\( \alpha_\beta \)-BOUNDED SETS AND \( \alpha_\beta \)-TOPOLOGICALLY NILPOTENT ELEMENTS

Hariwan Z. Ibrahim and Alias B. Khalaf

Abstract. The aim of this paper is to define and discuss the properties of \( \alpha_\beta \)-boundedness, \( \alpha_\beta \)-topological divisor of zero and \( \alpha_\beta \)-topologically nilpotent elements.

Keywords: operations, \( \alpha_\beta \)-open set, rings, \( \alpha_\beta \)-boundedness, \( \alpha_\beta \)-topologically nilpotent

1. Introduction

Given a topological space \((G, \tau)\), Njastad [16] was the first one to talk about \( \alpha \)-open sets and proved that they form a topology finer than \( \tau \). Ibrahim [6] introduced a strong form of \( \alpha \)-open sets called \( \alpha_\beta \)-open sets, where \( \beta \) is an operation on the family of all \( \alpha \)-open sets of \( G \). Later Khalaf and Ibrahim [10, 11, 12, 13, 14] continued studying the properties of such open sets and also introduced \( \alpha(\beta, \beta) \)-topological abelian groups, \( \alpha(\beta, \beta) \)-topological rings and \( \alpha(\beta, \gamma) \)-topological modules. In recent years, topological algebra was applied in both harmonic analysis and complex fractional calculus [7, 8, 4]. One of the central definitions of the present paper is that one of \( \alpha_\beta \)-bounded set for an \( \alpha(\beta, \beta) \)-topological ring and for an \( \alpha(\beta, \gamma) \)-topological module. Several properties are proven, like hereditariness, stability by topological closure, stability by taking unions and sums. Hereditariness and stability by unions lead me to think to the definition of boundedness proposed by S. T. Hu, [5]. We recall some of the well known definitions and results which can be found in most of text books of abstract algebra we refer to [1], [2], [3], [9], [15] and [17].

2. Preliminaries

Let \( A \) be a subset of a topological space \((G, \tau)\). We denote the interior and the closure of a set \( A \) by \( Int(A) \) and \( Cl(A) \), respectively. A subset \( A \) of a topological space \((G, \tau)\) is called \( \alpha \)-open [16] if \( A \subseteq Int(Cl(Int(A))) \). By \( \alpha O(G, \tau) \), we denote

Received April 04, 2017; accepted July 07, 2017
2010 Mathematics Subject Classification. Primary: 54A05, 54A10; Secondary: 54C05

435
the family of all $\alpha$-open sets of $G$. An operation $\beta : \alpha O(G, \tau) \rightarrow P(G)$ [6] is a
mapping from $\alpha O(G, \tau)$ in to power set $P(G)$ of $G$ satisfying the condition, $V \subseteq V^\beta$
for each $V \in \alpha O(G, \tau)$, where $V^\beta$ denotes the value of $\beta$ at $V$. We call the mapping $\beta$
an operation on $\alpha O(G, \tau)$. A subset $A$ of $G$ is called an $\alpha_\beta$-open set [6] if for each
point $x \in A$, there exists an $\alpha$-open set $U$ of $G$ containing $x$ such that $U^\beta \subseteq A$.
The complement of an $\alpha_\beta$-open set is said to be $\alpha_\beta$-closed. We denote the set of all
$\alpha_\beta$-open sets of $(G, \tau)$ by $\alpha O(G, \tau)_{_\beta}$. The $\alpha_\beta$-closure [6] of a subset $A$ of $G$ with
an operation $\beta$ on $\alpha O(G)$ is denoted by $\alpha_\beta Cl(A)$ and is defined to be the intersection of
all $\alpha_\beta$-closed sets containing $A$. An operation $\beta$ on $\alpha O(G, \tau)$ is said to be $\alpha$-regular
if for every $\alpha$-open sets $U$ and $V$ of each $x \in G$, there exists an $\alpha$-open set $W$ of $x$
such that $W^\beta \subseteq U^\beta \cap V^\beta$.

Definition 2.1. [10] Let $(G, \tau)$ be a topological space and $x \in G$, then a subset $N$ of $G$ is said to be $\alpha_\beta$-neighbourhood of $x$, if there exists an $\alpha_\beta$-open set $U$ in $G$
such that $x \in U \subseteq N$.

Definition 2.2. [6] A topological space $(G, \tau)$ with an operation $\beta$ on $\alpha O(G)$ is
said to be $\alpha_\beta T_2$ if for any two distinct points $x, y \in G$, there exist two $\alpha_\beta$-open sets $U$
and $V$ containing $x$ and $y$, respectively, such that $U \cap V = \phi$.

Definition 2.3. [11] A function $f : (G, \tau) \rightarrow (G', \tau')$ is said to be $\alpha_\beta$-open if
for any $\alpha_\beta$-open set $A$ of $(G, \tau)$, $f(A)$ is $\alpha_\beta'$-open in $(G', \tau')$.

