : n \in \mathbb{N}, T_n : X \longrightarrow CB(X) \text{ and } g : X \longrightarrow X\Big\}$.

Definition 1.6. Let $(X,\ d)$ be a metric space. Let $\Big\{T_n : X \longrightarrow X : n \in \mathbb{N}\Big\}$ be a sequence of mappings and $g : X \longrightarrow X$. The coincidence point sets $C(g,\ T_n)$ of the sequence of pair of mappings $\{(g,\ T_n) : n \in \mathbb{N}\}$ are stable if $\lim_{n\to\infty} H\Big(C(g,\ T_n),\ C(g,\ T)\Big) = 0$, where $T = \lim\limits_{n\to\infty} T_n$ and H is the Hausdorff metric.

Definition 1.7. Let $(X,\ d)$ be a metric space. Let $\Big\{T_n : X \longrightarrow CB(X) : n \in \mathbb{N}\Big\}$ be a sequence of multi-valued mappings and $g : X \longrightarrow X$. The coincidence point sets $C(g,\ T_n)$ of the sequence of pair of mappings $\{(g,\ T_n) : n \in \mathbb{N}\}$ are stable if $\lim_{n\to\infty} H\Big(C(g,\ T_n),\ C(g,\ T)\Big) = 0$, where $T = \lim\limits_{n\to\infty} T_n$ and H is the Hausdorff metric.

Note: In definition 1.6, if one treats each T_n $(n = 1, 2, 3, ...)$ as a multi-valued mapping in which case $T_n x$ is a singleton set for every $x \in X$, then the definition 1.6 reduces to definition 1.7. So definition 1.6 is a special case of definition 1.7.

Note: In both of the definitions 1.6 and 1.7, if one considers $g : X \longrightarrow X$ to be identity mapping, then both the definitions 1.6 and 1.7 reduce to the definition of stability of fixed point sets of a sequence of mappings $\Big\{T_n : X \longrightarrow X : n \in \mathbb{N}\Big\}$ and $\Big\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\Big\}$ respectively.

As mentioned earlier, the present work has two parts. The first part is a coincidence point result for a single-valued and a multi-valued mappings. The second part is the analysis of stability of coincidence point sets.

We note some features of the present work in the following.

- We consider nonlinear single and multi-valued mappings.

- We use in our theorem a rational inequality.
- We define multi-valued α - admissibility with respect to single-valued mappings.
- We define stability of coincidence point sets for single and multi-valued mapping pairs.
- The stability of coincidence point sets is established.
- Hausdorff distance is used.
- An illustrative example is discussed.

2. Main Results

Theorem 2.1. *Let $(X,\ d)$ be a complete metric space, $T : X \longrightarrow C(X)$ a multi-valued mapping, $g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. Suppose that*

(i) $Tx \subseteq g(X)$ *for every* $x \in X$,
(ii) g *and* T *are continuous,*
(iii) *The pair of mappings* $(g,\ T)$ *is compatible,*
(iv) T *is multi-valued* $\alpha -$ *admissible with respect to the mapping* g.

Let $\varphi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be a continuous and nondecreasing function with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ and $\varphi(t) < t$ for each $t > 0$. Suppose that for all $x,\ y \in X$,

$$\alpha(gx,\ gy)\ H(Tx,\ Ty) \leq \varphi\Big(\mathcal{U}(x,\ y)\Big), \tag{2.1}$$

where

$$\mathcal{U}(x,\ y) = max\left\{d(gx,\ gy),\ \frac{D(gy,\ Ty)\ [1 + D(gx,\ Tx)]}{1 + d(gx,\ gy)}, \frac{D(gy,\ Tx)\ [1 + D(gx,\ Ty)]}{1 + d(gx,\ gy)}\right\}. \tag{2.2}$$

If there exist $x_0 \in X$ and $y_1 \in Tx_0$ such that $\alpha(gx_0,\ y_1) \geq 1$, then $C(g,\ T)$ is non-empty.

