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SUFFICIENT CONDITIONS FOR NONUNIFORM WAVELET FRAMES ON LOCAL FIELDS *

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Abstract. The main objective of this paper is to establish a set of sufficient conditions for nonuniform wavelet frames on local fields of positive characteristic. The conditions proposed are stated in terms of the Fourier transforms of the wavelet system's generating functions.

Keywords: Fourier transform, wavelet frame, p-adic field.

1. Introduction

A field K equipped with a topology is called a local field if both the additive K^+ and multiplicative groups K^* of K are locally compact abelian groups. By a local field, we mean a locally compact, non-discrete, complete and totally disconnected field. The p-adic fields and p-series fields (Vilenkin p-groups), as well as their finite algebraic extensions, are the only examples of such fields. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). If K is of characteristic zero, then K is either p-adic field $\mathbb{Q}_p$ for some prime p, or a finite algebraic extension of such a field. If the characteristic of K is positive, then K is a field of formal power series over a finite field $GF(p^c)$. If $c = 1$, then it is a p-series field. If $c > 1$, then K is an algebraic extension of degree c of a p-series field. For more details about local fields, we refer to the monograph [11].

Despite the fact that the structures and metrics of p-adic fields and local fields of positive characteristic are comparable, their wavelet and multiresolution analysis (MRA) theory are quite different. The notion of MRA on local fields of positive characteristic was introduced by Jiang et al.[6]. In fact, they brought up

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a technique for constructing orthogonal wavelets on local fields and established a necessary and sufficient condition for the solution of refinement equation to generate an MRA for $L^2(K)$. Recently, Shah and Abdullah [9] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the core space V_0 is no longer a group, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$, where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathfrak{D}$ in the locally compact abelian group K^+. More precisely, this set is of the form $\Lambda = \{0, u(r)/N\} + \mathcal{Z}$, where $N \geqslant 1$ is an integer and r is an odd integer such that r and N are relatively prime. They call this a *nonuniform multiresolution analysis* on local fields of positive characteristic. As a consequence of this generalization, they obtain a necessary and sufficient condition for the existence of associated wavelets and extension of Cohen's theorem.

The notion of frames was first introduced by Duffin and Shaeffer [3] in connection with some deep problems in nonharmonic Fourier series, and more particularly with the question of determining when a family of exponentials $\left\{e^{i\alpha_n t} : n \in \mathbb{Z}\right\}$ is complete for $L^2[a, b]$. Frames did not generate much interest outside non- harmonic Fourier series until the seminal work by Daubechies et al.[2]. But in recent times, there has been a great interest in the applications of redundant wavelet systems (wavelet frames). In comparison with wavelet bases, wavelet frames provide more informative mathematical representations, and efficient and fast implementations with the same fundamental structure as orthonormal wavelet bases (See [1]). In addition to this, researchers have come up with the new mathematical transforms to analyze chirp signals namely the fractional wavelet transforms, the short-time fractional Fourier transform, the ridgelet transform and so on. For more about fractional wavelet transforms and their applications, we refer to [4, 5, 7]. The concept of wavelet frames on local fields was introduced by Shah and Abdullah in [10], where they established some characterizations of uniform wavelet frames on local fields via Fourier transforms and in particular, demonstrate how to build the Parseval wavelet frames for $L^2(K)$.

Concerning the construction of nonuniform wavelet frames on local fields, Shah [8] has given a simple scheme for constructing nonuniform wavelet frames in $L^2(K)$ using the basic concepts of operator theory and Fourier transforms. Motivating by the fundament works in [8, 9], we will drive some new sufficient conditions for the nonuniform wavelet systems to be frames on local fields by means of some basic equations in the frequency domain.

This article is organized as follows. In Section 2, we provide a brief introduction to Fourier analysis on local fields. Section 3 is devoted to the investigation of some sufficient conditions for nonuniform wavelet systems to be frames in $L^2(K)$.