Definition 2.4. [6] A mapping $f : (G, \tau) \rightarrow (G', \tau')$ is said to be $\alpha_\beta$-continuous
if for each $x$ of $G$ and each $\alpha_\beta'$-open set $V$ containing $f(x)$, there exists an $\alpha_\beta$-open
set $U$ such that $x \in U$ and $f(U) \subseteq V$.

Corollary 2.1. [12] A function $f : G \rightarrow G'$ is $\alpha_\beta$-continuous if and only if
$f^{-1}(V)$ is $\alpha_\beta$-open in $G$, for every $\alpha_\beta'$-open set $V$ in $G'$.

Definition 2.5. [13] Let $(G, +)$ be abelian group and $\tau$ be a topology on $G$. A
triple $(G, +, \tau)$ is said to be an $\alpha_\beta$-topological group if the following conditions
are satisfied:

1. For any two elements $a, b \in G$ and $U \in \alpha O(G, \tau)_{_\beta}$ such that $a + b \in U$, there
exist $V, W \in \alpha O(G, \tau)_{_\beta}$ with $a \in V$, $b \in W$ and $V + W \subseteq U$.

2. For any element $a \in G$ and $U \in \alpha O(G, \tau)_{_\beta}$ such that $-a \in U$, there exists
$V \in \alpha O(G, \tau)_{_\beta}$ with $a \in V$ and $-V \subseteq U$.

Definition 2.6. [14] Let $(R, +, \cdot)$ be a ring and $(R, \tau)$ be a topological space.
Then, $(R, +, \cdot, \tau)$ is called an $\alpha_\beta$-topological ring if the following conditions are
satisfied:

1. $(R, +, \tau)$ is $\alpha_\beta$-topological group.
2. For each elements \(a, b \in R\) and \(U \in \alpha O(R, \tau)_\beta\) such that \(a \cdot b \in U\), there exist \(V, W \in \alpha O(R, \tau)_\beta\) with \(a \in V\), \(b \in W\) and \(V \cdot W \subseteq U\).

**Definition 2.7.** [14] Let \((R, +, \cdot, \tau)\) be an \(\alpha(\beta, \gamma)\)-topological ring. A left \(R\)-module \(M\) is called an \(\alpha(\beta, \gamma)\)-topological left \(R\)-module if on \(M\) is specified a topology such that \(M\) is an \(\alpha(\gamma, \gamma)\)-topological abelian group and the following condition is satisfied:

For any \(r \in R\) and \(m \in M\) and arbitrary \(\alpha_\gamma\)-open set \(U\) containing the element \(r \cdot m\) in \(M\), there exist an \(\alpha_\beta\)-open set \(V\) containing the element \(r\) in \(R\) and an \(\alpha_\gamma\)-open set \(W\) the element \(m\) in \(M\) such that \(V \cdot W \subseteq U\).

**Proposition 2.1.** [13] Let a family \(B_0\) of subsets of an \(\alpha(\beta, \gamma)\)-topological abelian group \(G\) be a basis of \(\alpha_\beta\)-neighborhoods of zero in \(G\) and \(\beta\) be an \(\alpha\)-regular operation on \(\alpha O(G)\). Then, the following conditions are satisfied:

1. \(0 \in \bigcap_{V \in B_0} V\).
2. For any subsets \(U\) and \(V\) from \(B_0\), there exists a subset \(W \in B_0\) such that \(W \subseteq U \cap V\).
3. For any subset \(U \in B_0\), there exists a subset \(V \in B_0\) such that \(V + V \subseteq U\).
4. For any subset \(U \in B_0\), there exists a subset \(V \in B_0\) such that \(-V \subseteq U\).

Besides, if \(a \in G\), then \(B_a = \{a + V | V \in B_0\}\) is a basis of \(\alpha_\beta\)-neighborhoods of the element \(a\).

**Proposition 2.2.** [14] Let \(R\) be an \(\alpha(\beta, \gamma)\)-topological ring, \(B_0\) be a basis of \(\alpha_\gamma\)-neighborhoods of zero of an \(\alpha(\beta, \gamma)\)-topological \(R\)-module \(M\) and \(\gamma\) be an \(\alpha\)-regular operation on \(\alpha O(M)\). Then conditions (1) to (4) of Proposition 2.1, are satisfied together with the following conditions:

1. For any subset \(U \in B_0\), there exists a subset \(V \in B_0\) and an \(\alpha_\beta\)-neighborhood \(W\) of zero in \(R\) such that \(W \cdot V \subseteq U\).
2. For any subset \(U \in B_0\) and any element \(r \in R\), there exists a subset \(V \in B_0\) such that \(r \cdot V \subseteq U\).
3. For any subset \(U \in B_0\) and any element \(a \in M\), there exists an \(\alpha_\beta\)-neighborhood \(W\) of zero in \(R\) such that \(W \cdot a \subseteq U\).

**Corollary 2.2.** [13] Let \(U\) and \(V\) be \(\alpha_\beta\)-neighborhoods of zero of \(\alpha(\beta, \gamma)\)-topological abelian group \(G\) such that \(V + V \subseteq U\), then \(\alpha_\beta Cl(V) \subseteq U\).