Proof. By the condition, there exist $x_0 \in X$ and $y_1 \in Tx_0$ such that $\alpha(gx_0,\ y_1) \geq 1$. Since $Tx_0 \subseteq g(X)$ and $y_1 \in Tx_0$, there exists $x_1 \in X$ such that $gx_1 = y_1$. So $\alpha(gx_0,\ gx_1) \geq 1$. By Lemma 1.3, for $y_1 = gx_1 \in Tx_0$ there exists $y_2 \in Tx_1$ such that

$$d(y_1,\ y_2) \leq \alpha(gx_0,\ gx_1)\ H(Tx_0,\ Tx_1).$$

Since $Tx_1 \subseteq g(X)$ and $y_2 \in Tx_1$, there exists $x_2 \in X$ such that $gx_2 = y_2$. So from the above inequality, we have

$$d(gx_1,\ gx_2) \leq \alpha(gx_0,\ gx_1)\ H(Tx_0,\ Tx_1).$$

Applying (2.1) and using the monotone property of φ, we have

$$\begin{aligned} d(gx_1,\ gx_2) &\le \alpha(gx_0,\ gx_1)\ H(Tx_0,\ Tx_1) \\ &\le \varphi\Big(\max\Big\{d(gx_0,\ gx_1),\ \frac{D(gx_1,\ Tx_1)\ [1+D(gx_0,\ Tx_0)]}{1+d(gx_0,\ gx_1)}, \\ &\qquad \frac{D(gx_1,\ Tx_0)\ [1+D(gx_0,\ Tx_1)]}{1+d(gx_0,\ gx_1)}\Big\}\Big) \\ &\le \varphi\Big(\max\Big\{d(gx_0,\ gx_1),\ \frac{d(gx_1,\ gx_2)\ [1+d(gx_0,\ gx_1)]}{1+d(gx_0,\ gx_1)}, \\ &\qquad \frac{d(gx_1,\ gx_1)\ [1+d(gx_0,\ gx_2)]}{1+d(gx_0,\ gx_1)}\Big\}\Big) \\ &\le \varphi\Big(\max\Big\{d(gx_0,\ gx_1),\ d(gx_1,\ gx_2)\Big\}\Big). \end{aligned}$$

Therefore,

$$d(gx_1,\ gx_2) \le \varphi\Big(\max\Big\{d(gx_0,\ gx_1),\ d(gx_1,\ gx_2)\Big\}\Big). \tag{2.3}$$

Suppose that $d(gx_0,\ gx_1) < d(gx_1,\ gx_2)$. Then $d(gx_1,\ gx_2) > 0$ and it follows by (2.3) and a property of φ that

$$d(gx_1,\ gx_2) \le \varphi\Big(d(gx_1,\ gx_2)\Big) < d(gx_1,\ gx_2),$$

which is a contradiction. Hence $d(gx_1,\ gx_2) \le d(gx_0,\ gx_1)$. Then it follows from (2.3) that

$$d(gx_1,\ gx_2) \le \varphi\Big(d(gx_0,\ gx_1)\Big). \tag{2.4}$$

Since T is α-admissible with respect to the mapping g and $gx_1 \in Tx_0$, $gx_2 \in Tx_1$ with $\alpha(gx_0,\ gx_1) \ge 1$, we have that $\alpha(gx_1,\ gx_2) \ge 1$. By Lemma 1.3, for $y_2 = gx_2 \in Tx_1$ there exists $y_3 \in Tx_2$ such that

$$d(y_2,\ y_3) \le \alpha(gx_1,\ gx_2)\ H(Tx_1,\ Tx_2).$$

Since $Tx_2 \subseteq g(X)$ and $y_3 \in Tx_2$, there exists $x_3 \in X$ such that $gx_3 = y_3$. Then we have