2. Preliminaries on Local Fields

Let K be a fixed local field with the ring of integers $\mathfrak{D} = \{t \in K : |t| \leqslant 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dt for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimeadean norm $|\cdot| : K \to \mathbb{R}^+$ satisfying

(a) $|t| = 0$ if and only if $t = 0$;

(b) $|s\,t| = |s||t|$ for all $s, t \in K$;

(c) $|s + t| \leqslant \max\{|s|, |t|\}$ for all $s, t \in K$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{t \in K : |t| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in K. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of K such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{t \in K : |t| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in K^* and if $t \neq 0$, then can write $t = \mathfrak{p}^n s, s \in \mathfrak{D}^*$. Moreover, if $\mathcal{Q} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $t \in K$ can be expressed uniquely as $t = \sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{Q}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{t \in K : |t| < q^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [11]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $u \in \mathfrak{B}^k$, then $\chi_u(t) = \chi(u, t), t \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [11], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

Definition 2.1. The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\omega)$ and defined by

$$\hat{f}(\omega) = \int_K f(t)\overline{\chi_\omega(t)}\, dt. \tag{2.1}$$

Note that

$$\hat{f}(\omega) = \int_K f(t)\, \overline{\chi_\omega(t)}\, dt = \int_K f(t)\chi(-\omega t)\, dt.$$

The properties of Fourier transform on local field K are much similar to those of on the real line. In fact, the Fourier transform have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leqslant \|f\|_1$.
- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(K)$ is defined by

$$\hat{f}(\omega) = \lim_{k\to\infty} \hat{f}_k(\omega) = \lim_{k\to\infty} \int_{|t|\leqslant q^k} f(t)\overline{\chi_\omega(t)}\, dt, \tag{2.2}$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(t)\overline{\chi_{u(n)}(t)}\, dt. \tag{2.3}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(t)$ is called the Fourier series of f. From the standard L^2-theory for compact abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(t)\big|^2 dt = \sum_{n\in\mathbb{N}_0} \Big|\hat{f}\big(u(n)\big)\Big|^2. \tag{2.4}$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^\infty$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \quad n = a_0 + a_1 p + \dots + a_{c-1}p^{c-1}, \quad 0 \le a_k < p, \quad \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = \left(a_0 + a_1\zeta_1 + \dots + a_{c-1}\zeta_{c-1}\right)\mathfrak{p}^{-1}. \tag{2.5}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \dots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \le b_k < q, k = 0, 1, 2, \dots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \dots + u(b_s)\mathfrak{p}^{-s}. \tag{2.6}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leqslant s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) :$

$k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geqslant 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} e^{2\pi i/p}, & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.7}$$

3. Sufficient Conditions for Nonuniform Wavelet Frames

Let $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$, where $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of $\mathfrak{D}$ in K^+. For an integer $N \geqslant 1$ and an odd integer r with $1 \leq r \leq qN - 1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathcal{Z} = \left\{\frac{u(r)k}{N} + u(n) : n \in \mathbb{N}_0, k = 0, 1\right\}. \tag{3.1}$$

It is easy to verify that Λ is not a group on local field K, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$. This set can be treated as the translation set. For $j \in \mathbb{Z}$ and $\lambda \in \Lambda$, we define the dilation operator D_j and the translation operator T_λ as follows:

$$D_j f(t) = (qN)^{j/2} f\big((\mathfrak{p}^{-1}N)^j t\big) \quad \text{and} \quad T_\lambda f(t) = f(t - \lambda), \quad f \in L^2(K).$$

We are now in a position to define the concept of nonuniform wavelets on local fields. For a given $\psi \in L^2(\mathbb{R})$, define the nonuniform wavelet system

$$\mathcal{W}(\psi, j, \lambda) = \Big\{\psi_{j,\lambda}(t) =: (qN)^{j/2} \psi\left(\left(\mathfrak{p}^{-1}N\right)^j t - \lambda\right);\ j \in \mathbb{Z},\ \lambda \in \Lambda\Big\}. \tag{3.2}$$

It is worth noticing that, when $N = 1$, one recovers from the above nonuniform system the usual wavelet system on local fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathcal{Z} \subset \Lambda$.

In frequency domain, system (3.2) can be written as

$$\hat{\psi}_{j,\lambda}(\omega) = (qN)^{-j/2} \hat{\psi}\big((\mathfrak{p}^{-1}N)^{-j}\omega\big)\overline{\chi_\lambda\big((\mathfrak{p}^{-1}N)^{-j}\omega\big)}. \tag{3.3}$$

The nonuniform wavelet system (3.2) is called a *nonuniform wavelet frame*, if there exist positive numbers $A, B > 0$, so that

$$A\|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \left|\langle f, \psi_{j,\lambda}\rangle\right|^2 \leqslant B\|f\|_2^2, \quad \text{for all } f \in L^2(K). \tag{3.4}$$

The largest A and the smallest B that can be used in the above inequalities are called the lower and upper frame bounds. The wavelet frame is called tight if the lower and upper frame bounds are same.