**Proposition 2.3.** [13] Let \(G\) be \(\alpha(\beta, \gamma)\)-topological abelian group, \(G'\) be \(\alpha(\beta', \gamma')\)-topological abelian group and \(f : G \to G'\) be a homomorphic mapping of \(G\) to \(G'\). Then:
1. $f$ is an $\alpha_{(\beta,\beta')}$-continuous if and only if $f^{-1}(U')$ is an $\alpha_\beta$-neighborhood of zero in $G$ for any $\alpha_{\beta'}$-neighborhood $U'$ of zero in $G'$.

2. $f$ is an $\alpha_{(\beta,\beta')}$-open if and only if $f(U)$ is an $\alpha_{\beta'}$-neighborhood of zero in $G'$ for any $\alpha_{\beta}$-neighborhood $U$ of zero in $G$.

**Corollary 2.3.** [14] Let $R$ be an $\alpha_{(\beta,\beta)}$-topological ring, $a \in R$ and $\beta$ an $\alpha$-regular operation on $aO(R)$. Let also $B_0(R)$ be a basis of $\alpha_\beta$-neighborhoods of zero in $R$. Then, the element $a$ has a basis of $\alpha_{\beta}$-neighborhoods consisting of $\alpha_{\beta}$-closed neighborhoods.

**Proposition 2.4.** [14] Let $R$ be an $\alpha_{(\beta,\beta)}$-topological ring, $M$ an $\alpha_{(\beta,\gamma)}$-topological $R$-module, $Q$ a subset in $R$ and $B$ a subset in $M$. Then $\alpha_\gamma Cl(Q \cdot B) \supseteq \alpha_\beta Cl(Q) \cdot \alpha_\gamma Cl(B)$.

### 3. $\alpha_\beta$-Bounded Sets

**Definition 3.1.** Let $R$ be an $\alpha_{(\beta,\beta)}$-topological ring, $M$ be an $\alpha_{(\beta,\gamma)}$-topological $R$-module. A subset $S \subseteq M$ is called $\alpha_\beta$-bounded if for any $\alpha_\gamma$-neighborhood $U$ of zero in $M$, there exists an $\alpha_\beta$-neighborhood $V$ of zero in $R$ such that $V \cdot S \subseteq U$. An $\alpha_{(\beta,\gamma)}$-topological $R$-module $M$ is called $\alpha_\beta$-bounded if $M$ is an $\alpha_\beta$-bounded subset of the module $M$.

**Definition 3.2.** A subset $S$ of the $\alpha_{(\beta,\beta)}$-topological ring $R$ is called $\alpha_\beta$-bounded from left (right) if $S$ is an $\alpha_\beta$-bounded subset of the $\alpha_{(\beta,\beta)}$-topological left (right) $R$-module $R(\cdot)$, that is, for any $\alpha_\beta$-neighborhood $U$ of zero in $R$, there exists an $\alpha_\beta$-neighborhood $V$ of zero in $R$ such that $V \cdot S \subseteq U$ (respectively, $S \cdot V \subseteq U$). A subset $S$ of the $\alpha_{(\beta,\beta)}$-topological ring $R$, $\alpha_\beta$-bounded from left and from right, is called $\alpha_\beta$-bounded. An $\alpha_{(\beta,\beta)}$-topological ring $R$ is called $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) if $R$ is an $\alpha_\beta$-bounded from left (respectively, $\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) subset of the ring $R$.

**Corollary 3.1.** Any finite subset $Q$ of an $\alpha_{(\beta,\beta)}$-topological ring $R$ is $\alpha_\beta$-bounded.

**Proof.** Considering $R$ as left and right $\alpha_{(\beta,\beta)}$-topological $R$-modules, we get that $Q$ is $\alpha_\beta$-bounded from left and from right in $R$. □

**Proposition 3.1.** Let $S$ be a subring of an $\alpha_{(\beta,\beta)}$-topological ring $R$ and $A$ be an $\alpha_\beta$-bounded subset of an $\alpha_{(\beta,\gamma)}$-topological $R$-module $M$. Let $N$ be some $S$-submodule in $M$ containing $A$, then $A$ is an $\alpha_\beta$-bounded subset of $S$-module $N$.

**Proof.** Let $U$ be an $\alpha_\gamma$-neighborhood of zero in $N$. Then $U = V \cap N$ for a certain $\alpha_\gamma$-neighborhood $V$ of zero in $M$. Since $A$ is an $\alpha_\beta$-bounded subset in $M$, then there exists an $\alpha_\beta$-neighborhood $W$ of zero in $R$ such that $W \cdot A \subseteq V$. Then $W \cap S$ is an $\alpha_\beta$-neighborhood of zero in $S$ and $(W \cap S) \cdot A \subseteq W \cdot A \cap S \cdot A \subseteq V \cap N = U$. □
Corollary 3.2. Let $A$ be an $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) subset of an $\alpha_{(\beta,\gamma)}$-topological ring $R$ and $Q$ be a subring of $R$, which contains $A$. Then $A$ is an $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) subset of the ring $Q$.