$$d(gx_2,\ gx_3) \le \alpha(gx_1,\ gx_2)\ H(Tx_1,\ Tx_2).$$

Applying (2.1) and using the monotone property of φ, we have

$$\begin{aligned} d(gx_2,\ gx_3) &\le \alpha(gx_1,\ gx_2)\ H(Tx_1,\ Tx_2) \\ &\le \varphi\Big(\max\Big\{d(gx_1,\ gx_2),\ \frac{D(gx_2,\ Tx_2)\ [1+D(gx_1,\ Tx_1)]}{1+d(gx_1,\ gx_2)}, \\ &\qquad \frac{D(gx_2,\ Tx_1)\ [1+D(gx_1,\ Tx_2)]}{1+d(gx_1,\ gx_2)}\Big\}\Big) \\ &\le \varphi\Big(\max\Big\{d(gx_1,\ gx_2),\ \frac{d(gx_2,\ gx_3)\ [1+d(gx_1,\ gx_2)]}{1+d(gx_1,\ gx_2)}, \\ &\qquad \frac{d(gx_2,\ gx_2)\ [1+d(gx_1,\ gx_3)]}{1+d(gx_1,\ gx_2)}\Big\}\Big) \end{aligned}$$

$$\leq \varphi\Big(\max\Big\{d(gx_1,\ gx_2),\ d(gx_2,\ gx_3)\Big\}\Big).$$

Therefore,

$$d(gx_2,\ gx_3) \leq \varphi\Big(\max\Big\{d(gx_1,\ gx_2),\ d(gx_2,\ gx_3)\Big\}\Big). \tag{2.5}$$

Suppose that $d(gx_1,\ gx_2) < d(gx_2,\ gx_3)$. Then $d(gx_2,\ gx_3) > 0$ and it follows by (2.5) and a property of φ that

$$d(gx_2,\ gx_3) \leq \varphi\Big(d(gx_2,\ gx_3)\Big) < d(gx_2,\ gx_3),$$

which is a contradiction. Hence $d(gx_2,\ gx_3) \leq d(gx_1,\ gx_2)$. Then it follows from (2.5) that

$$d(gx_2,\ gx_3) \leq \varphi\Big(d(gx_1,\ gx_2)\Big). \tag{2.6}$$

Since T is α-admissible with respect to the mapping g and $gx_2 \in Tx_1$, $gx_3 \in Tx_2$ with $\alpha(gx_1,\ gx_2) \geq 1$, we have that $\alpha(gx_2,\ gx_3) \geq 1$. Maintaining this process, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$,

$$y_{n+1} = gx_{n+1} \in Tx_n, \tag{2.7}$$

$$\alpha(gx_n,\ gx_{n+1}) \geq 1 \tag{2.8}$$

and

$$d(gx_{n+1},\ gx_{n+2}) \leq \varphi\Big(d(gx_n,\ gx_{n+1})\Big). \tag{2.9}$$

Applying (2.9) repeatedly and using the monotone property of φ, we have

$$d(gx_{n+1},\ gx_{n+2}) \leq \varphi\Big(d(gx_n,\ gx_{n+1})\Big) \leq \varphi^2\Big(d(gx_{n-1},\ gx_n)\Big) \leq ... \leq \varphi^{n+1}\Big(d(gx_0,\ gx_1)\Big). \tag{2.10}$$

Then by a property of φ, we have

$$\sum_n d(gx_n,\ gx_{n+1}) \leq \sum_n \varphi^n\Big(d(gx_0,\ gx_1)\Big) < \infty,$$

which shows that $\{gx_n\}$ is a Cauchy sequence in X. As the metric space $(X,\ d)$ is complete, there exists a point $p \in X$ such that

$$gx_n \longrightarrow p \ \text{ as } \ n \longrightarrow \infty. \tag{2.11}$$

By (2.7) and (2.11), we have

$$y_n \longrightarrow p \ \text{ as } \ n \longrightarrow \infty. \tag{2.12}$$

Here $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $y_{n+1} \in Tx_n$ for $n \in \mathbb{N}$ and $\lim_{n \longrightarrow \infty} gx_n = y_n = p$, where $p \in X$. So applying the compatibility condition of the pairs of functions g and T, we have

$$\lim_{n \longrightarrow \infty} D(gy_{n+1},\ Tgx_n) = 0.$$

Since T and g are continuous, using (2.11) and (2.12), we have from the above inequality that $D(gp,\ Tp) = 0$. Since $Tp \in C(X)$, Tp is compact and hence Tp is closed, that is, $Tp = \overline{Tp}$, where $\overline{Tp}$ denotes the closure of Tp. Now, $D(gp,\ Tp) = 0$ implies that $gp \in \overline{Tp} = Tp$, that is, $p \in C(g,\ T)$. Hence $C(g,\ T)$ is non-empty.