The following lemma will be used later in the proofs of the main results. For a proof, we refer to [8].

Lemma 3.1. *Suppose $f \in L^2(K)$ such that $\hat{f}$ has compact support. If $\psi \in L^2(K)$ and*

$$\sup_{\omega\in K}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2<\infty.$$

Then

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2=\int_K\left|\hat{f}(\omega)\right|^2\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2d\omega+R_n(f), \tag{3.5}$$

where $R_n(f)=R_0+R_1+\cdots+R_{qN-1}$ and for $0\leqslant n\leqslant qN-1$, R_n is given by

$$R_n=\frac{1}{qN}\sum_{j\in\mathbb{Z}}\sum_{m\neq n}\int_K\left\{\overline{\hat{f}\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(n)\Big)\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(n)\Big)}\right.$$

$$\left.\times\,\hat{f}\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(m)\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(m)\Big)}\left[1+\chi\left(\frac{u(r)u(m-n)}{N}\right)\right]\right\}d\omega. \tag{3.6}$$

We now establish our first sufficient condition for the system (3.2) to be a wavelet frame for $L^2(K)$. For this, we set

$$\mathcal{Q}=\Big\{(\mathfrak{p}^{-1}N)k+n:k\in\mathbb{N}_0,1\leqslant\ell\leqslant qN-1\Big\}$$

$$\Delta_\psi(m)=\sup_{\omega\in\mathfrak{C}^{-1}\setminus\mathfrak{D}}\sum_{j\in\mathbb{Z}}\left|\beta_\psi\Big(m,\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|,$$

where $\mathfrak{C}^{-1}=\{t\in K:|t|\leqslant(qN)\}$ and

$$\beta_\psi(m,\omega)=\sum_{k\in\mathbb{N}_0}\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{k}\omega\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{k}\big(\omega+u(m)\big)\Big)}.$$

Theorem 3.2. *Suppose $\psi \in L^2(K)$ such that*

$$A_1=\inf_{\omega\in\mathfrak{C}^{-1}\setminus\mathfrak{D}}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2-\sum_{r\neq s\in\mathcal{Q}}\Big[\Delta_\psi\big(u(r-s)\big)\cdot\Delta_\psi\big(-u(r-s)\big)\Big]^{1/2}>0,$$

$$B_1=\sup_{\omega\in\mathfrak{C}^{-1}\setminus\mathfrak{D}}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2-\sum_{r\neq s\in\mathcal{Q}}\Big[\Delta_\psi\big(u(r-s)\big)\cdot\Delta_\psi\big(-u(r-s)\big)\Big]^{1/2}<+\infty.$$

Then $\big\{\psi_{j,\lambda}:j\in\mathbb{Z},\,\lambda\in\Lambda\big\}$ is a wavelet frame for $L^2(K)$ with bounds A_1 and B_1.