Proof. The statement follows from Proposition 3.1, if we consider $R$ as a left or right $\alpha_{(\beta,\gamma)}$-topological $R$-module and $Q$ as a left or right its topological $Q$-submodule.

Remark 3.1. Let $R$ be an $\alpha_{(\beta,\gamma)}$-topological ring, $S$ be an $\alpha_\beta$-bounded subset of the $\alpha_{(\beta,\gamma)}$-topological $R$-module $M$. If $S_1 \subseteq S$, then $S_1$ is an $\alpha_\beta$-bounded subset of the $\alpha_{(\beta,\gamma)}$-topological $R$-module $M$.

Remark 3.2. Let $S$ be an $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) subset of an $\alpha_{(\beta,\gamma)}$-topological ring $R$ and $S_1 \subseteq S$. Then $S_1$ is an $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) subset of the $\alpha_{(\beta,\gamma)}$-topological ring $R$.

Proposition 3.2. Let $M$ be an $\alpha_{(\beta,\gamma)}$-topological module over an $\alpha_{(\beta,\gamma)}$-topological ring $R$, $B$ be $\alpha_\beta$-bounded subsets of $M$ and $\gamma$ be an $\alpha$-regular operation on $\alpha O(M)$. Then $\alpha_\gamma \text{Cl}(B)$ is $\alpha_\beta$-bounded.

Proof. Let $U$ be an $\alpha_\gamma$-neighborhood of zero in $M$, then by Proposition 2.2, there is an $\alpha_\gamma$-neighborhood $V$ of zero in $M$ such that $V + V \subseteq U$, and so by Corollary 2.2, we have $\alpha_\gamma \text{Cl}(V) \subseteq U$ and $\alpha_\gamma \text{Cl}(V)$ is an $\alpha_\gamma$-closed $\alpha_\gamma$-neighborhood of zero in $M$. Since $B$ is an $\alpha_\beta$-bounded subset of $R$-module $M$, then there exists an $\alpha_\beta$-neighborhood $W$ of zero in $R$ such that $W \cdot B \subseteq V$. Then,

$$W \cdot \alpha_\gamma \text{Cl}(B) \subseteq \alpha_\beta \text{Cl}(W) \cdot \alpha_\gamma \text{Cl}(B) \subseteq \alpha_\gamma \text{Cl}(W \cdot B) \subseteq \alpha_\gamma \text{Cl}(V) \subseteq U,$$

that is, $\alpha_\gamma \text{Cl}(B)$ is $\alpha_\beta$-bounded. \(\square\)

Corollary 3.3. The $\alpha_\beta$-closure of an $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded) subset of an $\alpha_{(\beta,\gamma)}$-topological ring is a subset $\alpha_\beta$-bounded from left ($\alpha_\beta$-bounded from right, $\alpha_\beta$-bounded), where $\beta$ is $\alpha$-regular operation on $\alpha O(R)$.

Proof. The proof results from Proposition 3.2. \(\square\)

Proposition 3.3. Let $Q$ be an $\alpha_\beta$-bounded from left subset of an $\alpha_{(\beta,\gamma)}$-topological ring $R$ and $S$ be an $\alpha_\beta$-bounded subset of an $\alpha_{(\beta,\gamma)}$-topological $R$-module $M$, then $Q \cdot S$ is an $\alpha_\beta$-bounded subset of the $R$-module $M$.

Proof. Let $U$ be an $\alpha_\gamma$-neighborhood of zero in $M$ and $V$ be an $\alpha_\beta$-neighborhood of zero in $R$ such that $V \cdot S \subseteq U$. We can choose an $\alpha_\beta$-neighborhood $W$ of zero in $R$ such that $W \cdot Q \subseteq V$. Then,

$$W \cdot (Q \cdot S) = (W \cdot Q) \cdot S \subseteq V \cdot S \subseteq U,$$
that is, \( Q \cdot S \) is an \( \alpha_\beta \)-bounded subset of the \( \alpha(\beta, \gamma) \)-topological \( R \)-module \( M \). \( \square \)

**Proposition 3.4.** Let \( M \) be an \( \alpha(\beta, \gamma) \)-topological module over an \( \alpha(\beta, \beta) \)-topological ring \( R \). \( B_1 \) and \( B_2 \) be \( \alpha_\beta \)-bounded subsets of \( M \). \( \beta \) be an \( \alpha \)-regular operation on \( \alpha O(R) \) and \( \gamma \) be an \( \alpha \)-regular operation on \( \alpha O(M) \). Then

1. \( B_1 + B_2 \) is \( \alpha_\beta \)-bounded.
2. \( B_1 \cup B_2 \) is \( \alpha_\beta \)-bounded.