In the next the theorem we consider $g(X)$ to be closed in the metric space $(X,\ d)$ and also consider a order condition involving α. Due to this considerations we need not require the following assumptions which we consider in theorem 2.1 :

(a) the continuity assumption of g and T,

(b) the compatibility assumption of the pairs $(g,\ T)$.

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1 (except the hypothesis (ii) and (iii)), suppose that $g(X)$ is a closed subset of $(X,\ d)$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,\ x_{n+1}) \geq 1$ for all n and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n(l)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(l)},\ x) \geq 1$ for all l. Then $C(g,\ T)$ is non-empty.*

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. Then $\{gx_n\}$ satisfies (2.7), (2.8) and (2.9). Arguing similarly as in the proof of Theorem 2.1, we prove that $\{gx_n\}$ is a Cauchy sequence in $g(X) \subseteq X$ and satisfies (2.11), that is,

$$gx_n \longrightarrow p \ \text{ as } \ n \longrightarrow \infty.$$

As $g(X)$ is closed, we have $p \in g(X)$. So there exists $u \in X$ such that $gu = p$. Therefore

$$gx_n \longrightarrow gu \ \text{ as } \ n \longrightarrow \infty. \tag{2.13}$$

Using (2.7), (2.8) and (2.13), we have a subsequence $\{gx_{n(l)}\}$ of $\{gx_n\}$ such that

$$\alpha(gx_{n(l)},\ gu) \geq 1, \quad \text{for all } l \geq 1.$$

Since $gx_{n(l)+1} \in Tx_{n(l)}$, for all $l \geq 1$, from (2.1) and (2.2), we get

$$\begin{aligned} D(gx_{n(l)+1},\ Tu) &\leq H(Tx_{n(l)},\ Tu) \leq \alpha(gx_{n(l)},\ gu)\ H(Tx_{n(l)},\ Tu) \\ &\leq \varphi\Big(\max\Big\{d(gx_{n(l)},\ gu),\ \frac{D(gu,\ Tu)\ \big[1 + D(gx_{n(l)},\ Tx_{n(l)})\big]}{1 + d(gx_{n(l)},\ gu)}, \\ &\qquad \frac{D(gu,\ Tx_{n(l)})\ \big[1 + D(gx_{n(l)},\ Tu)\big]}{1 + d(gx_{n(l)},\ gu)}\Big\}\Big) \end{aligned}$$

$$\leq \varphi\Big(\max\Big\{d(gx_{n(l)},\ gu),\ \frac{D(gu,\ Tu)\ \Big[1+d(gx_{n(l)},\ gx_{n(l)+1})\Big]}{1+d(gx_{n(l)},\ gu)},$$
$$\frac{d(gu,\ gx_{n(l)+1})\ \Big[1+D(gx_{n(l)},\ Tu)\Big]}{1+d(gx_{n(l)},\ gu)}\Big\}\Big).$$

Taking limit $l \longrightarrow \infty$ in the above inequality, using (2.13) and the continuity of φ, we have

$$D(gu,\ Tu) \leq \varphi\Big(D(gu,\ Tu)\Big).$$

Suppose that $D(gu,\ Tu) \neq 0$. Then from the above inequality and by a property of φ, we have

$$D(gu,\ Tu) \leq \varphi\Big(D(gu,\ Tu)\Big) < D(gu,\ Tu),$$

which is a contradiction. Hence $D(gu,\ Tu) = 0$. Then arguing similarly as in the proof of Theorem 2.1, $u \in C(g,\ T)$, that is, $C(g,\ T)$ is non-empty.