Proof. By Lemma 3.1, we have

$$
\begin{aligned}
\big|R_n(f)\big| &\leqslant \frac{1}{qN}\sum_{m=0}^{qN-1}\sum_{j\in\mathbb{Z}}\int_K\left|\overline{\hat f(\omega)}\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|\left\{\sum_{n\neq m}\left|\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(n-m)\Big)\right.\right.\\
&\times\left.\left.\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(n-m)\Big)}\right|\right\}d\omega\\
&= \frac{1}{qN}\sum_{r=0}^{qN-1}\sum_{j\in\mathbb{Z}}\int_K\left|\overline{\hat f(\omega)}\right|\left\{\sum_{k\in\mathbb{N}_0}\sum_{r\neq s\in\mathcal{Q}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right.\right.\\
&\times\left.\left.\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j+k}u(r-s)\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+\big(\mathfrak{p}^{-1}N\big)^{k}u(r-s)\Big)}\right|\right\}d\omega\\
&= \frac{1}{qN}\sum_{r=0}^{qN-1}\int_K\left|\overline{\hat f(\omega)}\right|\left\{\sum_{k\in\mathbb{N}_0}\sum_{r\neq s\in\mathcal{Q}}\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j+k}\omega\Big)\right.\right.\\
&\times\left.\left.\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(r-s)\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j+k}\omega+\big(\mathfrak{p}^{-1}N\big)^{k}u(r-s)\Big)}\right|\right\}d\omega\\
&= \frac{1}{qN}\sum_{r=0}^{qN-1}\int_K\left|\overline{\hat f(\omega)}\right|\left\{\sum_{j\in\mathbb{Z}}\sum_{r\neq s\in\mathcal{Q}}\left|\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(r-s)\Big)\right.\right.\\
&\times\left.\left.\sum_{k\in\mathbb{N}_0}\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j+k}\omega\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{k}\big((\mathfrak{p}^{-1}N)^{-j}\omega+u(r-s)\big)\Big)}\right|\right\}d\omega\\
&= \frac{1}{qN}\sum_{r=0}^{qN-1}\int_K\left|\overline{\hat f(\omega)}\right|\left\{\sum_{j\in\mathbb{Z}}\sum_{r\neq s\in\mathcal{Q}}\left|\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(r-s)\Big)\right.\right.\\
&\times\left.\left.\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|\right\}d\omega\\
&= \frac{1}{qN}\sum_{r=0}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{r\neq s\in\mathcal{Q}}\int_K\left\{\left|\hat f(\omega)\right|\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^{1/2}\right\}\\
&\times\left\{\left|\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(r-s)\Big)\right|\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^{1/2}\right\}d\omega\\
&\leqslant \frac{1}{qN}\sum_{r=0}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{r\neq s\in\mathcal{Q}}\left\{\int_K\left|\hat f(\omega)\right|^2\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|d\omega\right\}^{1/2}\\
&\times\left\{\int_K\left|\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^{j}u(r-s)\Big)\right|^2\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|d\omega\right\}^{1/2}\\
&\leqslant \frac{1}{qN}\sum_{r=0}^{qN-1}\sum_{r\neq s\in\mathcal{Q}}\left\{\sum_{j\in\mathbb{Z}}\int_K\left|\hat f(\omega)\right|^2\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|d\omega\right\}^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left\{\sum_{j\in\mathbb{Z}}\int_K\left|\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j u(r-s)\Big)\right|^2\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|d\omega\right\}^{1/2}\\
&=\ \frac{1}{qN}\sum_{r=0}^{qN-1}\sum_{r\neq s\in\mathcal{Q}}\left\{\sum_{j\in\mathbb{Z}}\int_K\left|\hat f(\omega)\right|^2\left|\beta_\psi\Big(u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|d\omega\right\}^{1/2}\\
&\qquad\times\left\{\sum_{j\in\mathbb{Z}}\int_K\left|\hat f(\omega)\right|^2\left|\beta_\psi\Big(-u(r-s),\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|d\omega\right\}^{1/2}\\
&\leqslant\ \frac{1}{qN}\sum_{r=0}^{qN-1}\sum_{r\neq s\in\mathcal{Q}}\left\{\int_K\left|\hat f(\omega)\right|^2\Delta_\psi\big(u(r-s)\big)d\omega\right\}^{1/2}\\
&\qquad\times\left\{\int_K\left|\hat f(\omega)\right|^2\Delta_\psi\big(-u(r-s)\big)d\omega\right\}^{1/2}\\
&=\ \frac{1}{qN}\sum_{r=0}^{qN-1}\int_K\left|\hat f(\omega)\right|^2 d\omega\sum_{r\neq s\in\mathcal{Q}}\Big[\Delta_\psi\big(u(r-s)\big)\cdot\Delta_\psi\big(-u(r-s)\big)\Big]^{1/2}.
\end{aligned}
$$

Consequently, it follows that

$$
\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2\geqslant\int_K\left|\hat f(\omega)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2\right.
$$

$$
\left.-\sum_{r\neq s\in\mathcal{Q}}\Big[\Delta_\psi\big(u(r-s)\big)\cdot\Delta_\psi\big(-u(r-s)\big)\Big]^{1/2}\right\}d\omega \tag{3.7}
$$

and

$$
\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2\leqslant\int_K\left|\hat f(\omega)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2\right.
$$

$$
\left.+\sum_{r\neq s\in\mathcal{Q}}\big[\Delta_\psi\big(u(r-s)\big)\cdot\Delta_\psi\big(-u(r-s)\big)\big]^{1/2}\right\}d\omega. \tag{3.8}
$$