**Proof.** Let \( W \) be an \( \alpha_\gamma \)-neighborhood of zero such that \( W + W \subseteq U \), where \( U \) is \( \alpha_\gamma \)-neighborhood of zero in \( M \). Let \( V_1 \) and \( V_2 \) be \( \alpha_\beta \)-neighborhoods of zero in \( R \) such that \( V_1 \cdot B_1 \subseteq W \) and \( V_2 \cdot B_2 \subseteq W \). Then

1. \( (V_1 \cap V_2) \cdot (B_1 \cup B_2) \subseteq V_1 \cdot B_1 \cup V_2 \cdot B_2 \subseteq W \subseteq U \).
2. \( (V_1 \cap V_2) \cdot (B_1 + B_2) \subseteq V_1 \cdot B_1 + V_2 \cdot B_2 \subseteq W + W \subseteq U \).

Consequently, the union or the sum of finitely many \( \alpha_\beta \)-bounded subsets of the \( \alpha(\beta, \gamma) \)-topological module is \( \alpha_\beta \)-bounded. \( \square \)

**Remark 3.3.** If \( B \) and \( C \) are left (right) \( \alpha_\beta \)-bounded subsets of an \( \alpha(\beta, \beta) \)-topological ring \( R \) and \( \beta \) be an \( \alpha \)-regular operation on \( \alpha O(R) \), then \( B \cup C \) and \( B + C \) are left (right) \( \alpha_\beta \)-bounded.

**Corollary 3.4.** Let each of the subsets \( Q_i \) for \( i = 1, 2, \ldots, n \) of an \( \alpha(\beta, \beta) \)-topological ring \( R \) be \( \alpha_\beta \)-bounded from left (\( \alpha_\beta \)-bounded from right, \( \alpha_\beta \)-bounded), then the subset \( Q_1 \cdot Q_2 \cdot \ldots \cdot Q_n \) is \( \alpha_\beta \)-bounded from left (\( \alpha_\beta \)-bounded from right, \( \alpha_\beta \)-bounded).

**Proof.** The proof is clear. \( \square \)

**Proposition 3.5.** Let \( R \) be an \( \alpha(\beta, \beta) \)-topological ring, \( M \) be \( \alpha(\beta, \gamma) \)-topological \( R \)-module and \( M' \) be \( \alpha(\beta, \gamma) \)-topological \( R \)-module. Let \( f \) be an \( \alpha(\gamma, \gamma) \)-continuous homomorphism from the module \( M \) to the module \( M' \) and a subset \( N \) is \( \alpha_\beta \)-bounded in \( M \). Then the subset \( f(N) \) is \( \alpha_\beta \)-bounded in \( M' \).

**Proof.** Let \( U' \) be an \( \alpha_\gamma \)-neighborhood of zero in the \( R \)-module \( M' \). Due to Proposition 2.3, \( f^{-1}(U') \) is an \( \alpha_\gamma \)-neighborhood of zero in the \( R \)-module \( M \). Since \( N \) is an \( \alpha_\beta \)-bounded subset, then there exists an \( \alpha_\beta \)-neighborhood \( V \) of zero in the ring \( R \) such that \( V \cdot N \subseteq f^{-1}(U') \). Then,

\[
V \cdot f(N) = f(V \cdot N) \subseteq f(f^{-1}(U')) \subseteq U',
\]

that is, \( f(N) \) is an \( \alpha_\beta \)-bounded subset in \( M' \). \( \square \)
Proposition 3.6. Let $R$ be an $\alpha(\beta,\beta)$-topological ring and $R'$ be an $\alpha(\beta',\beta')$-topological ring, $f : R \to R'$ be $\alpha(\beta,\beta')$-continuous and $\alpha(\beta',\beta')$-open homomorphism from the ring $R$ to the ring $R'$. Let a subset $S$ be $\alpha\beta$-bounded from left ($\alpha\beta$-bounded from right, $\alpha\beta$-bounded) in the ring $R$, then the subset $f(S)$ is $\alpha\beta$-bounded from left ($\alpha\beta$-bounded from right, $\alpha\beta$-bounded) in the ring $R'$.

Proof. Let $U'$ be an $\alpha\beta'$-neighborhood of zero in the ring $R'$. Then, due to Proposition 2.3, $f^{-1}(U')$ is an $\alpha\beta$-neighborhood of zero in the ring $R$. Since the subset $S$ of the ring $R$ is $\alpha\beta$-bounded from left, then there exists an $\alpha\beta$-neighborhood $V$ of zero in $R$ such that $V \cdot S \subseteq f^{-1}(U')$. Due to Proposition 2.3, $f(V)$ is an $\alpha\beta'$-neighborhood of zero in the ring $R$. Then,

$$f(V) \cdot f(S) = f(V \cdot S) \subseteq f(f^{-1}(U')) \subseteq U', $$

that is, the subset $f(S)$ is $\alpha\beta$-bounded from left in $R'$.

When the subset $S$ of the ring $R$ is $\alpha\beta$-bounded from right or $\alpha\beta$-bounded, the proof is analogous. □

Definition 3.3. An element $a$ of an $\alpha(\beta,\beta)$-topological ring $R$ is called a left (right) $\alpha\beta$-topological divisor of zero if there exists a subset $S \subseteq R$ such that:

1. $0 \not\in \alpha\beta Cl(S)$.
2. $0 \in \alpha\beta Cl(a \cdot S)$ (respectively, $0 \in \alpha\beta Cl(S \cdot a)$).