Example 2.1. Let $X = \Big[2,\ \infty\Big)$ and "d" be the usual metric on X.

Let $T : X \longrightarrow C(X)$ be defined as follows:

$$Tx = \begin{cases} \Big[x + \frac{1}{x} - \frac{1}{3},\ 3\Big], & \text{if } \ 2 \leq x \leq 3 \\ \{x\}, & \text{if } \ x > 3. \end{cases}$$

Let $g : X \longrightarrow X$ be defined as follows:

$$gx = \begin{cases} x + \frac{1}{x} - \frac{1}{3}, & \text{if } \ 2 \leq x \leq 3 \\ x, & \text{if } \ x > 3. \end{cases}$$

Let $\alpha : X \times X \to [0,\ \infty)$ be defined as

$$\alpha(x,\ y) = \begin{cases} 2, & \text{if } \ x = y > 3 \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Let φ: $[0,\ \infty) \longrightarrow [0,\ \infty)$ be defined as follows:

$$\varphi(t) = k\ t,\ \text{ where } t \in [0,\ \infty) \text{ and } \frac{1}{2} \leq k < 1.$$

Then all the conditions of the Theorems 2.1 and 2.2 are satisfied and here $C(g,\ T) = \Big[2,\ \infty\Big)$.

3. Stability of coincidence point sets

In this section, we investigate the stability of coincidence point sets of the set valued contractions mentioned in section 2.

Theorem 3.1. *Let $(X,\ d)$ be a complete metric space, $T_i : X \longrightarrow C(X),\ i = 1,\ 2$ be two multi-valued mappings, $g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$. Let the following conditions be satisfied.*

- (i) *$T_i x \subseteq g(X)$ for every $x \in X$, where $i \in \{1,\ 2\}$,*
- (ii) *Each T_i $(i=1,2)$ and g are continuous,*
- (iii) *Each T_i $(i=1,2)$ is multivalued α–admissible with respect to the mapping g,*
- (iv) *Each pair mappings $(g,\ T_i)$, where $i \in \{1,\ 2\}$, is compatible,*
- (v) *Let $M_i = Sup\ \{d(x,\ gx) : x \in C(g,\ T_i)\}$, where $i \in \{1,\ 2\}$, exist.*

Suppose that each pair $(g,\ T_i)$, $i = 1, 2$ satisfies 2.1 and 2.2 of theorem 2.1, where $\varphi : [0,\ \infty) \to [0,\ \infty)$ is a continuous and nondecreasing function with $\Phi(t) = \sum_{n=1}^{\infty} \varphi^n(t) < \infty$, $\Phi(t) \to 0$ as $t \to 0$ and $\varphi(t) < t$ for each $t > 0$. Also suppose that for any $x \in C(g,\ T_1)$, we have $\alpha(gx,\ y) \geq 1$ whenever $y \in T_2 x$; and for any $x \in C(g,\ T_2)$, we have $\alpha(gx,\ y) \geq 1$ whenever $y \in T_1 x$. Then $H\Big(C(g,\ T_1),\ C(g,\ T_2)\Big) \leq \Phi(K) + R$, where $K = \sup_{x \in X} H(T_1 x,\ T_2 x)$ and $R = Max\ \{M_i : i = 1,\ 2\}$.

Proof. By theorem 2.1, the sets $C(g,\ T_1)$ and $C(g,\ T_2)$ are non-empty. Let $x_0 \in C(g,\ T_1)$, that is, $gx_0 \in T_1 x_0$. Following the Lemma 1.2, there exists $y_1 \in T_2 x_0$ such that

$$d(gx_0,\ y_1) \leq H(T_1 x_0,\ T_2 x_0).$$

Since $x_0 \in C(g,\ T_1)$ and $y_1 \in T_2 x_0$, by the condition of the theorem, we have $\alpha(gx_0,\ y_1) \geq 1$. As $T_2 x_0 \subseteq g(X)$ and $y_1 \in T_2 x_0$, there exists $x_1 \in X$ such that $gx_1 = y_1$. So $\alpha(gx_0,\ gx_1) \geq 1$. Therefore, the above inequality reduces to the following inequality