Taking infimum in (3.7) and supremum in (3.8), we obtain

$$
A_1\|f\|_2^2\leqslant\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2\leqslant B_1\|f\|_2^2.
$$

The proof of Theorem 3.2 is complete. □

Before we state our next sufficient condition, we introduce some notations. Similar to the a-adic number, we call an element $\alpha \in K$, a qN-adic number if it has the form $\alpha = (\mathfrak{p}^{-1}N)^j(\lambda - \sigma)$, $j \in \mathbb{Z}$, $\lambda \neq \sigma \in \Lambda$. With this concept, define the following sets:

$$\Gamma = \Big\{\alpha \in K : \text{there exists } (j,\lambda) \in \mathbb{Z}\times\Lambda \text{ such that } \alpha = (qN)^j(\lambda-\sigma); \lambda \neq \sigma\Big\}, \tag{3.9}$$

and for all $\alpha \in \Gamma$, define

$$I(\alpha) = \Big\{(j,\lambda) \in \mathbb{Z}\times\Lambda : \alpha = (qN)^j(\lambda-\sigma); \lambda \neq \sigma\Big\}, \tag{3.10}$$

$$\Delta_\alpha^+(\omega) = \textstyle\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega + (\lambda-\sigma)\Big)} \tag{3.11}$$

$$\Delta_\alpha^-(\omega) = \textstyle\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega - (\lambda-\sigma)\Big)}. \tag{3.12}$$

Theorem 3.3. *Suppose $\psi \in L^2(K)$ such that*

$$\begin{aligned} A_2 &= \inf_{\omega\in\mathfrak{C}^{-1}\setminus\mathfrak{D}} \Big\{\Delta_0^+(\omega) - \sum_{\alpha\in\Gamma\setminus\{0\}} \big|\Delta_\alpha^+(\omega)\big|\Big\} > 0,\\ B_2 &= \sup_{\omega\in\mathfrak{C}^{-1}\setminus\mathfrak{D}} \sum_{\alpha\in\Gamma\setminus\{0\}} \big|\Delta_\alpha^+(\omega)\big| < +\infty. \end{aligned}$$

Then $\big\{\psi_{j,\lambda} : j \in \mathbb{Z},\ \lambda \in \Lambda\big\}$ is a wavelet frame for $L^2(K)$ with bounds A_2 and B_2.

Proof. From the definition of $\Delta_\alpha^+(\omega)$, it follows that $\Delta_0^+(\omega) = \sum_{j\in\mathbb{Z}} \Big|\hat{\psi}\big((\mathfrak{p}^{-1}N)^{-j}\omega\big)\Big|^2$. From Lemma 3.1, we can re-estimate $R_n(f)$ as

$$\begin{aligned}
\left|R_n(f)\right| &= \left|\frac{1}{4N}\sum_{m=0}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{n\neq m}\int_K \left\{\overline{\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j u(m)\Big)}\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(m)\Big)\right.\right.\\
&\times\quad \left.\left.\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j u(n)\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(n)\Big)}\left(1+\chi\left(\frac{ru(n-m)}{N}\right)\right)\right\}d\omega\right|\\
&\leqslant \frac{1}{qN}\sum_{m=0}^{qN-1}\sum_{j\in\mathbb{Z}}\sum_{n\neq m}\int_K \left|\overline{\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j u(m)\Big)}\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j u(n)\Big)\right|\\
&\times\quad \left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(m)\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+u(n)\Big)}\right|d\omega\\
&= \sum_{\alpha\in\Gamma\setminus\{0\}}\ \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)}\int_K \left|\overline{\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j \lambda\Big)}\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j \sigma\Big)\right|\\
&\times\quad \left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+\lambda\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+\sigma\Big)}\right|d\omega\\
&\leqslant \sum_{\alpha\in\Gamma\setminus\{0\}}\ \sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)}\int_K \left|\overline{\hat f(\omega)}\hat f\Big(\omega+\big(\mathfrak{p}^{-1}N\big)^j (\lambda-\sigma)\Big)\right|\\
&\times\quad \left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+(\lambda-\sigma)\Big)}\right|d\omega\\
&= \sum_{\alpha\in\Gamma\setminus\{0\}}\int_K \left|\overline{\hat f(\omega)}\hat f\big(\omega+u(\alpha)\big)\right|\\
&\times\quad \left\{\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\overline{\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+(\lambda-\sigma)\Big)}\right|\right\}d\omega \ \big(\text{By (3.10)}\big)\\
&= \sum_{\alpha\in\Gamma\setminus\{0\}}\int_K \left|\overline{\hat f(\omega)}\hat f\big(\omega+u(\alpha)\big)\right|\left|\Delta_\alpha^+(\omega)\right|d\omega \qquad \big(\text{By (3.11)}\big)
\end{aligned}$$