An element $a$ is called an $\alpha\beta$-topological divisor of zero if it is a left and right $\alpha\beta$-topological divisor of zero, that is, there exist subsets $S_1 \subseteq R$ and $S_2 \subseteq R$ such that $0 \not\in \alpha\beta Cl(S_1)$ and $0 \not\in \alpha\beta Cl(S_2)$, but as well $0 \in \alpha\beta Cl(a \cdot S_1)$ and $0 \in \alpha\beta Cl(S_2 \cdot a)$.

Remark 3.4. In an $\alpha\beta T_2$ $\alpha(\beta,\beta)$-topological ring $R$ any left (right) divisor of zero is a left (right) $\alpha\beta$-topological divisor of zero.

Indeed, if $a$ is a left divisor of zero in $R$ and $0 \neq b$ is such that $a \cdot b = 0$, then, the subset $\{b\}$ is $\alpha\beta$-closed in $R$ and $0 \not\in \{b\} = \alpha\beta Cl(\{b\})$. It is evident that $0 \in \alpha\beta Cl(a \cdot \{b\}) = \{0\}$, that is, $a$ is a left $\alpha\beta$-topological divisor of zero in $R$.

The following example shows that the condition that $R$ is $\alpha\beta T_2$ is necessary for the above remark.

Example 3.1. Consider the ring $\mathbb{Z}_4$. Let $\tau$ be the discrete topology on $\mathbb{Z}_4$. For each $A \in \alpha O(\mathbb{Z}_4, \tau)$, we define $\beta$ on $\alpha O(\mathbb{Z}_4, \tau)$ by $A^\beta = \mathbb{Z}_4$. Since $\mathbb{Z}_4$ is not $\alpha\beta T_2$, so the element $2$ is divisor of zero in $\mathbb{Z}_4$, but it is not $\alpha\beta$-topological divisor of zero because $0 \not\in \alpha\beta Cl(S) = \mathbb{Z}_4$ for any subset $S$ of $\mathbb{Z}_4$.

Proposition 3.7. Let $a$ be a left (right) $\alpha\beta$-topological divisor of zero in an $\alpha(\beta,\beta)$-topological ring $R$. Then for any $b \in R$ the element $b \cdot a$ is a left $\alpha\beta$-topological divisor of zero in $R$ (respectively, the element $a \cdot b$ is a right $\alpha\beta$-topological divisor of zero in $R$).
Proof. Let $S \subseteq R$, $0 \notin \alpha_\beta Cl(S)$ and $0 \in \alpha_\beta Cl(a \cdot S)$. Then $0 \in b \cdot \alpha_\beta Cl(a \cdot S) \subseteq \alpha_\beta Cl(b \cdot a \cdot S)$, that is, $b \cdot a$ is a left $\alpha_\beta$-topological divisor of zero in $R$. Analogously is considered the case when $a$ is a right $\alpha_\beta$-topological divisor of zero in $R$. \hfill \Box

**Proposition 3.8.** Let $R$ be an $\alpha(\beta, \beta)$-topological ring and $a, b \in R$. If $a \cdot b$ is a left (right) $\alpha_\beta$-topological divisor of zero in $R$, then either $a$ or $b$ is a left (right) $\alpha_\beta$-topological divisor of zero.

Proof. Let $a \cdot b$ be a left $\alpha_\beta$-topological divisor of zero in $R$ and $S \subseteq R$ be such that $0 \notin \alpha_\beta Cl(S)$, and $0 \in \alpha_\beta Cl((a \cdot b) \cdot S)$. If $0 \in \alpha_\beta Cl(b \cdot S)$, then $b$ is a left $\alpha_\beta$-topological divisor of zero. If $0 \notin \alpha_\beta Cl(b \cdot S)$, then, taking into account that $0 \in \alpha_\beta Cl((a \cdot b) \cdot S) = \alpha_\beta Cl(a \cdot (b \cdot S))$, we get that $a$ is a left $\alpha_\beta$-topological divisor of zero. Analogously is considered the case when $a \cdot b$ is a right $\alpha_\beta$-topological divisor of zero in $R$. \hfill \Box

4. $\alpha_\beta$-Topologically Nilpotent Elements

**Definition 4.1.** A subset $S$ of an $\alpha(\beta, \beta)$-topological ring $R$ is called $\alpha_\beta$-topologically nilpotent if for any $\alpha_\beta$-neighborhood $U$ of zero in $R$ there exists a natural number $n_0$ such that $S^{(n)} \subseteq U$ for all $n \geq n_0$. An element $a \in R$ is called $\alpha_\beta$-topologically nilpotent if the one-element set $\{a\}$ of the ring $R$ is $\alpha_\beta$-topologically nilpotent, that is, if for any $\alpha_\beta$-neighborhood $U$ of zero in $R$ there exists a natural number $n_0$ such that $a^n \in U$ for all $n \geq n_0$.