$$d(gx_0,\ gx_1) \leq H(T_1 x_0,\ T_2 x_0). \tag{3.1}$$

Since $gx_1 = y_1 \in T_2 x_0$, following the Lemma 1.3 there exists $y_2 \in T_2 x_1$ such that

$$d(gx_1,\ y_2) \leq \alpha(gx_0,\ gx_1) H(T_2 x_0,\ T_2 x_1).$$

Since $T_2 x_1 \subseteq g(X)$ and $y_2 \in T_2 x_1$, there exists $x_2 \in X$ such that $gx_2 = y_2$. Then the above inequality becomes

$$d(gx_1,\ gx_2) \leq \alpha(gx_0,\ gx_1) H(T_2 x_0,\ T_2 x_1).$$

Arguing similarly as in the proof of Theorem 2.1, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$

$$y_{n+1} = gx_{n+1} \in T_2 x_n, \tag{3.2}$$

and (2.8), (2.9) and (2.10) are satisfied. Arguing similarly as in the proof of Theorem 2.1, we prove that $\{gx_n\}$ is a Cauchy sequence X and there exists a point $p \in X$ such that (2.11) and (2.12) are satisfied and also $gp \in \overline{T_2p} = T_2p$, that is, $p \in C(g,\ T_2)$, that is, $C(g,\ T_2)$ is non-empty.

From (3.1) and the the definition of K, we have

$$d(gx_0,\ gx_1) \leq H(T_1x_0,\ T_2x_0) \leq K = \sup_{x\in X} H(T_1x,\ T_2x). \tag{3.3}$$

By triangle inequality and (2.10), we have

$$d(gx_0,\ p) \leq d(gx_{n+1},\ p) + \sum_{i=0}^{n} d(gx_i,\ gx_{i+1}) \leq d(gx_{n+1},\ p) + \sum_{i=0}^{n} \varphi^i\Big(d(gx_0,\ gx_1)\Big).$$

Taking limit $n \longrightarrow \infty$ in the above inequality, using (2.11), (3.3) and the properties of φ, we have

$$d(gx_0,\ p) \leq \sum_{i=0}^{\infty} \varphi^i\Big(d(gx_0,\ gx_1)\Big) \leq \sum_{i=0}^{\infty} \varphi^i(K) = \Phi(K). \tag{3.4}$$

By triangle inequality and (3.4), we have

$$d(x_0,\ p) \leq d(x_0,\ gx_0) + d(gx_0,\ p) \leq d(x_0,\ gx_0) + \Phi(K) \leq \Phi(K) + R.$$

Thus, given arbitrary $x_0 \in C(g,\ T_1)$, we can find $p \in C(g,\ T_2)$ for which

$$d(x_0,\ p) \leq \Phi(K) + R.$$

Similarly, we can prove that for arbitrary $z_0 \in C(g,\ T_2)$, there exists a $w \in C(g,\ T_1)$ such that $d(z_0,\ w) \leq \Phi(K) + R$. Hence, we conclude that

$$H\Big(C(g,\ T_1),\ C(g,\ T_2)\Big) \leq \Phi(K) + R.$$

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1 (except the hypothesis (ii) and (iv)), suppose that $g(X)$ is a closed subset of $(X,\ d)$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,\ x_{n+1}) \geq 1$ for all n and $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n(l)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(l)},\ x) \geq 1$ for all l. Then $H\Big(C(g,\ T_1),\ C(g,\ T_2)\Big) \leq \Phi(K) + R$, where K and R are as defined in Theorem 3.1.*

Proof. From Theorem 2.2, the sets $C(g,\ T_1)$ and $C(g,\ T_2)$ are non-empty. Let $y_0 \in C(g,\ T_1)$, that is, $gy_0 \in T_1y_0$. Arguing similarly as in the proof of Theorem 2.2 and Theorem 3.1, we have the required proof.