$$\text{(3.13)}\qquad \leqslant \sum_{\alpha\in\Gamma\setminus\{0\}}\left\{\int_K\left|\hat f(\omega)\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2}\left\{\int_K\left|\hat f\big(\omega+u(\alpha)\big)\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2}$$

(3.14)

$$\leqslant\left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\int_K\left|\hat f(\omega)\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2}\left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\int_K\left|\hat f\big(\omega+u(\alpha)\big)\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2},$$

Since

$$u(\alpha)=u\big((qN)^j(\lambda-\sigma)\big)=(\mathfrak{p}^{-1}N)^j(\lambda-\sigma),\ \text{ for } (j,\lambda)\neq(j,\sigma)\in I(\alpha),$$

we deduce that

$$
\begin{aligned}
&\Delta^{+}_{\alpha}(\omega)\\
=&\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega+(\lambda-\sigma)\Big)}\\
=&\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}(\omega-u(\alpha))\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}(\omega-u(\alpha))+(\lambda-\sigma)\Big)}\\
=&\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega-\big(\mathfrak{p}^{-1}N\big)^{-j}u(\alpha)\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega-\big(\mathfrak{p}^{-1}N\big)^{-j}u(\alpha)+(\lambda-\sigma)\Big)}\\
=&\sum_{(j,\lambda)\neq(j,\sigma)\in I(\alpha)} \hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega-(\lambda-\sigma)\Big)\overline{\hat{\psi}\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)} \quad \text{(By (3.10))}\\
=&\ \overline{\Delta^{-}_{\alpha}(\omega)} \qquad \text{(By (3.14))}.
\end{aligned}
$$

Therefore

$$
\sum_{\alpha\in\Gamma\setminus\{0\}}\big|\Delta^{+}_{\alpha}(\omega)\big| = \sum_{\alpha\in\Gamma\setminus\{0\}}\big|\Delta^{-}_{\alpha}(\omega)\big|. \tag{3.15}
$$

Replacing $\omega+\alpha$ by ω in (3.14) and using (3.15), we have

$$
\begin{aligned}
|R_n(f)| &\leqslant \left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\int_K \Big|\hat{f}(\omega)\Big|^2\big|\Delta^{+}_{\alpha}(\omega)\big|\,d\omega\right\}^{1/2}\left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\int_K \Big|\hat{f}(\omega)\Big|^2\Big|\overline{\Delta^{-}_{\alpha}(\omega)}\Big|\,d\omega\right\}^{1/2}\\
&= \int_K \Big|\hat{f}(\omega)\Big|^2\left\{\sum_{\alpha\in\Gamma\setminus\{0\}}\big|\Delta^{+}_{\alpha}(\omega)\big|\right\}d\omega. \qquad (3.16)
\end{aligned}
$$

By Lemma 3.1 and inequality (3.16), we obtain

$$
\int_K \Big|\hat{f}(\omega)\Big|^2\left\{\Delta^{+}_{0}(\omega)-\sum_{\alpha\in\Gamma\setminus\{0\}}\big|\Delta^{+}_{\alpha}(\omega)\big|\right\}d\omega \leqslant \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\big|\langle f,\psi_{j,\lambda}\rangle\big|^2, \tag{3.17}
$$

and

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\big|\langle f,\psi_{j,\lambda}\rangle\big|^2 &\leqslant \int_K \Big|\hat{f}(\omega)\Big|^2\left\{\Delta^{+}_{0}(\omega)+\sum_{\alpha\in\Gamma\setminus\{0\}}\big|\Delta^{+}_{\alpha}(\omega)\big|\right\}d\omega\\
&= \int_K \Big|\hat{f}(\omega)\Big|^2\left\{\sum_{\alpha\in\Gamma}\big|\Delta^{+}_{\alpha}(\omega)\big|\right\}d\omega. \qquad (3.18)
\end{aligned}
$$