**Remark 4.1.** Since for any subsets $S_1$ and $S$ of a ring $R$ from the inclusion $S_1 \subseteq S$ follows that $S_1^{(n)} \subseteq S^{(n)}$ for any $n \in \mathbb{N}$, then any subset of an $\alpha_\beta$-topologically nilpotent subset of an $\alpha(\beta, \beta)$-topological ring is an $\alpha_\beta$-topologically nilpotent subset, too. In particular, any element of an $\alpha_\beta$-topologically nilpotent subset is $\alpha_\beta$-topologically nilpotent.

**Proposition 4.1.** Let $K$ be a skew field endowed with $\alpha_\beta T_2$ ring $\alpha(\beta, \beta)$-topology, $\beta$ be an $\alpha$-regular operation on $\alpha O(K)$ and $a$ be a non-zero $\alpha_\beta$-topologically nilpotent element in $K$. Then the element $a^{-1}$ is not $\alpha_\beta$-topologically nilpotent.

Proof. Assume the contrary, that is, that $a^{-1}$ is an $\alpha_\beta$-topologically nilpotent element. Since the skew field $K$ is $\alpha_\beta T_2$, there exists an $\alpha_\beta$-neighborhood $U$ of zero in $K$ such that $1 \notin U$. We can choose an $\alpha_\beta$-neighborhood $V$ of zero in $K$ such that $V \cdot V \subseteq U$. It is clear that $1 \notin V$. Since the elements $a$ and $b^{-1}$ are $\alpha_\beta$-topologically nilpotent, then there exist natural numbers $n_1$ and $n_2$ such that $a^n \in V$ for $n \geq n_1$ and $a^{-n} \in V$ for $n \geq n_2$. Putting $n_0 = \max\{n_1, n_2\}$, we get that $1 = a^{n_0} \cdot a^{-n_0} \in V \cdot V \subseteq U$, which contradicts the choice of the $\alpha_\beta$-neighborhood $U$. \hfill \Box

**Proposition 4.2.** Let $R$ be an $\alpha(\beta, \beta)$-topological ring with the unitary element and $a$ be an invertible element in $R$. Then, the element $x \in R$ is $\alpha_\beta$-topologically nilpotent if and only if the element $a \cdot x \cdot a^{-1}$ is $\alpha_\beta$-topologically nilpotent.
Proof. Let \( x \) be an \( \alpha\beta \)-topologically nilpotent element and \( U \) be an \( \alpha\beta \)-neighborhood of zero in \( R \). We can choose an \( \alpha\beta \)-neighborhood \( V \) of zero in \( R \) such that \( a \cdot V \cdot a^{-1} \subseteq U \). Let \( n_0 \) be a natural number such that \( x^n \in V \) for \( n \geq n_0 \). Then \( (a \cdot x \cdot a^{-1})^n = a \cdot x^n \cdot a^{-1} \in a \cdot V \cdot a^{-1} \subseteq U \) for \( n \geq n_0 \), that is, \( a \cdot x \cdot a^{-1} \) is an \( \alpha\beta \)-topologically nilpotent element.

Conversely, let \( a \cdot x \cdot a^{-1} \) be an \( \alpha\beta \)-topologically nilpotent element. Then, as it was shown above, the element \( x = a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot a = a^{-1} \cdot (a \cdot x \cdot a^{-1}) \cdot (a^{-1})^{-1} \) is \( \alpha\beta \)-topologically nilpotent. \( \square \)

**Proposition 4.3.** Let \( S \) be an \( \alpha\beta \)-topologically nilpotent subset of an \( \alpha(\beta,\beta) \)-topological ring \( R \) and \( \beta \) an \( \alpha \)-regular operation on \( \alpha O(R) \). Then, the subset \( \alpha\beta Cl(S) \) is an \( \alpha\beta \)-topologically nilpotent subset.

**Proof.** Let \( S \) be an \( \alpha\beta \)-topologically nilpotent subset and \( U \) be an \( \alpha\beta \)-neighborhood of zero in \( R \). Due to Corollary 2.3, there exists an \( \alpha\beta \)-closed \( \alpha\beta \)-neighborhood \( V \) of zero in \( R \) such that \( V \subseteq U \). Let \( n_0 \) be a natural number such that \( S^{(n)} \subseteq V \) for all \( n \geq n_0 \). Then
\[
(\alpha\beta Cl(S))^{(n)} \subseteq \alpha\beta Cl(S^{(n)}) \subseteq \alpha\beta Cl(V) = V \subseteq U \quad \text{(see Proposition 2.4),}
\]
hence, \( \alpha\beta Cl(S) \) is an \( \alpha\beta \)-topologically nilpotent subset. \( \square \)

**Proposition 4.4.** Let \( f \) be an \( \alpha(\beta,\beta) \)-continuous homomorphism of an \( \alpha(\beta,\beta) \)-topological ring \( R \) onto an \( \alpha(\beta',\beta') \)-topological ring \( R' \) and let \( S \) be an \( \alpha\beta \)-topologically nilpotent subset of \( R \). Then \( f(S) \) is an \( \alpha\beta \)-topologically nilpotent subset of the ring \( R' \).