Lemma 3.1. *Let $(X,\ d)$ be a complete metric space. Let $\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\}$ be a sequence of multi-valued mappings, $g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\ \infty)$.*

Let each T_n ($n \in \mathbb{N}$) be multi-valued α - admissible with respect to the mapping g. Suppose that if $\{x_n\}$ and $\{y_n\}$ are two sequences in X with $x_n \to a$ and $y_n \to b$ as $n \to \infty$, then

$$\alpha(x_n,\ y_n) \geq 1 \text{ for every } n \in \mathbb{N} \implies \alpha(a,\ b) \geq 1. \tag{3.5}$$

Suppose that the sequence $\{T_n\}$ is uniformly convergent to a multi-valued mapping $T : X \longrightarrow C(X)$. If for every $n \in \mathbb{N}$, the pair $(g,\ T_n)$ satisfies (2.1) and (2.2), then T is multi-valued α-admissible with respect to g and the pair $(g,\ T)$ satisfies (2.1) and (2.2), where the function $\varphi : [0,\ \infty) \to [0,\ \infty)$ is continuous.

proof. For every $n \in \mathbb{N}$, the pair $(g,\ T_n)$ satisfies (2.1) and (2.2), we have

$$\alpha(gx,\ gy)\ H(T_nx,\ T_ny) \leq \varphi\Big(\max\Big\{d(gx,\ gy),\ \frac{D(gy,\ T_ny)\ [1 + D(gx,\ T_nx)]}{1 + d(gx,\ gy)},\ \frac{D(gy,\ T_nx)\ [1 + D(gx,\ T_ny)]}{1 + d(gx,\ gy)}\Big\}\Big).$$

Since the sequence $\{T_n\}$ is uniformly convergent to T and φ is continuous, taking limit $n \to \infty$ in the above inequality, we get

$$\alpha(gx,\ gy)\ H(Tx,\ Ty) \leq \varphi\Big(\max\Big\{d(gx,\ gy),\ \frac{D(gy,\ Ty)\ [1 + D(gx,\ Tx)]}{1 + d(gx,\ gy)},\ \frac{D(gy,\ Tx)\ [1 + D(gx,\ Ty)]}{1 + d(gx,\ gy)}\Big\}\Big),$$

which shows that the pair $(g,\ T)$ satisfies (2.1)and (2.2).

Now, we shall prove that T is multi-valued α-admissible with respect to the mapping g. Let $\alpha(gx,\ gy) \geq 1$, for some $x,\ y \in X$. Let $a \in Tx$ and $b \in Ty$ be arbitrary. Since $T_n \to T$ uniformly, there exist two sequences $\{x_n\}$ in $\{T_nx\}$ and $\{y_n\}$ in $\{T_ny\}$ such that $x_n \to a$ and $y_n \to b$ as $n \to \infty$. Since for every $n \in \mathbb{N}$, T_n is multi-valued α-admissible with respect to g and $\alpha(gx,\ gy) \geq 1$, it follows from definition 1.3 that $\alpha(x_n,\ y_n) \geq 1$, for every $n \in \mathbb{N}$. Then by (3.5), it follows that $\alpha(a,\ b) \geq 1$. Thus for $x,\ y \in X$, we have

$$\alpha(gx,\ gy) \geq 1 \Longrightarrow \alpha(a,\ b) \geq 1, \text{ where } a \in Tx \text{ and } b \in Ty.$$

Hence T is multi-valued α-admissible with respect to the mapping g.

In the following we present our stability results.