Taking infimum over $\mathfrak{C}^{-1}\setminus\mathfrak{D}$ in (3.17) and supremum over $\mathfrak{C}^{-1}\setminus\mathfrak{D}$ in (3.18), we obtain

$$A_2\|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2 \leqslant B_2\|f\|_2^2.$$

The proof Theorem 3.3 is complete. □

With the notations in (3.10) and (3.12), define the new sets as

$$\text{(3.19)}\quad \Pi_\alpha^+ = \sup\left\{\,\left|\Delta_\alpha^+(\omega)\right| : \omega\in\mathfrak{C}^{-1}\backslash\mathfrak{D}\right\},\quad \Pi_\alpha^- = \sup\left\{\,\left|\Delta_\alpha^-(\omega)\right| : \omega\in\mathfrak{C}^{-1}\backslash\mathfrak{D}\right\}.$$

Theorem 3.4. *Suppose $\psi\in L^2(K)$ such that*

$$A_3 = \inf_{\omega\in\mathfrak{C}^{-1}\backslash\mathfrak{D}}\left\{\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2\right\} - \sum_{\alpha\in\Gamma\backslash\{0\}}\Big[\Pi_\alpha^+\Pi_\alpha^-\Big]^{1/2} > 0$$

$$B_3 = \sup_{\omega\in\mathfrak{C}^{-1}\backslash\mathfrak{D}}\left\{\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2\right\} + \sum_{\alpha\in\Gamma\backslash\{0\}}\Big[\Pi_\alpha^+\Pi_\alpha^-\Big]^{1/2} < +\infty.$$

Then $\{\psi_{j,\lambda} : j\in\mathbb{Z},\, \lambda\in\Lambda\}$ is a wavelet frame for $L^2(K)$ with bounds A_3 and B_3.

Proof. Using (3.14) in (3.13), we have

$$\begin{aligned}\left|R_n(f)\right| &\leqslant \sum_{\alpha\in\Gamma\backslash\{0\}}\left\{\int_K\left|\hat f(\omega)\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2}\left\{\int_K\left|\hat f(\omega+u(\alpha))\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2}\\ &= \sum_{\alpha\in\Gamma\backslash\{0\}}\left\{\int_K\left|\hat f(\omega)\right|^2\left|\Delta_\alpha^+(\omega)\right|d\omega\right\}^{1/2}\left\{\int_K\left|\hat f(\omega)\right|^2\left|\overline{\Delta_\alpha^-(\omega)}\right|d\omega\right\}^{1/2}\\ &\leqslant \sum_{\alpha\in\Gamma\backslash\{0\}}\Big[\Pi_\alpha^+\Pi_\alpha^-\Big]^{1/2}\int_K\left|\hat f(\omega)\right|^2.\end{aligned}$$

Proceeding similarly as in Theorem 3.2, we have

$$\int_K\left|\hat f(\omega)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2 - \sum_{\alpha\in\Gamma\backslash\{0\}}\Big[\Pi_\alpha^+\Pi_\alpha^-\Big]^{1/2}\right\}d\omega \leqslant \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2,$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2 \leqslant \int_K\left|\hat f(\omega)\right|^2\left\{\sum_{j\in\mathbb{Z}}\left|\hat\psi\Big(\big(\mathfrak{p}^{-1}N\big)^{-j}\omega\Big)\right|^2 + \sum_{\alpha\in\Gamma\backslash\{0\}}\Big[\Pi_\alpha^+\Pi_\alpha^-\Big]^{1/2}\right\}d\omega.$$

These two inequalities imply that

$$A_3\|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\psi_{j,\lambda}\rangle\right|^2 \leqslant B_3\|f\|_2^2.$$

The proof of Theorem 3.4 is complete. □

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