**Proof.** Let \( U' \) be an \( \alpha\beta \)-neighborhood of zero in \( R' \), then \( f^{-1}(U') \) is an \( \alpha\beta \)-neighborhood of zero in \( R \). There exists a natural number \( n_0 \) such that \( S^{(n)} \subseteq f^{-1}(U') \) for all \( n \geq n_0 \). Then \( (f(S))^{(n)} = f(S^{(n)}) \subseteq f(f^{-1}(U')) = U' \) for all \( n \geq n_0 \), that is, the subset \( f(S) \) is \( \alpha\beta \)-topologically nilpotent in \( R' \). \( \square \)

**Proposition 4.5.** Let \( S \) be an \( \alpha\beta \)-bounded from left (right) subset of an \( \alpha(\beta,\beta) \)-topological ring \( R \), then the following statements are equivalent:

1. \( S \) is an \( \alpha\beta \)-topologically nilpotent subset.
2. \( S^{(k)} \) is an \( \alpha\beta \)-topologically nilpotent subset for any natural number \( k \).
3. There exists a natural number \( k_0 \) such that \( S^{(k_0)} \) is an \( \alpha\beta \)-topologically nilpotent subset.

**Proof.** It is evident that (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

Let us show that (3) \( \Rightarrow \) (1). Let \( S^{(k_0)} \) be an \( \alpha\beta \)-topologically nilpotent subset in \( R \) and \( U \) be an \( \alpha\beta \)-neighborhood of zero in \( R \). Due to Corollary 3.4, the subsets
$S^{(2)}, S^{(3)}, ..., S^{(k_0-1)}$ are $\alpha_\beta$-bounded from left. Therefore, we can choose an $\alpha_\beta$-neighborhood $V$ of zero in $R$ such that $V \cdot S^{(i)} \subseteq U$ for $i = 1, 2, ..., k_0 - 1$. Since the subset $S^{(k_0)}$ is $\alpha_\beta$-topologically nilpotent, there exists a natural number $n_0$ such that $(S^{(k_0)})^{(n)} \subseteq V$ for all $n \geq n_0$. Let $m \geq n_0 \cdot k_0$. Then $m = k_0 \cdot q + r$, where $q \geq n_0$ and $0 \leq r < k_0$. Thus,

$$S^{(m)} = S^{(k_0 \cdot q + r)} = (S^{(k_0)})^{(q)} \cdot S^{(r)} \subseteq V \cdot S^{(r)} \subseteq U,$$

that is, $S$ is an $\alpha_\beta$-topologically nilpotent subset.

**Corollary 4.1.** Let $a$ be an element of an $\alpha_{(\beta,\beta)}$-topological ring $R$. Then the following conditions are equivalent:

1. $a$ is an $\alpha_\beta$-topologically nilpotent element.
2. $a^k$ is an $\alpha_\beta$-topologically nilpotent element for any natural number $k$.
3. $a^{k_0}$ is an $\alpha_\beta$-topologically nilpotent element for a certain natural number $k_0$.

**Proof.** The statement results from Proposition 4.5 and from the $\alpha_\beta$-boundedness of the one-element subset $\{a\}$ (see Corollary 3.1).

**Proposition 4.6.** Let $T$ be the subset of all $\alpha_\beta$-topologically nilpotent elements of an $\alpha_{(\beta,\beta)}$-topological ring $R$ and $\beta$ an $\alpha$-regular operation on $\alpha O(R)$. Then, the following statements are equivalent:

1. $T$ is an $\alpha_\beta$-open subset.
2. There exists an $\alpha_\beta$-open $\alpha_\beta$-neighborhood $U$ of zero in $R$ consisting of $\alpha_\beta$-topologically nilpotent elements.

**Proof.** It is evident that (1) $\Rightarrow$ (2).

Let us show that (2) $\Rightarrow$ (1). Let $t \in T$ and $n_0$ be a natural number such that $t^{n_0} \in U$, where $U$ is $\alpha_\beta$-open $\alpha_\beta$-neighborhood of zero in $R$. We can choose an $\alpha_\beta$-neighborhood $V$ of the element $t$ such that

$$V^{(n_0)} = V \cdot V \cdot ... \cdot V \subseteq U,$$

Then $v^{n_0}$ is an $\alpha_\beta$-topologically nilpotent element, for any $v \in V$. Due to Corollary 4.1, any element from $V$ is $\alpha_\beta$-topologically nilpotent, that is, $V \subseteq T$, which means that the subset $T$ is $\alpha_\beta$-open.
REFERENCES

12. A. B. KHALAF and H. Z. IBRAHIM, $\alpha$-connectedness and some properties of $\alpha_{(\gamma,\beta)}$-continuous functions, Accepted in The First International Conference of Natural Science (ICNS) from 11th – 12th July (2016), Charmo University.

Hariwan Z. Ibrahim
Department of Mathematics
Faculty of Education
University of Zakho
Kurdistan-Region, Iraq
hariwan_math@yahoo.com

Alias B. Khalaf
Department of Mathematics
Faculty of Science
University of Duhok
Kurdistan-Region, Iraq
aliasbkhalaf@gmail.com