Theorem 3.3. *Let $(X,\ d)$ be a complete metric space. Let $\Big\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\Big\}$ be a sequence of multi-valued mappings, $g : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow$*

$[0,\ \infty)$. *Suppose that the sequence* $\{T_n\}$ *is uniformly convergent to a multi-valued mapping* $T : X \longrightarrow C(X)$. *Suppose that the following hold:*

(i) $T_n x \subseteq g(X)$ *for every* $x \in X$ *and* $n \in \mathbb{N}$,

(ii) *each* $T_n (n \in \mathbb{N})$ *and* g *are continuous,*

(iii) *each* $T_n (n \in \mathbb{N})$ *is multi-valued* α*–admissible with respect to the mapping* g,

(iv) *each pair of mappings* (g, T_n), *where* $n \in \mathbb{N}$, *is compatible,*

(v) *each pair of mappings* (g, T_n), *where* $n \in \mathbb{N}$, *satisfies* (2.1) *and* (2.2), *where the conditions upon* φ *are the same as in Theorem* 3.1,

(vi) *if* $\{x_n\}$ *and* $\{y_n\}$ *are two sequences in* X *with* $x_n \to a$ *and* $y_n \to b$ *as* $n \to \infty$, *then* $\alpha(x_n,\ y_n) \geq 1$ *for every* $n \in \mathbb{N} \implies \alpha(a,\ b) \geq 1$,

(vii) *for any* $x \in C(g, T_n)(n \in \mathbb{N})$ *there exists* $y \in Tx$ *such that*$\alpha(gx,\ y) \geq 1$; *also, for any* $x \in C(g, T)$ *there exists* $y \in T_n x (n \in \mathbb{N})$ *such that* $\alpha(gx, y) \geq 1$,

$(viii)$ $M_n = Sup\Big\{d(x, gx) : x \in C(g,\ T_n) \bigcup C(g,\ T)\Big\}$ *and* $M_n \to 0$ *as* $n \to \infty$.

Then $\lim_{n\to\infty} H\Big(C(g,\ T_n),\ C(g,\ T)\Big) = 0$, *that is, the coincidence point sets of the sequence of pair of mappings* $\{(g,\ T_n)\}$ *are stable.*

proof. Since $\{T_n : X \longrightarrow C(X) : n \in \mathbb{N}\}$ is a sequence of continuous multi-valued mappings and is uniformly convergent to a multi-valued mapping $T : X \longrightarrow C(X)$, T is also continuous. By Lemma 3.1, T is multi-valued α-admissible with respect to g and the pair $(g,\ T)$ satisfies (2.1) and (2.2). Let $K_n = \sup_{x\in X} H(T_n x,\ Tx)$. Since the sequence $\{T_n\}$ is uniformly convergent to T on X,

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty} \sup_{x\in X} H(T_n x,\ Tx) = 0. \tag{3.6}$$

From Theorem 3.1, we get

$$H\Big(C(g,\ T_n),\ C(g,\ T)\Big) \leq \Phi(K_n) + M_n, \quad \text{for every } n \in \mathbb{N}.$$

Since φ is continuous and $\Phi(t) \to 0$ as $t \to 0$, using (3.6) and the condition (viii) of the theorem, we have

$$\lim_{n\to\infty} H\Big(C(g,\ T_n),\ C(g,\ T)\Big) \leq \lim_{n\to\infty} \Big[\Phi(K_n) + M_n\Big] = 0,$$

that is,

$$\lim_{n\to\infty} H\Big(C(g,\ T_n),\ C(g,\ T)\Big) = 0.$$

Hence the proof is completed.

Theorem 3.4. *In addition to the hypotheses of Theorem 3.3 (except the hypothesis (ii) and (iv)), suppose that* $g(X)$ *is a closed subset of* $(X,\ d)$ *and if* $\{x_n\}$ *is* *a sequence in* X *such that* $\alpha(x_n,\ x_{n+1}) \geq 1$ *for all* n *and* $x_n \longrightarrow x$ *as* $n \longrightarrow \infty$, *then there exists a subsequence* $\{x_{n(l)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(l)},\ x) \geq 1$ *for all* l. *Then* $\lim_{n\to\infty} H\Big(C(g,\ T_n),\ C(g,\ T)\Big) = 0$, *that is, the coincidence point sets of the sequence of pair of mappings* $\{(g,\ T_n)\}$ *are stable.*

proof. Arguing similarly as in the proof of theorem 3.3 and using theorem 3.2, we have the required proof.

Competing interests. The authors declare that they have no competing interests.

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